Polar degree of hypersurfaces B3 - Part- 3

Dirk Siersma

Utrecht University

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Content Polar Degree lecture B3 part 3

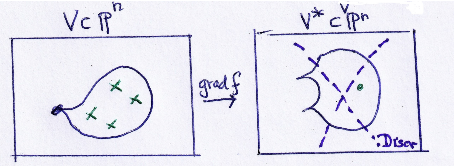
Part 3 of Lecture B3 consists of two sections:

- Polar degree, history and examples,
- Generalized Dimca-Papadima Formula
  A general reference for this second section is:
  D. Siersma, M. Tibăr, *Polar degree of hypersurfaces with 1-dimensional singularities* >> `https://arxiv.org/pdf/2105.05743.pdf` <<CLICK

Polar degree, history and examples

We recall from part 1 of this lecture B3 the definition::

Let $V \subset \mathbb{P}^n$ be a complex projective hypersurface of degree $d \geq 2$, defined by a homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, $n \geq 2$, of degree d. The *polar degree* pol(V) is defined as the *topological degree* of the gradient mapping: (sometimes called Gauss map): $\mathrm{grad} f : \mathbb{P}^n \setminus \operatorname{Sing} V \to \mathbb{P}^n$



If the degree is non-zero then the image has full dimension. The target space contains a discriminant hypersuface where the gradient map is not covering map The image of $V \setminus \operatorname{Sing} V$ is V^*, the dual of V..

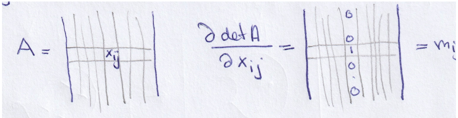
Historically the focus has been on the algebraic properties of the definition. The topological approach started around 2000.

Examples

By using the algebraic defintion one can check the following examples:

1. Smooth hypersurface $f = x_0^d + \cdots x_n^d$ has $\mathrm{pol}(V) = (d-1)^n$
2. Projective cone $f(x_0, x_1, \cdots, x_n) = h(x_1, \cdots, x_n)$ has $\mathrm{pol}(V) = 0$.
3. Hankel type determinant $f = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & 0 \\ y_4 & y_5 & 0 & 0 \end{vmatrix}$ has $\mathrm{pol}(V) = 1$.
4. The determinantal hypersurface $f = \det(A) = 0$ in $\mathbb{P}^{n^2-1}$ has $\mathrm{pol}(V) = 1$. Note that $\mathrm{grad} \det A$ is an invertible mapping due to Cramer's rule:

$$A \cdot \mathrm{grad} \det A = (detA) \cdot I$$



pol$(V) = 0$; Gradient map has lower dimensional image

Otto Hesse (1851 and 1856) claimed that if the Hessian determinant $\det(\frac{\delta^2 f}{\delta x_i \delta x_j})$ is identically zero,(,which equivalent to: *the polar degree is zero) if and only if the hypersurface V is a projective cone*; that is eliimnation of one variable is possible. His proof was considered as not 'rigourous'.

In 1875 M. Pasch showed that Hesse's claim was true for ternairy and quartic cubics.

Gordan and Noether (1876) disproved it , but showed that it is true up to birational transformations.
In fact any relation $F(\frac{\delta f}{\delta x_0}, \cdots, \frac{\delta f}{\delta x_n}) = 0$ gives pol$(V) = 0$
NB. The Hesse-case coresponds to a linear relation with constant coefficients:

$$z_0 \frac{\delta f}{\delta x_0} + \cdots z_n \frac{\delta f}{\delta x_n} = 0$$

A typical example with a 2-dimensional singular set is:

$$f = x_3^{d-1} x_0 + x_3^{d-2} x_4 x_1 + x_4^{d-1} x_2 = 0 \ \ ; d \geq 3$$

Exercise:: Compute the singular set and the relation F in this case.

pol(V) = 1 ; Homaloidal polynomials

In this case the gradient map is a birational map. The name *Cremona transformation* is used for these maps. They form an important class of maps in algebraic geometry and are well studied.
We mention some examples:

1. The determinantal hypersuface
2. Determinants of generic sub-Hankel matrices
3. The smooth quadratic hypersurface
4. The (generic) arrangement of hyperplanes $z_0 z_1 z_2 z_3 = 0$ in $\mathbb{P}^3$.
5. For hypersurfaces with isolated singularities: the list of Dolgacev (2000), cf part 1 of this lecture B3.

For the algebraic approach we refer especially to [CRS]:
C. Cilberto, F. Russo, A. Simis, *Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian*, Adv. Math. 218 (2008), no 6, 1759-1805.

General Impression: Homaloidal hypersurfaces with isolated singularities occur only in $\mathbb{P}^3$ (Dolgacev's list). For $\mathbb{P}^n$ they have severe singularties (increasing with the dimension).

Semi-continuity of pol(V) under deformations

The following result holds for any singular locus and any value of pol(V):

Proposition

The polar degree is lower semi-continuous in deformations of fixed degree d. More precisely, if f_s is a deformation of $f_0 := f$ of constant degree, then $\mathrm{pol}(V_s) \geq \mathrm{pol}(V)$ *for $s \in \mathbb{C}$ close enough to 0, where* $V_s := \{f_s = 0\}$.

proof: Let $b \in \mathbb{P}^n$ be a regular value for $\mathrm{grad} f : \mathbb{P}^n \setminus \mathrm{Sing}\, V \to \mathbb{P}^n$ and let $(\mathrm{grad} f)^{-1}(b) = \{a_1, \cdots, a_k\}$, $k \geq 0$. There exist disjoint compact neighborhoods U_i of a_i and U' of b such that $\mathrm{grad} f : (U_i, a_i) \to (U', b)$ is a diffeomorphism. Next take s so close to 0 such that $\mathrm{grad} f_s | U_i$ are still diffeomorphisms, and that $\mathrm{grad} f_s(U_i)$ still contains b in its interior. Let $W = \cap_{i=1}^k \mathrm{grad} f_s(U_i)$ and $Z_i = (\mathrm{grad} f_s)^{-1}(W) \cap U_i$. The restriction $\mathrm{grad} f_s : \bigsqcup_i Z_i \to W$ is a diffeomorphism on each component Z_i, and has topological degree pol(V). Moreover b is still a regular value for this restriction, but perhaps not anymore for the full map $\mathrm{grad} f_s : \mathbb{P}^n \setminus \mathrm{Sing}\, V_s \to \mathbb{P}^n$. Arbitrarily close to b there exist points b' which are regular values for $\mathrm{grad} f_s$. Then the number of counter-images $\#(\mathrm{grad} f_s)^{-1}(b')$ is pol(V_s) and $(\mathrm{grad} f_s)^{-1}(b')$ contains at least one point in each Z_i. This shows the inequality $\mathrm{pol}(V_s) \geq \mathrm{pol}(V)$.

Generalized Dimca-Papadima Formula

In the presense of 0- and 1-dimensional singularities we will prove the Dimca-Papadima formula and its generalization via the vanishing homolgy defined in lecture B2.

Proposition

Let $V \in \mathbb{P}^n$ *with only isolated singularities with Milnor numbers* $\mu_p(V)$. *Then:*

$$\mathrm{pol}(V) = (d-1)^n - \sum_p \mu_p(V)$$

Let $V \subset \mathbb{P}^n$ *be a hypersurface of degree* d *with a 1-dimensional singular set. Then:*

$$\mathrm{pol}(V) = (d-1)^n - \sum_{p \in \Sigma^{\mathrm{is}}} \mu_p(V) - \sum_{i=1}^{r} c_i \mu_i^{\pitchfork} + (-1)^n \sum_{q \in Q} (\chi(\mathcal{A}_q) - 1)$$

where $c_i = 2g_i + \gamma_i + (d+1)\deg \Sigma_i^c - 2$, *where* g_i *is the genus of the normalization* $\tilde{\Sigma}_i^c$ *of* Σ_i^c, *and where* $\deg \Sigma_i^c$ *denotes the degree of* Σ_i^c *as a reduced curve.*

Proof: We show the second formula; the proof of the first is included as a special case. Recall $\mathrm{pol}(V) = b_{n-1}(V - H)$ for generic H. Moreover (since the homology is concentrated) :
$\chi(V - H) = 1 + (-1)^{n-1}\,\mathrm{pol}(V)$ and $\chi(V_\epsilon - H) = 1 + (-1)^{n-1}\,\mathrm{pol}(V_\epsilon)$ and

$$\mathrm{pol}(V) - \mathrm{pol}(V_\epsilon) = (-1)^{n-1}\{\chi(V - H) - \chi(V_\epsilon - H)\}$$

In the lecture B2 (part3) we looked at the pair (V_Δ, V_ϵ) and showed that:

$$\chi(V) - \chi(V_\epsilon) = \chi^\Upsilon(V) = (-1)^{n+1}\sum_{i=1}^{\rho}\chi(\Sigma^*)\mu_i^\pitchfork - \sum_{q\in Q}\tilde{\chi}(\mathcal{A}_q) + (-1)^{n+1}\sum_{r\in R}\mu_r.$$

$$\chi(V\cap H) - \chi(V_\epsilon \cap H) = \chi^\Upsilon(V\cap H) = (-1)^{n-1}\sum_{a\in sing(V\cap H)}\mu_a(V\cap H)$$

Note $\chi(\Sigma^*) = 2g_i + \gamma_I + \nu_i - 2$, where ν_i is the number of axis points on Σ_i and $\mu_a(V\cap H) = \mu_i^\pitchfork$ if $a \in \Sigma_i$. Next apply

$$\chi(V - H) = \chi(V) - \chi(V \cap H),$$

also for V_ϵ and combine the formula's above.
Excercise: Give the details.

Polar degree and vanishing Euler characteristic

The above proof shows also that polar degree is related to the vanishing homology via the vanishing Euler-characteristic $\chi^{\Upsilon}(V) = \chi(V) - \chi(V_\epsilon) = \chi^{\Upsilon}(V)$:

Proposition

For a projective hypersurface V of arbitrary dimension of its singular set and a generic hyperplane H:

$$\mathrm{pol}(V) = (d-1)^n - (-1)^{n-1}\{\chi^{\Upsilon}(V) - \chi^{\Upsilon}(V \cap H)\}.$$

Examples and excercise: Cubic surfaces

The classification of reduced surfaces by Bruce and Wall is a good source of examples, where the above fomula can be used. We leave the compuations as an excercise.

For *isolated singularities*, we give the number of singularities and their types.
$\mathrm{pol}(V) = 8$: the smooth cubic
$\mathrm{pol}(V) = 7 : A_1$.
$\mathrm{pol}(V) = 6 : 2A_1$ or A_2.
$\mathrm{pol}(V) = 5 : 3A_1$ or A_1A_2 or A_3.
$\mathrm{pol}(V) = 4 : 4A_1$ or $A_2 2A_1$ or A_3A_1 or $2A_2$ or A_4 or D_4.
$\mathrm{pol}(V) = 3 : A_3 2A_1$ or $A_1 2A_2$ or A_4A_1 or A_5 or D_5.
$\mathrm{pol}(V) = 2 : 3A_2$ or A_5A_1 or E_6.
$\mathrm{pol}(V) = 1$: no homaloidal surfaces.
$\mathrm{pol}(V) = 0 : \tilde{E}_6$, which is a cone.

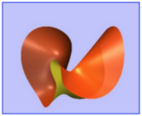
Pictures of cubic surfaces

It is interesting to look at pictures of cubic surfaces. You can draw the real parts by yourselves using software SURFER `https://www.imaginary.org/program/surfer` or find them on the web. Here are allready examples.

Surfaces with polar degree 2

Class XX: one E_6 singularity

In Cayley's notation: U8



$W X^2 + X Z^2 + Y^3$

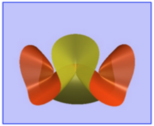
Class XIX: one A_1 and one A_5 singularity

In Cayley's notation: B6 + C2

$W X Z + Y^2 Z + X^3$

Class XXI: three A_2 singularities

In Cayley's notation: 3 B3



$W X Z + Y^3$

Next *irreducible* cubics with *nonisolated singularities* :
(CN) cone over a nodal curve,
(CC) cone over a cuspidal curve
both with $\mathrm{pol}(V) = 0$ because they are cones, and two other cases:
(E1) $x_0^2 x_2 + x_1^2 x_3$; $\mathrm{pol}(V) = 2$
(E2) $x_0^2 x_2 + x_0 x_1 x_3 + x_1^3$; $\mathrm{pol}(V) = 1$,
where the singular set is a projective line with two special points of type D_∞ in the first case, and a single special point of type $J_{2,\infty}$ in the second case
Among the *reducible* cubics there are only the following three cases with non-zero polar degree:

(QP) The union of a smooth quadratic with a general hyperplane: $\mathrm{pol}(V) = 2$,
(QT) The union of a smooth quadratic with a tangent hyperplane: $\mathrm{pol}(V) = 1$,
(CP) The union of a quadratic cone and a general hyperplane: $\mathrm{pol}(V) = 1$.

All the other reducible cubics are cones and thus have $\mathrm{pol}(V) = 0$.

From all these results:

Proposition

There are only three homaloidal cubic surfaces, all with nonisolated singularities.

J.W. Bruce, C.T.C.Wall, *On the Classification of Cubic Surfaces*, J. London Math. Soc (2), 19 (1979), 245-256.

C. Cilberto, F. Russo, A. Simis, *Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian*, Adv. Math. 218 (2008), no 6, 1759-1805.

A. Dimca, S. Papadima, *Hypersurface complements, Milnor fibers and higher homotopy groups of arrangements.* Ann. of Math. (2) 158 (2003), no. 2, 473-507.

I. Dolgachev, *Polar Cremona transformations.* Michigan Math. J. 48 (2000), 191-202.

J. Huh, *Milnor numbers of projective hypersurfaces with isolated singularities.* Duke Math. J. 163 (2014), no. 8, 1525-1548.

D. Siersma, M. Tibăr, *Polar degree of hypersurfaces with 1-dimensional singularities*
`https://arxiv.org/pdf/2105.05743.pdf` <<CLICK