Polar degree of hypersurfaces
B3- Part- 2

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Classification of hypersurfaces with isolated singularities with polar degree 1 or 2

In this Part 2 of lecture B3 we continu where we ended in part1. We will show the completeness of Dolgacev's list for homaloidal hypersurfaces (Huh's theorem) and give a proof of f Huh's conjecture for $\mathrm{pol}(V) = 2$

The main reference for this part 2 is:

D. Siersma, J. Steenbrink, M.Tibar,
On Huh's conjectures for the polar degree
Journal of Algebraic Geometry 30 (2021), 189-203
ArXiv version $>$ https://arxiv.org/pdf/1805.08175.pdf

Classification of hypersurfaces with isolated singularities with polar degree 1 or 2

In this section we give a simultanious proof of two theorems:

Theorem (Huh 2014)

A projective hypersurface $V \subset \mathbb{P}^n$ with only isolated singularities and $\mathrm{pol}(V) = 1$ *is one of the following, after a linear change of homogeneous coordinates:*

List of homaloidal hypersurfaces

(i) $(n \geq 2, d = 2)$ *a smooth quadric:*

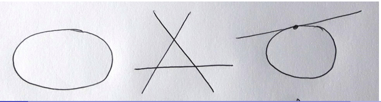
$$f = x_0^2 + \cdots + x_n^2 = 0.$$

(ii) $(n = 2, d = 3)$ *the union of three non-concurrent lines:*

$$f = x_0 x_1 x_2 = 0, \qquad (3A_1).$$

(iii) $(n = 2, d = 3)$ *the union of a smooth conic and one of its tangents:*

$$f = x_0(x_1^2 + x_0 x_2) = 0, \qquad (A_3).$$



Theorem (Siersma-Steenbrink-Tibar, 2018)

The hypersurfaces $V \subset \mathbb{P}^n$ *with isolated singularities and* $\mathrm{pol}(V) = 2$ *are only those in Huh's conjectural list.*
In particular there are no such hypersurfaces for $n > 3$.

Three normal cubic surfaces:

1. $(n = 3, d = 3)$ *a normal cubic surface containing a single line:*

$$f = x_0x_1^2 + x_1x_2^2 + x_1x_3^2 + x_2^3 = 0, \qquad (E_6).$$

2. $(n = 3, d = 3)$ *a normal cubic surface containing two lines:*

$$f = x_0x_1x_2 + x_0x_3^2 + x_1^3 = 0, \qquad (A_5, A_1).$$

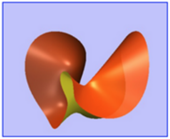
3. $(n = 3, d = 3)$ *a normal cubic surface containing three lines and three binodes:*

$$f = x_0x_1x_2 + x_3^3 = 0, \qquad (A_2, A_2, A_2).$$

Surfaces with polar degree 2

Class XX: one E_6 singularity

In Cayley's notation: U8



$W X^2 + X Z^2 + Y^3$

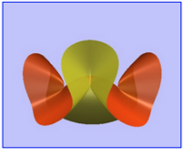
Class XIX: one A_1 and one A_5 singularity

In Cayley's notation: B6 + C2

$W X Z + Y^2 Z + X^3$

Class XXI: three A_2 singularities

In Cayley's notation: 3 B3



$W X Z + Y^3$

Huh (2014) verified the case $n = 3$ and $d = 3, 4$. He also indicated that Radu Laza has verified the conjecture in case $n = 3$ and $d = 3$. There was no general principle for doing more.

Nine plane curves, of degrees 3, 4 and 5:

1. $(n=2, d=5)$ *two smooth conics meeting at a single point and their common tangent:*

$$f = x_0(x_1^2 + x_0x_2)(x_1^2 + x_0x_2 + x_0^2) = 0, \qquad (J_{2,4}).$$

2. $(n=2, d=4)$ *two smooth conics meeting at a single point:*

$$f = (x_1^2 + x_0x_2)(x_1^2 + x_0x_2 + x_0^2) = 0, \qquad (A_7).$$

3. $(n=2, d=4)$ *a smooth conic, a tangent and a line passing through the tangency point:*

$$f = x_0(x_0 + x_1)(x_1^2 + x_0x_2) = 0, \qquad (D_6, A_1).$$

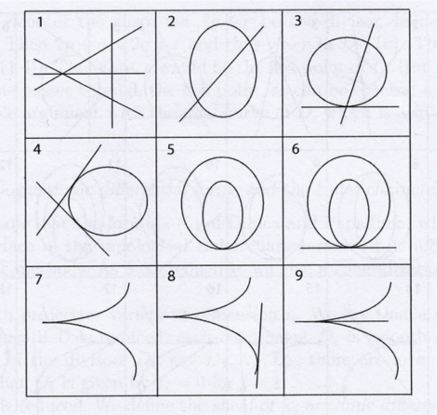
4. $(n=2, d=4)$ *a smooth conic and two tangent lines:*

$$f = x_0x_2(x_1^2 + x_0x_2) = 0, \qquad (A_1, A_3, A_3).$$

1. $(n=2, d=4)$ *three concurrent lines and a line not meeting the center point:*
$$f = x_0x_1x_2(x_0+x_1) = 0, \qquad (D_4, A_1, A_1, A_1).$$
2. $(n=2, d=4)$ *a cuspidal cubic and its tangent at the cusp:*
$$f = x_0(x_1^3 + x_0^2x_2) = 0, \qquad (E_7).$$
3. $(n=2, d=4)$ *a cuspidal cubic and its tangent at the smooth flex point:*
$$f = x_2(x_1^3 + x_0^2x_2) = 0, \qquad (A_2, A_5).$$
4. $(n=2, d=3)$ *a cuspidal cubic:*
$$f = x_1^3 + x_0^2x_2 = 0, \qquad (A_2).$$
5. $(n=2, d=3)$ *a smooth conic and a secant line:*
$$f = x_1(x_1^2 + x_0x_2) = 0, \qquad (A_1, A_1).$$

end of the list.

Curves with polar degree 2



$d=4$ $D_4 \oplus 3A_1$	$d=3$ $2A_1$	$d=4$ $D_6 \oplus A_1$
$d=4$ $2A_3 \oplus A_1$	$d=4$ A_7	$d=5$ $J_{2,4}$
$d=3$ A_2	$d=4$ $A_5 \oplus A_2$	$d=4$ E_7

The list of plane curves is included in the total list of $\mathrm{pol}(V) = 2$ curves found by Fasarella and Medeiros (2012).

Huh's main lemma: Bound from below

Lemma

For a hypersurface V with only isolated singularities, $p \in \mathrm{Sing}(V)$ and H generic through p one has:

$$\mathrm{pol}(V) \geq \mu_p(V \cap H)$$

unless V is a cone with apex p.

Huh's proof used slicing and polar methods. For our lecture we use a splitting principle, which will be explained in part 4 of this lecture B3:

$$\mathrm{pol}(V) = \alpha(V, H) + \beta(V, H)$$

where $\alpha(V, H) = \sum \alpha_p(V \cap H)$ for isolated singularities and all contributions are non-negative.

Corollary

For hypersurfaces with isolated singularities one has:

- *1. If* $\mathrm{pol}(V) = 1$ *then* V *has only* A_k*-singularities,*
- *2. If* $\mathrm{pol}(V) = 2$ *then* V *has only* A_k, D_k, E_*, J_* *singularities*

Proof.

If $\mu_p(V \cap H) = 1$ one can choose local coordinates such that the 2-jet in p becomes $x_0^2 + \cdots + x_{n-1}^2$.
By the classification of singularities (see the lecture A) it follows that the local singularity at p is of type A_k.
If $\mu_p(V \cap H) = 2$: similar, but more advanced. See the paper [SST]. □

1. *Singularities of corank* 2 *with nonzero* 3-*jet*. Besides the simple singularities $\boldsymbol{A}$, $\boldsymbol{D}$, E_6, E_7, E_8 there is a further infinite series of classes:

Notation	Normal form	Restrictions	Multiplicity μ	Modality m
$J_{k,0}$	$x^3 + bx^2y^k + y^{3k} + cxy^{2k+1}$	$k > 1,\ 4b^3 + 27 \neq 0$	$6k - 2$	$k - 1$
$J_{k,i}$	$x^3 + x^2y^k + ax^{3k+i}$	$k > 1,\ i > 0,\ a_0 \neq 0$	$6k - 2 + i$	$k - 1$
E_{6k}	$x^3 + y^{3k+1} + axy^{2k+1}$	$k \geqq 1$	$6k$	$k - 1$
E_{6k+1}	$x^3 + xy^{2k+1} + ay^{3k+2}$	$k \geqq 1$	$6k + 1$	$k - 1$
E_{6k+2}	$x^3 + y^{3k+2} + axy^{2k+2}$	$k \geqq 1$	$6k + 2$	$k - 1$

Deformation to $f_{n,d}$

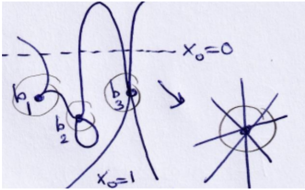
Let $H := \{x_0 = 0\} \subset \mathbb{P}^n$ be a generic hyperplane with respect to our hypersurface $V := \{f = 0\}$ and let $f = f_d + x_0 f_{d-1} + \cdots + x_0^d f_0$. We consider the family:

$$g_s(1, x_1, \cdots, x_n) := f_d + s f_{d-1} + \cdots + s^d f_0 \text{ on } \mathbb{C}^n = \mathbb{P}^n \setminus H.$$

where f_d is a general homogeneous polynomial, topologically equivalent to

$$f_{n,d} := x_1^d + \cdots + x_n^d.$$

The family g_s describes a deformation of $g_1 = f_{|\mathbb{C}^n}$ to $g_0 = f_d$, which deforms the hypersurface $V \setminus H$ to the hypersurface $\{f_d = 0\}$, with a single isolated singularity at 0, and which is topologically equivalent to the hypersurface $\{f_{n,d} = 0\}$.



We get a *multi-adjacency* of the local singularities at b_i to the singularity at O of $f_{n,d}$.

Intermezzo: Singularity invariants

One of the invariants of the singularities is its Milnor number. But there are also finer invariants. We first look at the intersectionform S on the homology of the Milnor fibre: This is the pairing given by the sequence of maps:

$$S : H_{n-1}(F) \otimes H_{n-1}(F) \to H_{n-1}(F, \partial F) \otimes H_{n-1}(F) \to \mathbb{Z}$$

The geometric meaning of S is the intersection of $(n-1)$cycles in the Milnor fibre, which has real dimension $2n-2$. The second map is a non-degenerate pairing (by Poincaré-duality).
In dimension $n = 3 mod\ 4$ the intersection form is symmetric and self-intersections are -2 . We consider this case below, other cases are almost similar.

The intersection form will give us three numbers: μ_-, μ_0, μ_+ which add up to μ and are resp. the dimensions of the negative, zero and positive eigenspaces. These numbers are studied in great detail in work of Brieskorn, Arnol'd, Gusein Zade, Ebeling. The numbers μ_-, μ_+ and the rank of S are semi-continuous in (multi)-adjacencies.
The simple singularities A_k, D_k, E_6, E_7, E_8 are the only negative definite forms. They are related to the famous Dynkin diagrams.

Continuation of proof

The multi-adjacency induces an inclusion of Milnor lattices:

$$\oplus_{b \in \mathrm{Sing}\, V} H_{n-1}(F_{b_i}) = \oplus \mathbb{Z}^{\mu_{b_i}} \longrightarrow \mathbb{Z}^{(d-1)^n} = H_{n-1}(x_1^d + \cdots + x_n^d = t)$$

with intersection matrices S_i resp S.
(See e.g. D. Siersma: Classification and Deformation of Singularities, page 74.)
Consider now the case $\mathrm{pol}(V) = 1$ We now that the singularities at b_i are of type A_k, the S_i is negative definite and the rank of S_i is equal to μ_{b_i}. For $f_{n,d}$ holds:

$$\mu_- = (d-1)^3 - 2\binom{d}{3} \;\; ; \;\; \mu_0 = 2\binom{d-1}{2} = d^2 - 3d + 2 \; ; \;\; \mu_+ = 2\binom{d-1}{3}$$

and especially $\mu_0(x_1^d + \cdots + x_n^d) \geq 2$ if $n = 2, d \geq 4$ or $n > 2, d \geq 3$.
Since $\mathrm{pol}(V) = 1$ we have $\sum rankS_i = (d-1)^n - 1$ but $\mathrm{rank}\; S \leq (d-1)^n - 2$.
The semi continuity of rank excludes all hypersurfaces, which are not in Dolgacev's list.

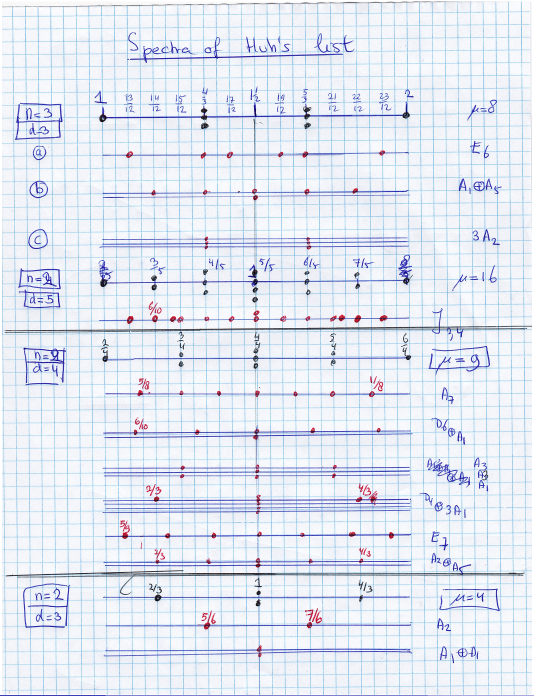
This finishes the proof in case $\mathrm{pol}(V) = 1$.
The argument above is different form Huh's argument. We will also use this for the case that $\mathrm{pol}(V) = 2$. The intersection form argument still works for $n = 3$, but for the general case we will use the semi-continuity of the spectrum.

Intermezzo: Spectrum numbers

The spectrum numbers of a singularity are related to the *eigenvalues of the monodromy*. They were well-studied by Steenbrink and Varchenko. Their definition uses the Hodge filtration of the homology of the Milnor fibre. The theory is rather complicated and we only schetch here how these number can be used in the proof of our theorem. For details see the paper [SST] and the references mentioned there. The *spectrum numbers* are defined by

$$s_i = log(\lambda_i) + n - \rho$$

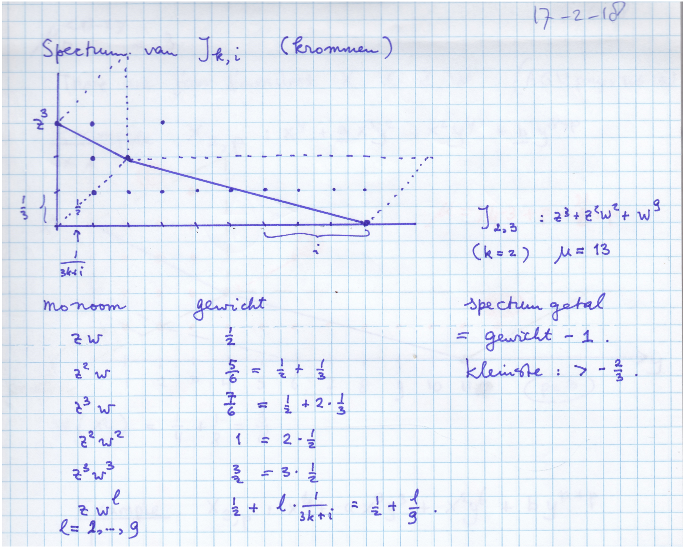
where λ_i is an eigenvalue of monodromy. The *log* is taken in the interval $(-1, 0]$ and ρ is the level in the Hodge filtration.
Spectrum numbers of quasi-homogeneous singularities or those with non-degenerate Newton diagram are computable. (Steenbrink-Varchenko). We list those, which appear in Huh's conjecture in the next slide. Each part of the list starts with the spectrum of $f_{n,d}$ and below them (with difference of Milnor numbers 2) one sees the spectra the combinations of singularities.


Spectra of Huh's list
n=3
d=3
μ=8
E6
A1⊕A5
3 A2
n=2
d=5
μ=16
J3,4
n=2
d=4
μ=9
A7
D6⊕A1
A3⊕A3⊕A1
D4⊕3A1
E7
A2⊕A5
n=2
d=3
μ=4
A2
A1⊕A1

End of the proof

The semi-continuity of the spectrum means that the spectrum numbers of the local singularities are well-devided between the spectrum numbers of $f_{n,d}$. Look e.g. at the list. We don't state this rule in detail.
The strategy of the proof is now to show that other combinations of singularities, where the Milnor numbers add up to $\text{pol}(V) - 2$ don't satisfy this rule. It is easy to rule out several big classes of hypersurfaces; in a few remaining cases one has to do some detailed computations.

You find a sample computation (made by my friend Steenbrink on the next slide). It is not the idea that you will understand this here. But anyhow you could recognize a Newton diagram and the generators of the Milnor Algebra. (NB. The text is written in Ducth, the language of The Netherrlands)



This end this 'proof''

Finiteness for isolated singularities

We could also solve:

Conjecture (Huh, 2014)

There is no projective hypersurface $V \subset \mathbb{P}^n$ *of polar degree* k *with only isolated singular points, for sufficiently large* n *and* $d = \deg V$.

A more precise statement?

Theorem (Finiteness Theorem, SST)

For any integer $k \geq 2$, *let* K_k *denote the set of pairs of integers* (n, d) *with* $n \geq 2$ *and* $d \geq 3$, *such that there exists a projective hypersurface* V *in* $\mathbb{P}^n$ *of degree* d *with isolated singularities and* $\mathrm{pol}(V) = k$.
Then K_k *is finite for any* $k \geq 2$.

The proof is also an application of spectrum numbers. We refer again to [SST].

This is the end of this part 2 of the lecture B3.

J. Huh, *Milnor numbers of projective hypersurfaces with isolated singularities.* Duke Math. J. 163 (2014), no. 8, 1525-1548.

D. Siersma, J. Steenbrink, M.Tibar, *On Huh's conjectures for the polar degree* Journal of Algebraic Geometry 30 (2021), 189-203 ArXiv version > https://arxiv.org/pdf/1805.08175.pdf

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>> https://webspace.science.uu.nl/~siers101/ClassificationDeformation2.pdf <<.