Course F2
Non-isolated singularities
the 1 dimensional case

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Singularities and Applications
on line 2021, updated july 2022

Outline

Introduction

Local theory of singularties

- Milnor fibration
- 1-dimensional singular set
- Homology via deformation
- Betti numbers
- Examples

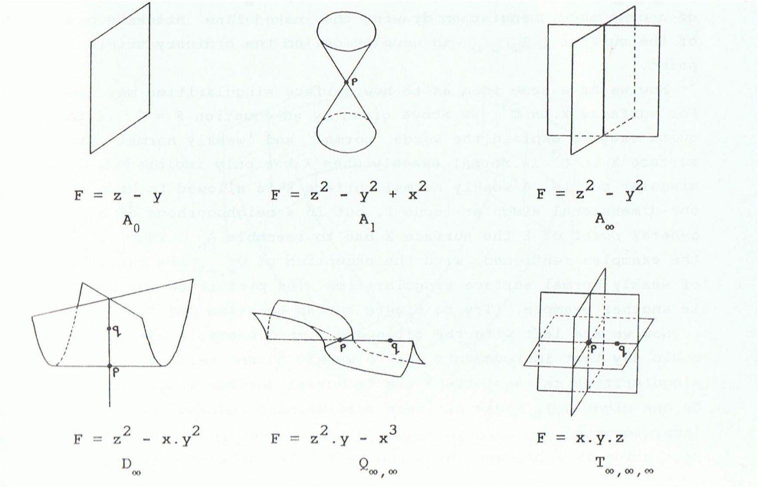
Global case: Projective hypersurfaces

- Vanishing Homology
- Concentration of VH and Betti numbers
- Examples

Joint work with Mihai Tibăr.

Introduction

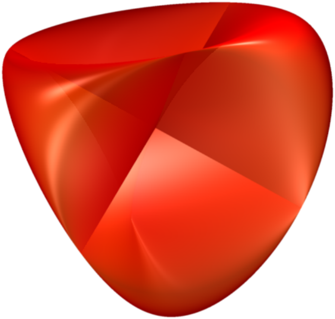
Some examples of singularities:



What are the Singular points ?

Introduction

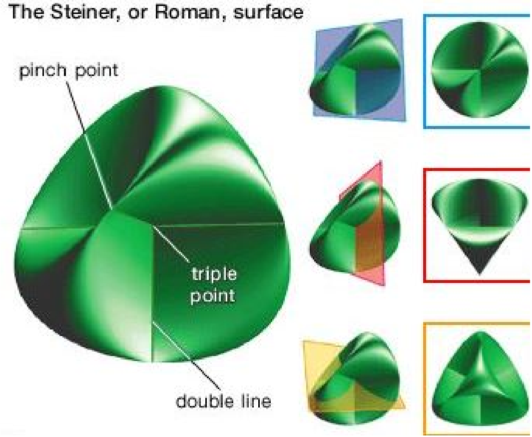
Roman Steiner Surface



Find the 3 axis of singular points. Do you see the Whitney umbrella's (D_∞-points) ? How many ?

Introduction

Roman Steiner Surface



Goals

The main goal of this lecture is to determine topologial invariants of the singular hypersurface, which come from the Milnor fibre of its smoothing.

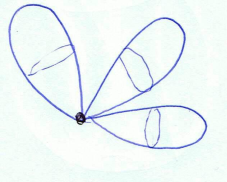
The lecture consits of several parts. In the first part we study the local case and focus on Milnor fibres and its interaction with the Milnor fibres and monodromies of transversal sections.

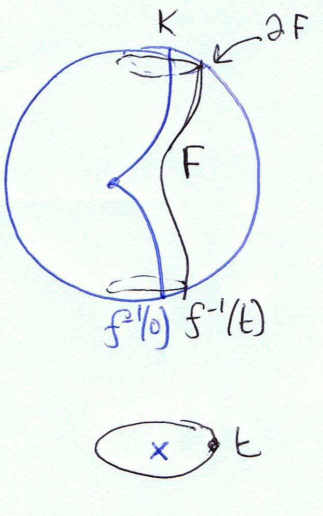
A main reference is the paper Dirk Siersma *The vanishing topology of non-isolated singularities*, in: New Developments in Singularity Theory (Cambridge 2000), pp. 447-472; NATO Sci. Ser. II Math. Phys. Chem. 21, Kluwer, 2001. CLICK!

This is an updated version of the notes for the course B2 from the 2021-CIMPA school on line. At several places the current manuscript refers to the notes and to other coursesof that year. Especially also to the 2021-course C of Laurentiu Maxim, which was reviewed in his 2022-course F3. All his material you can find on his webspace. CLICK!

Isolated singularities

We recall first some well known facts:
$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ holomorphic has a Milnor fibration.
with Milnor neigborhood:
$E = f^{-1}(\Delta) \cap B(O, r)$
and Milnor fibre:
$F = f^{-1}(t) \cap B(O, r)$
$F \cong S^n \vee \cdots \vee S^n$
Milnor number $\mu = b_n(F)$





Complex Morse Singularity

Complex polynomial:

$$f = z_0^2 + \cdots + z_n^2$$

also called A_1. The Milnor fibre F is the intersection of a ball with

$$z_0^2 + \cdots + z_n^2 = \delta$$

It contains the real n-sphere:

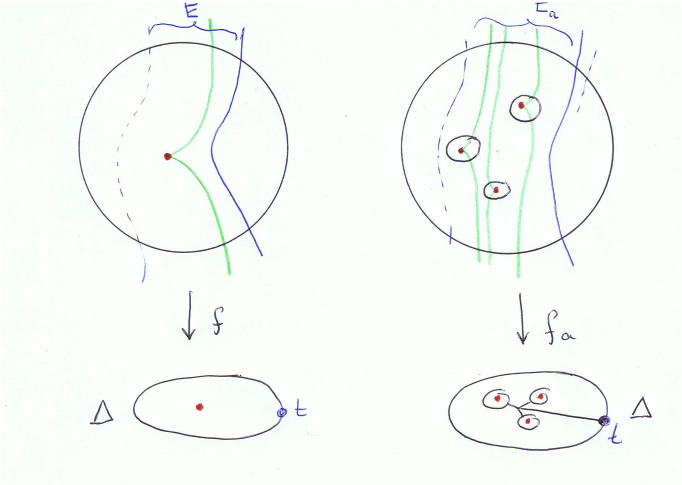
$$x_0^2 + \cdots + x_n^2 = \delta$$

In fact

$$F \cong S^n$$

By adding a $n+1$ cell F becomes contractible.

Milnor’s Bouquet theorem via deformation



$$H_*(E,F) \simeq H_*(F_{\bullet}, F_{\rightarrow}) = \bigoplus_{r \in R} H_*(F_{\circledcirc_r}, F_{\circ_r}) = \bigoplus_{r \in R} H_*(E_r, F_r) = \begin{cases} \mathbb{Z}^{\sum \mu_r} & * = n \\ 0 & * \neq n \end{cases}$$

Non-isolated Singularities

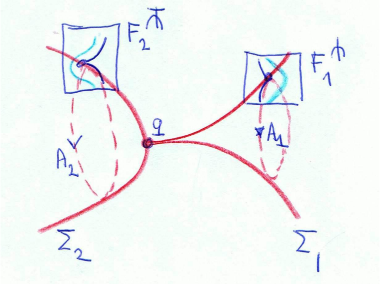
Kato-Matsumoto result for s-dimensional singular set Σ:

Proposition

$\tilde{H}_k(F) = 0$ *ouside the range:* $n - s \leq k \leq n$

In case $\dim \Sigma = 1$, the only non vanishing homology groups are:

- $H_n(F)$ (always free),
- $H_{n-1}(F)$, which can have torsion.



1-dimensional singular set

Let $\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_r$
$F_i^\pitchfork$ is the Milnor fiber of the restriction of f to the transversal hyperplane at some $x \in \Sigma_i \setminus \{0\}$, which is an isolated singularity, $\tilde{H}_*(F_i^\pitchfork)$ is concentrated in dimension $n-1$.

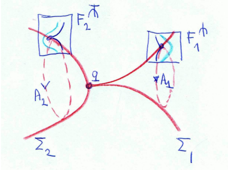
This defines a local system on $\Sigma_i \setminus \{0\}$ with fibre $\tilde{H}_{n-1}(F_i^\pitchfork) = \mathbb{Z}^{\mu_i^\pitchfork}$. On this group there acts the *local system monodromy* (also called *vertical monodromy*):

$$A_i : \tilde{H}_{n-1}(F_i^\pitchfork) \to \tilde{H}_{n-1}(F_i^\pitchfork).$$

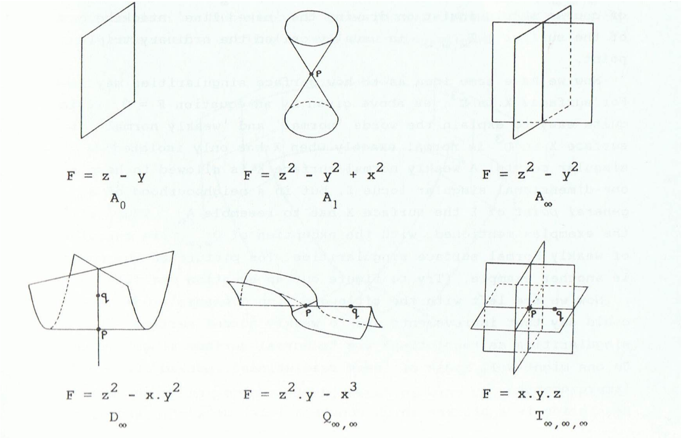
$$\partial F = \partial_1 F \cup \partial_2 F,$$

vanishing zone (near to Σ):

$$\partial_2 F = \bigsqcup_{i=1}^{r} \partial_2 F_i.$$



Examples



Vertical monodromies:

$$A_\infty : \; I \qquad D_\infty : \; -I \qquad Q_{\infty,\infty} : \; -I \qquad T_{\infty,\infty,\infty} : \; I, I, I$$

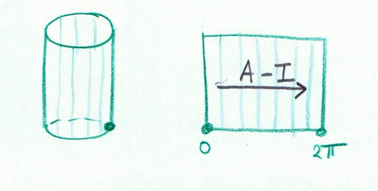
Wang Sequence

Each $\partial_2 F_i$ is fibered over the link of Σ_i with fiber $F_i^{\pitchfork}$. The Wang sequence of this fibration:

$$0 \to H_n(\partial_2 F_i) \to H_{n-1}(F_i^{\pitchfork}) \stackrel{A_i - I}{\to} H_{n-1}(F_i^{\pitchfork}) \to H_{n-1}(\partial_2 F_i) \to 0$$

So $\quad H_n(\partial_2 F) \; = \; \bigoplus_{i=1}^{r} \mathrm{Ker}(A_i - I)$ (free group)

and $H_{n-1}(\partial_2 F) \cong \bigoplus_{i=1}^{r} \mathrm{Coker}\,(A_i - I)$.



Local 6-term sequence

Exact sequence of $(F, \partial_2 F)$ reduces to:

$$0 \to H_{n+1}(F, \partial_2 F) \to H_n(\partial_2 F) \to H_n(F) \to$$
$$\to H_n(F, \partial_2 F) \to H_{n-1}(\partial_2 F) \to H_{n-1}(F) \to 0$$

Moreover by variation-isomorphisms and duality:

$H_{n+1}(F, \partial_2 F) \cong H_{n-1}(F)^{\text{free}}$ and $H_n(F, \partial_2 F) \cong H_n(F) \oplus H_{n-1}(F)^{\text{torsion}}$.

Note $H_n(\partial_2 F) = \bigoplus_{i=1}^{r} \mathrm{Ker}(A_i - I)$

and $H_{n-1}(\partial_2 F) \cong \bigoplus_{i=1}^{r} \mathrm{Coker}\,(A_i - I)$ play a crucial role.

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and $H_{n-1}(\partial_2 F) \cong \bigoplus_{i=1}^{r} \mathrm{Coker}\,(A_i - I)$ play a crucial role.

Corollary: $b_{n-1}(F) \leq \sum \mu_i^{\pitchfork}$

This is an important inequality !

Topological consequences

When is ∂F or K a topological sphere ?

8.1 PROPOSITION. *Let $n > 2$. The following are equivalent:*

1. *∂F is a topological sphere S^{2n-1}*
2. $\begin{cases} (a) & H_{n-1}(F) = 0 \\ (b) & \textit{The intersection form } S \textit{ on } H_n(F) \text{ has determinant } \pm 1 \end{cases}$
3. $\begin{cases} (a) & \det(A_i - 1) = \pm 1 \textit{ for all } i = 1, \ldots, r \\ (b) & \det(T_n - 1) = \pm 1 \end{cases}$

NB. For $n = 2$ replace (1) by: ∂F is a homology sphere.

NB. This generalizes Milnor's result for isolated singularities:
1. $\Longleftrightarrow$ 3.(b)

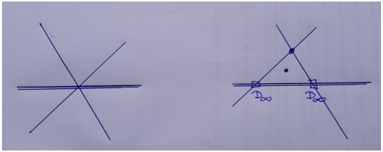
Randell showed: K is a homotopy (homology) sphere $\leftrightarrow$ T_q is an isomorhism for $q = n, n-1$.

From this: ∂F is a topological sphere iff K is a homotopy sphere and $\det(A_i - I) = \pm 1$

part 2: Homology via deformation

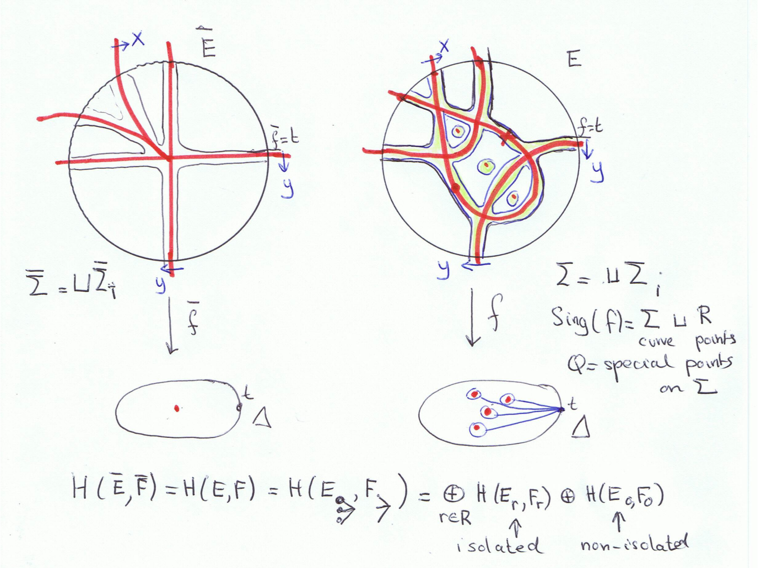
The goal of this part is to determine how admissible deformations will be helpfull to determine the toplogy of the Milnor fibre. A toy example is

$$f(x,y) = y^2(x^2 - (y-s)^2)$$



The main reference for this section is:
Dirk Siersma, Mihai Tibar: Milnor Fibre Homology via Deformation, CLICK ArXiv Math.AG 1512.02840, December 2015. In W.Decker et al. (eds), Singularities and Computer Algebra, Springer 2017, 305-322;

Vanishing Homology via deformation



$$H(E,F) = \oplus_{r\in R} H(E_r,F_r) \oplus H(E_0,F_0)$$

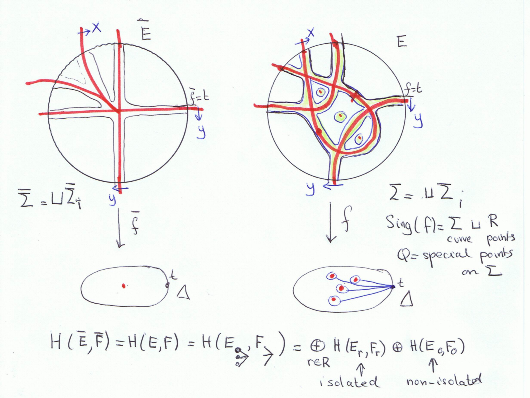
Let $\overline{f} : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with singular locus $\overline{\Sigma}$ of dimension 1. $\overline{f}$ has Milnor pair $(\overline{E}, \overline{F})$, $\overline{\Sigma} = \overline{\Sigma_1} \cup \ldots \cup \overline{\Sigma_r}$, etc.

Admissible Deformation $f_a : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with $f_0 = \overline{f}$.

- Fibres of f_a intersect the boundary of the Milnor sphere B transversally (stratified sence),
- $\Sigma \cap \partial B \cong \overline{\Sigma} \cap \partial B$, including transversal types and transversal monodromies.
- for all a small enough the fibration over the boundary of Δ is equivalent to the Milnor fibration of $\overline{f}$,
- $E \cong \overline{E}$ (contractible).

Note that $\overline{\Sigma} = \bigcup_{i \in \overline{I}} \overline{\Sigma}$ and $\Sigma = \bigcup_{i \in I} \Sigma_i$ can have a different number of irreducible components.

Additivity of Vanishing Homology

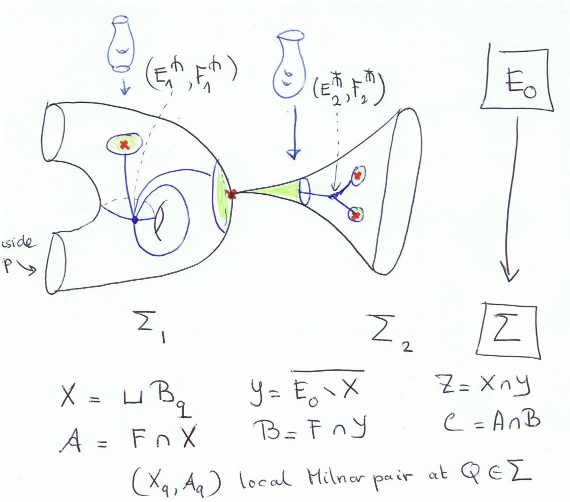


$H(E, F) = \oplus_{r \in R} H(E_r, F_r) \oplus H(E_0, F_0)$

We call this splitting: additivity of vanishing homology. It is done by excision and deformation retraction.

As a second step we can concentrate each contribution to a small neighborhood of the singular set.

'Fibration' over the singular set



Remind: surjectivity

$$H_{n-1}(F_i^{\pitchfork}) \twoheadrightarrow H_{n+1}(\partial_2 \mathcal{A}_q) = \oplus\,\mathrm{coker}(A_s - I) \twoheadrightarrow H_{n-1}(\mathcal{A}_q)$$

Homology of Milnor fibre

Theorem

- $H_*(F)$ *is concentrated in dimensions* $n-1$ *and* n*,*
- $\chi(E,F) = \sum_{q\in Q}\chi(\mathcal{X}_q,\mathcal{A}_q) + \sum_{i=1}^{\rho}\chi(\Sigma_i^*)\mu_i^{\pitchfork} + (-1)^{n+1}\sum_{r\in R}\mu_r.$
- *There is an surjection* $\oplus_{i=1}^{\rho} H_{n-1}(F_i^{\pitchfork}) \to H_{n-1}(F)$ *and* $H_{n-1}(F)$ *has a description as cokernel.*
- $b_{n-1}(F) \leq \sum_i \min_{q\in\Sigma_i^*} b_{n-1}(\mathcal{A}_q)$

Corrolary

- $b_{n-1}(F) \leq \sum_{i=1}^{\rho}\mu_i^{\pitchfork}$; ($\rho =$ *number after deformation!)*
- *(irred. case) If there is at least one q such that* $H_{n-1}(\mathcal{A}_q) = 0$ *then* $H_{n-1}(F) = 0$ *;* *concentration in dimension n only!*

About the proof

Use (relative) CW-decompositions of $\Sigma_i^* := \Sigma_i \setminus B$ and the transversal and local Milnor fibres. These are related to cells in only 2 dimensions. We concentrate on the vanishing homology near the 1-dimensonial part. Its vanishing homology corresponds to the union terms in the Mayer-Vietoris sequence:

$$
\begin{aligned}
&0 \to H_{n+1}(\mathcal{Z},\mathcal{C}) \to H_{n+1}(\mathcal{X},\mathcal{A}) \oplus H_{n+1}(\mathcal{Y},\mathcal{B}) \to H_{n+1}(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B})\\
&\to H_n(\mathcal{Z},\mathcal{C}) \xrightarrow{j} H_n(\mathcal{X},\mathcal{A}) \oplus H_n(\mathcal{Y},\mathcal{B}) \to H_n(\mathcal{X}\cup\mathcal{Y},\mathcal{A}\cup\mathcal{B}) \to 0.
\end{aligned}
$$

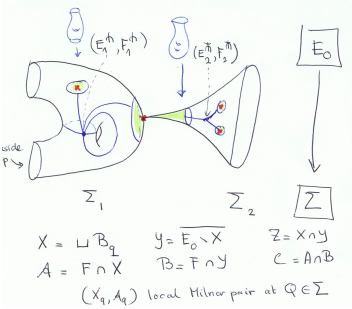
The $\chi(F)$-formula follows easily. Moreover: $H_{n-1}(F) = \operatorname{coker} j$

$$
\begin{aligned}
H_n(\mathcal{Z},\mathcal{C}) &= \oplus_{q\in Q} \oplus_{s\in S_q} H_n(\mathcal{Z}_s,\mathcal{C}_s),\\
H_n(\mathcal{X},\mathcal{A}) &= \oplus_{q\in Q} H_n(\mathcal{X}_q,\mathcal{A}_q).\\
H_n(\mathcal{Y},\mathcal{B}) &= \oplus \frac{\mathbb{Z}^{\mu_i^{\pitchfork}}}{\operatorname{Im} A_j - I} \text{ (j over } \Sigma_i^* \text{ loops)}
\end{aligned}
$$

First component is direct sum of local 6-term sequences. Both component of j are surjective!

About the Betti numbers

Next construct the vanishing homology, starting from $(E_i^{\pitchfork}, F_i^{\pitchfork})$.



Extend over Σ_i^* by adding extra cells for loops.

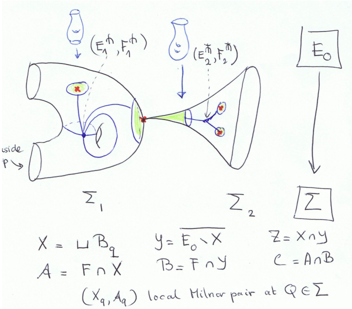
- Adding relations for genus and outside loops : $\mathrm{Im}(A_j - I)$,
- Adding relations for Q-point loops: $\mathrm{Im}(A_s - I),\ s \in S_q$

At this moment we have covered Σ_i^* and as a consequence

$$b_{n-1}(F) \leq \sum_{i=1}^{\rho} \min_{s,j} \dim \operatorname{coker}(A_* - I) \leq \sum_{i=1}^{\rho} \mu_i^{\pitchfork}.$$

About the Betti numbers

Extend over Σ_i^* by adding extra cells for loops.



- Adding relations for genus and outside loops : $\mathrm{Im}(A_j - I)$,
- Adding relations for Q-point loops: $\mathrm{Im}(A_s - I)$, $s \in S_q$
- We finally 'plug in' a contribution $H_{n-1}(\mathcal{A}_q)$ for each point $q \in Q$ and get: $b_{n-1}(F) \leq \sum_i \min_{q \in \Sigma_i^*} b_{n-1}(\mathcal{A}_q)$

Remind: surjectivity

$$H_{n-1}(F_i^\pitchfork) \twoheadrightarrow H_{n+1}(\partial_2 \mathcal{A}_q) = \oplus \operatorname{coker}(A_s - I) \twoheadrightarrow H_{n-1}(\mathcal{A}_q)$$

Transversal A_1 singularities

Let Σ be a 1-dimensional icis with transversal type A_1 . f_a a deformation with Σ_a smooth (equal to the Milnor fibre of the singular curve Σ) having only A_∞ and D_∞ and A_1 -singularites.

Theorem

The homotopy type of the Milnor fibre, F is a bouquet of spheres:

if $\#D_\infty > 0$, *then* $F \simeq S^n \vee \ldots \vee S^n$;
if $\#D_\infty = 0$, *then* $F \simeq S^{n-1} \vee S^n \vee \ldots \vee S^n$.
Moreover, $\tilde{b}_n(F) - \tilde{b}_{n-1}(F) = \mu(\Sigma) - 1 + 2\#D_\infty + \#A_1$

The same statement on the homotopy type of F is true in all cases where f allows a deformation with only A_∞, D_∞, and A_1 -singularities.
Classical case: Σ is a smooth line with transversal type A_1. All these isolated line singularities (except A_∞) have S^n- bouquets.

Theo de Jong's list of singularities

Σ is a smooth line, with generic transversal types

$$S \in \{A_1, A_2, A_3, D_4, E_6, E_7(n=2), E_8(n=2)\}.$$

There is a list of building block singularities, F_iS of type S.
Deformations with only F_iS points and A_1 points exist.

The Milnor fibre has the homotopy type of a bouquet:

$$F \simeq \bigvee_{\epsilon} S^{n-1} \vee \bigvee_{\mu+\epsilon} S^n$$

with

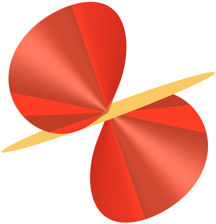
$$\mu = b_n(F) - b_{n-1}(F) = \sum \alpha_i h_i + \#A_1 - \mu^{\pitchfork},$$

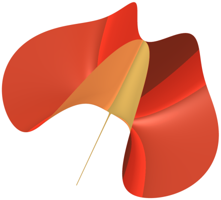
where h_i is the number of F_iS points and α_i and ϵ can be computed explicitly. Only in exceptional cases is $\epsilon \neq 0$ and in these cases ϵ is small; in fact 0, 1 or $\mu^{\pitchfork}$.

Transversal type A_3. Can be deformed into

F_1A_3 : $f = xz^2 + y^2z$; $F \overset{\text{ht}}{\simeq} S^1$

F_2A_3 : $f = xy^4 + z^2$; $F \overset{\text{ht}}{\simeq} S^2$





Transversal type A_3. Can be deformed into

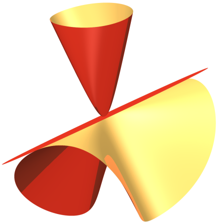
$F_1 A_3$: $f = xz^2 + y^2 z$; $F \overset{\text{ht}}{\simeq} S^1$

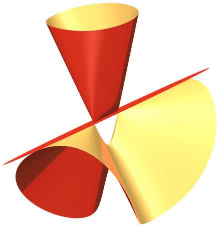
$F_2 A_3$: $f = xy^4 + z^2$; $F \overset{\text{ht}}{\simeq} S^2$

(a) $F \overset{\text{ht}}{\simeq} S^{n-1} \vee S^n \cdots \vee S^n$ if $\# F_2 A_3 = 0$,

(b) $F \overset{\text{ht}}{\simeq} S^n \vee \cdots \vee S^n$ else.

Deformation of type (a) $f_s = (x^k - s)z^2 + yz^2 + y^2 z$.





$\Sigma = 3$ axis with transversal A_1

$T_{\infty,\infty,\infty}$: $f = xyz$,
has $F \cong S^1 \times S^1$ and no admissible deformations.

$$f = x^2y^2 + y^2z^2 + z^2x^2.$$

There exist two totally different deformations of f_0.

- $f_s = x^2y^2 + y^2z^2 + z^2x^2 + sxyz$, . in which Σ_s consists of the three coordinate axes:

$$\#A_1 = 4 \quad \#D_\infty = 6 \quad \#T_{\infty\infty\infty} = 1$$

On each Σ_i we have $\#D_\infty = 2$. Consequence $b_1 = 0$.

- $f_a = (xy - a_2x - a_1y)^2 + (yz + a_3y - a_2z)^2 + (xz + a_3x - a_1z)^2$,

Here, Σ_a is a Milnor fibre of Σ, and $\mu(\Sigma) = 2$.

$$\#A_1 = 6 \quad \#D_\infty = 4 \quad \#T_{\infty\infty\infty} = 0 \quad \chi(\Sigma_s) = -1.$$

Since the deformation has only A_∞, D_∞, and A_1 -points, we conclude that $F \simeq S^2 \vee \ldots \vee S^2$, exactly 15 spheres.

Part 3: Projective hypersurfaces

In this 3rd part we will study the global topology of hypersurfaces with a 1 dimensional singular set.

The main reference for this part is the publication:
D. Siersma, M. Tibăr: *Projective hypersurfaces with 1-dimensional singularities*, Europ. J. Math. 3 (2017), 565-586 CLICK; arXiv: 11411.2640 Math.AG, november 2014.

There exists also another video related to this part of the lecture made during the School and Workshop on Singularities in geometry, topology, foliations and dynamics (Merida 2014). If you want to watch it you can CLICK HERE.

An overview of the topology of projective hypersurfaces with any dimensional singular set was treated in 2022-course F3 by Laurentiu Maxim. You can find his notes CLICK HERE. This is a short overview of his 2021-course C.

Projective hypersurfaces

What is the homology of a complex projective hypersurface $V \subset \mathbb{P}^{n+1}$ of degree d ?

Proposition

If V is smooth:

- $H_k(V, \mathbb{Z}) \cong H_k(\mathbb{P}^n, \mathbb{Z})$ *if* $k \neq n$*,*
- $H_n(V, \mathbb{Z}) = \mathbb{Z}^m$*,*
- $\chi(V) = m + 1 = n + 2 - \frac{1}{d}[1 + (-1)^{n+1}(d-1)^{n+2}]$*.*

The follows from the Lefschetz Hyperplane Theorem and the Poincare duality for smooth manifolds and direct computation of the Euler characteristic.

More details: See Course C of Laurentio Maxim, lecture 1b.

Projective hypersurfaces

What happens of V has singularities ?

The classical Lefschetz Hyperplane Theorem (LHT) implies: $H_k(V,\mathbb{Z}) \overset{\simeq}{\to} H_k(\mathbb{P}^{n+1},\mathbb{Z})$ for $j < n$ and an epimorphism for $j = n$, for any singular locus $\mathrm{Sing}\, V$.

V is a CW-complex of dimension $2n$. What are the remaining homology groups $H_k(V,\mathbb{Z})$ for $n \leq j \leq 2n$?

We will compare the homology of V with a smoothing V_ϵ and use the concept of vanishing homology.

In this course (B) we restrict to hypersurfaces with isolated singularities or with a1-dimensional singular set.
Course C contains the general case.

Vanishing Homology

Definition

Let f be a homogeneous polynomial of degree d, which defines V and let h_d general of degree d. Consider the pencil:

$$\pi : \mathbb{V}_\Delta = \{(x, \varepsilon) \in \mathbb{P}^{n+1} \times \Delta \mid f + \varepsilon h_d = 0\} \longrightarrow \Delta$$

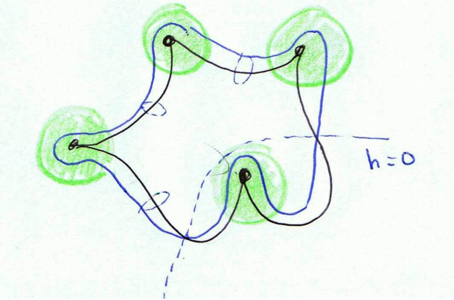
with generic fibre V_ε nonsingular for all $\varepsilon \in \Delta^*$, a small disk around $0 \in \mathbb{C}$. $\mathbb{V}_\Delta$ retracts to $V = V_0$.
We define:

$$H_*^\curlyvee(V) := H_*(\mathbb{V}_\Delta, V_\varepsilon; \mathbb{Z})$$

and call it the *vanishing homology* of V.

- The vanishing homology compares V to the smooth hypersurface V_ε of the same degree.
- It does not depend on the particular smoothing of degree d, thus is an invariant of V.
- It is also intermediate step towards computing the homology of singular hypersurfaces.

Vanishing Homology for hypersurfaces with isolated singularities



Additivity of vanishing homology:
$H_*(\mathbb{V}_\Delta, V_\varepsilon) \simeq \oplus_{r \in R} H_*(B_r, B_r \cap V_\varepsilon)$ concentrated !
$H_{n+1}(B_r, B_r \cap V_\varepsilon) \cong \mathbb{L}_r = \mathbb{Z}^{\mu(V,r)}$.

Lemma
$H_k^\Upsilon(V) = 0$ if $k \neq n+1$, $\quad H_{n+1}^\Upsilon(V) = \bigoplus_{r \in R} \mathbb{L}_r$.

From VH to homology for isolated singularities

The homology sequence of the pair $(\mathbb{V}_\Delta, V_\varepsilon)$ gives 5-terms exact sequence:

$$0 \to H_{n+1}(V_\varepsilon) \to H_{n+1}(V) \to \bigoplus_{r\in R} \mathbb{L}_r \xrightarrow{\Phi_n} \mathbb{L} \to H_n(V) \to 0$$

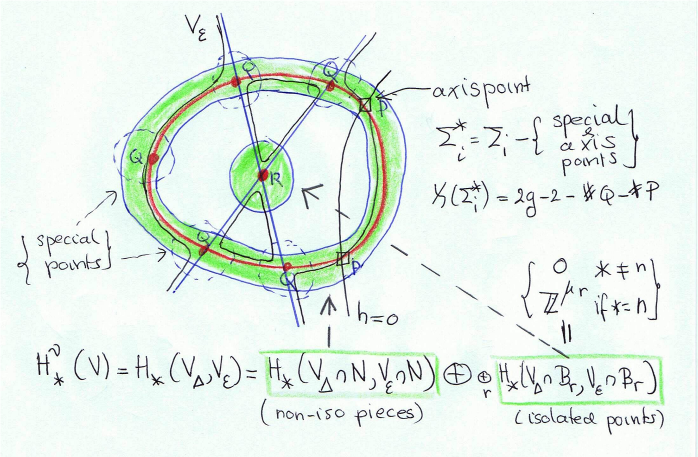
where $\mathbb{L} := H_n(V_\varepsilon)$ is the intersection lattice and the map Φ_n is identified to the boundary map $H_{n+1}(\mathbb{V}_\Delta, V_\varepsilon) \to H_n(V_\varepsilon)$.

Proposition

- $H_k(V) \simeq H_k(\mathbb{P}^n)$ *for* $k \neq n, n+1$,
- $H_{n+1}(V) \simeq H_{n+1}(\mathbb{P}^n) \oplus \ker \Phi_n$,
- $H_n(V) \simeq \operatorname{coker} \Phi_n$.

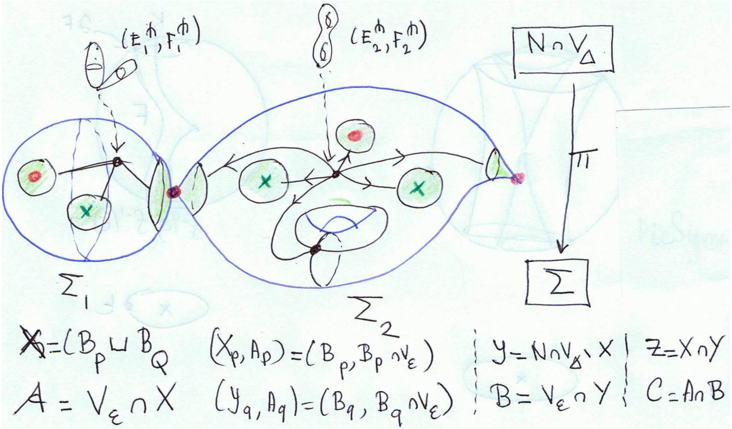
What about Φ_n ?

Additivity of Vanishing Homology: non isolated



0 and 1 dimensional singularities

‘Fibration’ over the singular set



Mayer-Vietoris decomposition $(\mathbb{V}_\Delta \cap N, V_\varepsilon \cap N)$

Remind:

$$H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) \subset H_{n+1}(\partial_2 \mathcal{A}_q) = \oplus \ker(A_s - I) \subset H_n(E_i^\pitchfork, F_i^\pitchfork)$$

Concentration of VH and Betti numbers

Theorem

- *$H_*^{\Upsilon}(V)$ is concentrated in dimensions $n+1$ and $n+2$,*
- *There is an embedding $H_{n+2}^{\Upsilon}(V) \subset \oplus_{i=1}^{\rho} H_n(E_i^{\pitchfork} F_i^{\pitchfork})$*
- $\chi^{\Upsilon}(V) =$
  $(-1)^{n+1} \sum_{i=1}^{\rho} \chi(\Sigma^*)\mu_i^{\pitchfork} - \sum_{q \in Q} \tilde{\chi}(\mathcal{A}_q) + (-1)^{n+1} \sum_{r \in R} \mu_r.$
- *(stated in the irreducible case)*
  $H_{n+2}^{\Upsilon}(V) = \bigcap_{q \in Q} H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) \cap \bigcap_{j \in G} \ker(A_j - I).$

Corrolary

- $b_{n+2}^{\Upsilon}(V) \leq \sum_{i=1}^{\rho} \mu_i^{\pitchfork}$
- *(irred. case) If there is at least one q such that*
  $H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) = 0$ *then* $b_{n+2}^{\Upsilon}(V) = 0$ *;*
  concentration in dimension $n+1$ only!

Remind: $H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) \cong H_{n-2}(\mathcal{A}_q)^{\text{free}}$

About the proof

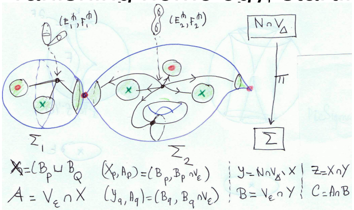
Use (relative) CW-decompositions of Σ_i^* and the transversal and local Milnor fibres. These are related to cells in only 2 dimensions.
A Mayer-Vietoris sequence-argument can be used to show concentration and the vanishing homology-6-term-sequence

$$0 \to H_{n+2}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \to H_{n+1}(\mathcal{Z}, \mathcal{C}) \to H_{n+1}(\mathcal{X}, \mathcal{A}) \oplus H_{n+1}(\mathcal{Y}, \mathcal{B})$$
$$\to H_{n+1}(\mathcal{X} \cup \mathcal{Y}, \mathcal{A} \cup \mathcal{B}) \to H_n(\mathcal{Z}, \mathcal{C}) \xrightarrow{j} H_n(\mathcal{X}, \mathcal{A}) \oplus H_n(\mathcal{Y}, \mathcal{B}) \to 0$$

The $\chi^{\Upsilon}(V)$-formula follows easily.

About the Betti numbers

Next construct the vanishing homology, starting from $(E_i^{\pitchfork}, F_i^{\pitchfork})$.



Extend over Σ_i^* by adding extra cells for loops.

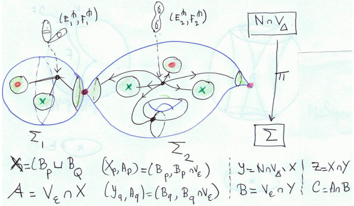
- Adding relations for genus loops, restrict to: $\ker(A_j - I),\ j \in G$
- Adding relations for Q-point loops, restrict to: $\ker(A_s - I),\ s \in S_q$
- Adding NO relations for axis point loops, since $A_p = I$

At this moment we have covered Σ_i^* and as a consequence

$$b_{n+2}^{\Upsilon}(V) \leq \sum_{i=1}^{\rho} \min_{s,j} \dim \ker(A_* - I) \leq \sum_{i=1}^{\rho} \mu_i^{\pitchfork}.$$

About the Betti numbers

Extend over Σ_i^* by adding extra cells for loops.



- Adding relations for genus loops, restrict to:
  $\ker(A_j - I),\ j \in G$
- Adding relations for Q-point loops, restrict to:
  $\ker(A_s - I),\ s \in S_q$
- Adding NO relations for axis point loops, since $A_p = I$
- We finally 'plug in' a contribution $H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q)$ for each point $q \in Q$ and get:
  $H^{\Upsilon}_{n+2}(V) = \bigcap_{q \in Q} H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) \cap \bigcap_{j \in G} \ker(A_j - I).$

$$H_{n+1}(\mathcal{A}_q, \partial_2 \mathcal{A}_q) \subset H_{n+1}(\partial_2 \mathcal{A}_q) = \bigoplus \ker(A_s - I) \subset H_n(E_i^{\pitchfork}, F_i^{\pitchfork})$$

No eigenvalue 1 implies vanishing homology 0

Corollary

If, for every $i \in \{1, \dots, \rho\}$, at least one of the transversal monodromies along the loops $\Gamma_i \subset \Sigma_i$ has no eigenvalue 1, then $H^{\curlyvee}_{n+2}(V) = 0$. □

Example

$V := \{x^2 z + y^2 w = 0\} \subset \mathbb{P}^3$ has

- $\operatorname{Sing} V = \mathbb{P}^1$ generic transversal type is A_1,
- three axis points,
- two special points q of type D_∞. The germ D_∞ is an *isolated line singularity*.
  Its Milnor fiber F is homotopy equivalent to the sphere S^2, the transversal monodromy is $-\mathrm{id}$.
- $H^{\curlyvee}_4(V) \simeq H_1(F) = 0$ and $\operatorname{rank} H^{\curlyvee}_3 = 5$.

No Special points and $\cdots$

In case Σ is irreducible:

$$H^{\Upsilon}_{n+2}(V) = \bigcap_{q\in Q} H_{n+1}(\mathcal{A}_q, \partial_2\mathcal{A}_q) \cap \bigcap_{j\in G} \ker(A_j - I).$$

Corollary

If there are no special points on Σ and the monodromy along every the genus loop is the identity, then $H^{\Upsilon}_{n+2}(V) \simeq H_{n-1}(F^{\pitchfork})$.

Example

This situation can be seen in $V := \{xy = 0\} \subset \mathbb{P}^3$ for which $H^{\Upsilon}_4(V) \simeq \mathbb{Z}$ and $\operatorname{rank} H^{\Upsilon}_3(V) = 1$.

Computing $H_{n+1}^{\curlyvee}(V)$ and not only its rank ?

Sometimes possible!

Example

$V := \{x^2z + y^3 + xyw = 0\} \subset \mathbb{P}^3$. Then

- $\mathrm{Sing}\, V = \mathbb{P}^1$, transversal type A_1,
- 3 axis points,
- a single point q of type $J_{2,\infty}$ with Milnor fiber a bouquet of 4 spheres S^2, and transversal monodromy the identity.

We get $H_4^{\curlyvee}(V) \simeq H_1(F) = 0$ and $\mathrm{rank}\, H_3^{\curlyvee}(V) = 6$.

In this case $H_3^{\curlyvee}(V) \simeq \mathbb{Z}^6$.
(no torsion)

Surface case

In case of surfaces $V \subset P^3$ we have:

$$H_4(V) \simeq \mathbb{Z}^r \text{ and } H_4^{\Upsilon}(V) \simeq \mathbb{Z}^{r-1},$$

where r is the number of irreducible components of V.

Corollary

$$r - 1 \leq \sum_{i=1}^{\rho} \mu_i^{\pitchfork}.$$

Absolute homology

Proposition

If $\dim \operatorname{Sing} V \leq 1$

$$H_k(V) \simeq H_k(V_\varepsilon) = H_k(\mathbb{P}^n) \text{ for } k \neq n, n+1, n+2.$$

Proof.

Long exact sequence of the pair $(\mathbb{V}_\Delta, V_\varepsilon)$ and concentration of vanishing homology in 2 dimensions.

□

Question: What about the remaining 8-terms of the sequence ?

Absolute homology

Proposition

(a) $b_{n+2}(V) \leq 1 + \sum_{i=1}^{\rho} \mu_i^{\pitchfork}$

(b) $b_n(V) \leq \dim \mathbb{L}$,

where $\mathbb{L} := H_n(V_\varepsilon)$ is the intersection lattice of the smooth hypersurface V_ε of degree d. In case n is even, this moreover yields:

(c) $H_{n+2}(V) \simeq \mathbb{Z} \oplus H^{\curlyvee}_{n+2}(V)$,

(d) $H_{n+1}(V) \simeq \ker \Phi_n$,

(e) $H_n(V) \simeq \operatorname{coker} \Phi_n$.

Special interest in cases, where $H^{\curlyvee}_{n+2}(V) = 0$, or when V is a $\mathbb{Z}$-homology manifold.

Final Message

This is the end of part of the course F2 (old B2)

It will be followed by F2- notes on
Polar Degree

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- D. Siersma, M. Tibăr: *Milnor fibre homology via deformation, arXiv: 1512.08240 Math.AG, december 2015.* CLICK*. In W.Decker et al. (eds), Singularities and Computer Algebra, Springer 2017, 305-322;*
- *D. Siersma, M. Tibăr: Projective hypersurfaces with 1-dimensional singularities, Europ. J. Math. 3 (2017), 565-586* CLICK*; arXiv: 11411.2640 Math.AG, november 2014.*

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List of Theo de Jong

S	F_1S	F_2S	F_3S
A_1	$xy^2 + z^2$		
A_2	$xy^3 + z^2$	$xz^2 + y^3$	
A_3	$xz^2 + y^2z$	$xy^4 + z^2$	
D_4	$xz^3 + y^2z$	$y^3 + z^3 + xw^2 + yzw$	
E_6	$xw^2 + y^2w + z^3$	$xy^4 + z^3 + y^3z$	$y^4 + xz^3 + y^2z^2$
E_7	$xz^3 + y^3z$	$xy^3z + z^3 + y^5$	
E_8	$xy^5 + z^3 + y^4z$	$y^5 + xz^3 + y^2z^2$	

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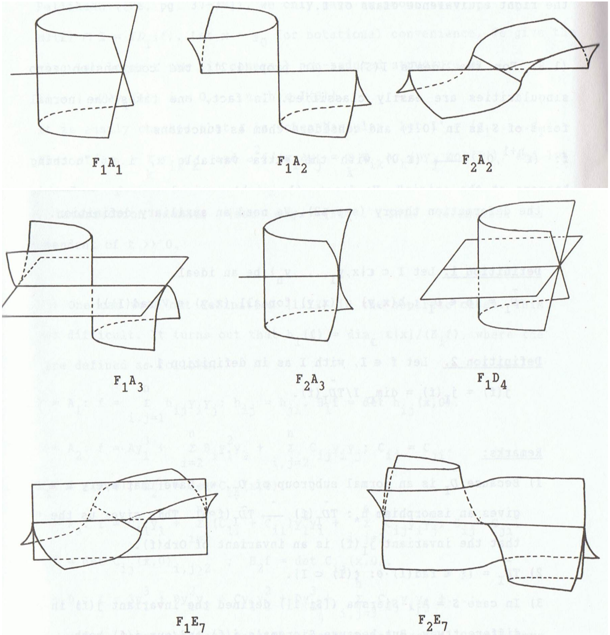
Pictures



Roman Steiner Surface