Polar degree of projective hypersurfaces

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Polar degree and topology

Let $V \subset \mathbb{P}^n$ be a complex projective hypersurface of degree $d \geq 2$, with arbitrary singularity set Sing V. It is defined by a homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, $n \geq 2$, of degree d.

The *polar degree* pol(V) is defined as the *topological degree* of the gradient mapping:

$$\mathrm{grad} f : \mathbb{P}^n \setminus \text{Sing } V \to \mathbb{P}^n. \tag{1}$$

It has been conjectured by Dolgachev [Do, 2000] that it depends only on the *reduced structure* of V (and not on the defining function f) thus the notation pol(V) makes sense.
This has been proved by Dimca and Papadima [DP, 2003].

Proof of Dolgachev's conjecture

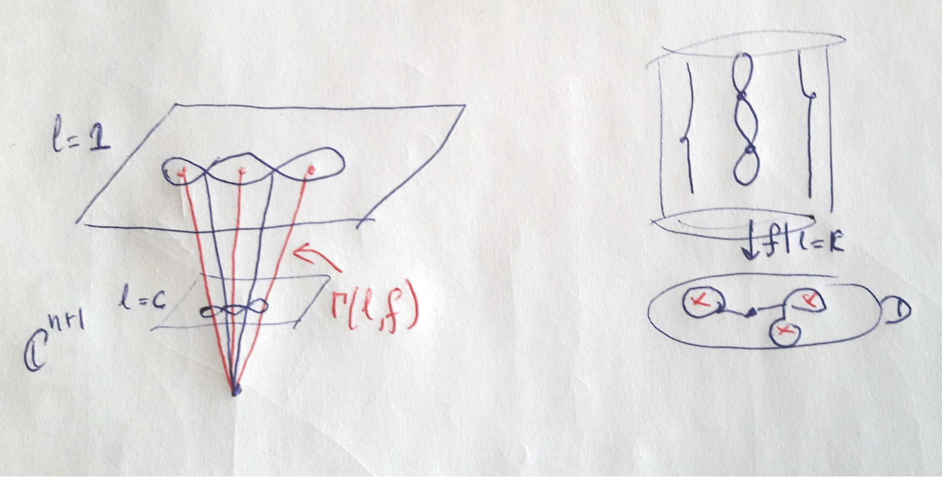
Theorem (Equivalent definitions for pol(V))

$\ell : \mathbb{C}^{n+1} \to \mathbb{C}$ linear function, identified to a point in $\mathbb{P}^n$. Let ℓ be general, in the sense that it is stratified-transversal to V after endowing V with a Whitney stratification. Then:

1. $\mathrm{pol}(V) := \#(\mathrm{grad} f)^{-1}(\ell)$, This is just the definition.

2. $\mathrm{pol}(V) = \mathrm{mult}_0 \Gamma(\ell, f)$, the multiplicity of the polar locus of the map $(l, f) : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^2, 0)$.

The polar curve $\Gamma(\ell, f)$ is here a homogeneous subset of $\mathbb{C}^{n+1} \Rightarrow$ union of lines, thus $\mathrm{mult}_0 \Gamma(\ell, f) =$ the number of these lines. This is also the number of points $x \in V \cap \{l \neq 0\}$ such that $\mathrm{grad} f(x) \in \mathbb{P}^n$ coincides with $\ell \in \mathbb{P}^n$, and this number is equal to $\#(\mathrm{grad} f)^{-1}(\ell)$.



l=1
l=c
$\mathbb{C}^{n+1}$
Γ(l,f)
f | l=c

3. $\mathrm{pol}(V) = \mathrm{rank}\ H_{n-1}(\mathbb{C}\mathrm{lk}_{\{f=0\}}(\{0\}))$
where $\mathbb{C}\mathrm{lk}_{\{f=0\}}(\{0\})$ denotes the *complex link* of the stratum $\{0\}$ of the hypersurface $\{f = 0\} \subset \mathbb{C}^{n+1}$. This is the local Milnor fibre of the function $\ell_| : (\{f = 0\}, 0) \to (\mathbb{C}, 0)$. See previous page $\longleftarrow$. Note that the complex link is a bouquet of spheres (cf the lecture in F2 about bouquet theorems) and that the number of spheres is equal to the number of critical points of f restricted to $\{l = c \neq 0\}$.

Since f is homogeneous, this complex link is homeomorphic to the affine subset $\{f = 0\} \cap \{\ell = 1\} \subset \mathbb{C}^{n+1}$. Denoting $H := \{\ell = 0\} \subset \mathbb{P}^n$, we thus have $\{f = 0\} \cap \{\ell = 1\} \overset{\text{homeo}}{\simeq} V \backslash H$, and we get:

4. $\mathrm{pol}(V) = \mathrm{rank}\ H_{n-1}(V \backslash H)$, [Dimca-Papadima, 2003]

This shows that $\mathrm{pol}(V)$ is topological, and hence proves Dolgachev's conjecture.

But already the definition 3. shows the same thing.

Polar degree = Top betti number of affine part

In fact we showed that for any $V \subset \mathbb{P}^n$, with possibly nonisolated singularities, and $H \subset \mathbb{P}^n$ a general hyperplane.

Proposition (Dimca and Papadima, 2003)

The homology $H_*(V \setminus H)$ is concentrated in dimension $n-1$, and

$$\mathrm{pol}(V) = \mathrm{rank}\ H_{n-1}(V \setminus H).$$

(nb. This is an equivalent formulation)

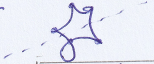
Hypersurfaces with at most isolated singularities

Proposition (Dimca-Papadima)

$$\mathrm{pol}(V) = (d-1)^n - \sum_{p \in \mathrm{Sing}\, V} \mu_p(V) \geq 0. \quad (2)$$

Proposition (Dimca-Papadima) Let $V \subset \mathbb{P}^n$ have only isolated singularities with Milnor numbers $\mu_p(V)$. Then

$$\mathrm{pol}(V) = (d-1)^n - \Sigma_p \mu_p(V)$$



Proof: Recall $\mathrm{pol}(V) = b_{n-1}(V \cap H)$ for H generic and $\chi(V - H) = 1 + (-1)^{n-1} \mathrm{pol}(V)$

$\chi(V_\varepsilon - H) = 1 + (-1)^{n-1} \mathrm{pol}(V_\varepsilon)$

We use "vanishing homology" and compare V and V_ε and get $\mathrm{Pol}(V) - \mathrm{Pol}(V_\varepsilon) = (-1)^{n-1}\{\chi(V - H) - \chi(V_\varepsilon - H)\}$

For isolated singularities:

$$\begin{cases} \chi(V) - \chi(V_\varepsilon) = (-1)^n \Sigma \mu_p \\ \chi(V \cap H) - \chi(V_\varepsilon \cap H) = 0 \end{cases}$$

use $\chi(V - H) = \chi(V) - \chi(V \cap H)$, idem for V_ε ⊠

→ Yet this does not help for bounding $\mathrm{pol}(V)$ from below.

The smooth hypersurface $V_{n,d}$ defined by the equation

$$x_0^d + \cdots + x_n^d = 0$$

realises the *maximum* polar number $\mathrm{pol}(V_{n,d}) = (d-1)^n$ for fixed n, in this category of "hypersurfaces with at most isolated singularities". We will see later that this is also the maximum for all cases.

Excercise Compute $\mathrm{pol}(V_{n,d})$ from the definition. Can you also use one of the equivalent definitions ? NB. The answer is used in the above proof of the Dimca-Papadima formula.

What are the hypersurfaces with $\mathrm{pol}(V) = 1$, called "homaloidal"? Dolgacev's made a conjectiural lis list for homaloidal hypersurfaces with isolated singularities. The above formula shows in particular that the smooth quadratic hypersurface $V_{n,2}$ is the only smooth V which is homaloidal.

Huh's main lemma: Bound from below

Lemma

For a hypersurface V with only isolated singularities, $p \in \mathrm{Sing}(V)$ and H generic through p one has:

$$\mathrm{pol}(V) \geq \mu_p(V \cap H)$$

unless V is a cone with apex p.

The (original) proof of Huh relies on *the theory of slicing by pencils with singularities in the axis* from [Ti1, Ti2, Ti3]. In our lecture we use another method:a splitting principle, which is explained in part 4 of the lectture B3 from 2021. The link is Click HERE. A main result is:

$$\mathrm{pol}(V) = \alpha(V, H) + \beta(V, H)$$

where $\alpha(V, H) = \sum \alpha_p(V \cap H)$ with isolated contributions, which are all non-negative.

$$\mathrm{pol}(V) = \alpha(V, H) + \beta(V, H)$$

where $\alpha(V, H) = \sum \alpha_p(V, H)$. Each term is defined a Lê-Milnor number of a (stratified) isolated singularity of the corresponding linear function on V.and all contributions are non-negative.
The proof works for any dimensional singular sets and uses the concept of admissible hyperplane, ie. hyperplane which is (stratified) transversal to$V$, and where the corresponding polar set is a curve.

In the case of isolated singularities of V we can make several short-cuts in the general proof. An important step is to show that admissible hyperplanes through a given singular point of V exist. This follows from our constrained polar curve theorem in [ST], which is a non-trivial statement.

In the isolated sinularity case the numbers $\alpha_p(V, H))$ are just the Milnor numbers $\mu_p(V \cap H)$. of the linear function (defining H), restricted to the germ of singular hypersurface (V, p).

We will give the proof of the $\alpha + \beta$ formula for isolated singularities on the blackboard.

Classification of hypersurfaces with isolated singularities with polar degree 1 or 2

We will show the completeness of Dolgacev's list for homaloidal hypersurfaces with isolated singularities (Huh's theorem) and give a proof of f Huh's conjecture for $\mathrm{pol}(V) = 2$. The case $\mathrm{pol}(V) = 1$ is included.

The main reference is:

D. Siersma, J. Steenbrink, M.Tibar,
On Huh's conjectures for the polar degree
Journal of Algebraic Geometry 30 (2021), 189-203
ArXiv version > https://arxiv.org/pdf/1805.08175.pdf

Classification of hypersurfaces with isolated singularities with polar degree 1 or 2

In this section we give a simultaneous proof of two theorems:

Theorem (Huh 2014)

A projective hypersurface $V \subset \mathbb{P}^n$ with only isolated singularities and $\mathrm{pol}(V) = 1$ *is one of the following, after a linear change of homogeneous coordinates:*

List of homaloidal hypersurfaces

(i) $(n \geq 2, d = 2)$ *a smooth quadric:*

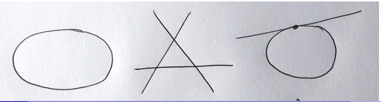
$$f = x_0^2 + \cdots + x_n^2 = 0.$$

(ii) $(n = 2, d = 3)$ *the union of three non-concurrent lines:*

$$f = x_0 x_1 x_2 = 0, \qquad (3A_1).$$

(iii) $(n = 2, d = 3)$ *the union of a smooth conic and one of its tangents:*

$$f = x_0(x_1^2 + x_0 x_2) = 0, \qquad (A_3).$$



Theorem (Siersma-Steenbrink-Tibar, 2018)

The hypersurfaces $V \subset \mathbb{P}^n$ *with isolated singularities and* $\mathrm{pol}(V) = 2$ *are only those in Huh's conjectural list.*
In particular there are no such hypersurfaces for $n > 3$.

Three normal cubic surfaces:

1. $(n = 3, d = 3)$ *a normal cubic surface containing a single line:*

$$f = x_0x_1^2 + x_1x_2^2 + x_1x_3^2 + x_2^3 = 0, \qquad (E_6).$$

2. $(n = 3, d = 3)$ *a normal cubic surface containing two lines:*

$$f = x_0x_1x_2 + x_0x_3^2 + x_1^3 = 0, \qquad (A_5, A_1).$$

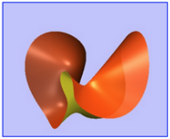
3. $(n = 3, d = 3)$ *a normal cubic surface containing three lines and three binodes:*

$$f = x_0x_1x_2 + x_3^3 = 0, \qquad (A_2, A_2, A_2).$$

Surfaces with polar degree 2

Class XX: one E_6 singularity

In Cayley's notation: U8



$W X^2 + X Z^2 + Y^3$

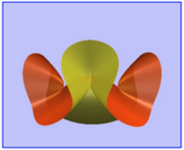
Class XIX: one A_1 and one A_5 singularity

In Cayley's notation: B6 + C2

$W X Z + Y^2 Z + X^3$

Class XXI: three A_2 singularities

In Cayley's notation: 3 B3



$W X Z + Y^3$

Huh (2014) verified the case $n = 3$ and $d = 3, 4$. He also indicated that Radu Laza has verified the conjecture in case $n = 3$ and $d = 3$. There was no general principle for doing more.

Nine plane curves, of degrees 3, 4 and 5:

1. $(n = 2, d = 5)$ *two smooth conics meeting at a single point and their common tangent:*

$$f = x_0(x_1^2 + x_0x_2)(x_1^2 + x_0x_2 + x_0^2) = 0, \qquad (J_{2,4}).$$

2. $(n = 2, d = 4)$ *two smooth conics meeting at a single point:*

$$f = (x_1^2 + x_0x_2)(x_1^2 + x_0x_2 + x_0^2) = 0, \qquad (A_7).$$

3. $(n = 2, d = 4)$ *a smooth conic, a tangent and a line passing through the tangency point:*

$$f = x_0(x_0 + x_1)(x_1^2 + x_0x_2) = 0, \qquad (D_6, A_1).$$

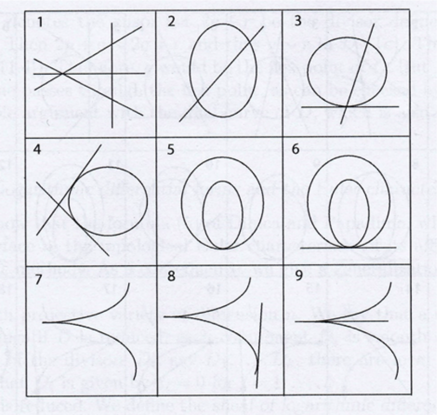
4. $(n = 2, d = 4)$ *a smooth conic and two tangent lines:*

$$f = x_0x_2(x_1^2 + x_0x_2) = 0, \qquad (A_1, A_3, A_3).$$

1. $(n=2, d=4)$ *three concurrent lines and a line not meeting the center point:*
$$f = x_0x_1x_2(x_0+x_1) = 0, \qquad (D_4, A_1, A_1, A_1).$$
2. $(n=2, d=4)$ *a cuspidal cubic and its tangent at the cusp:*
$$f = x_0(x_1^3 + x_0^2x_2) = 0, \qquad (E_7).$$
3. $(n=2, d=4)$ *a cuspidal cubic and its tangent at the smooth flex point:*
$$f = x_2(x_1^3 + x_0^2x_2) = 0, \qquad (A_2, A_5).$$
4. $(n=2, d=3)$ *a cuspidal cubic:*
$$f = x_1^3 + x_0^2x_2 = 0, \qquad (A_2).$$
5. $(n=2, d=3)$ *a smooth conic and a secant line:*
$$f = x_1(x_1^2 + x_0x_2) = 0, \qquad (A_1, A_1).$$

end of the list.

Curves with polar degree 2



$d=4$ $D_4 \oplus 3A_1$	$d=3$ $2A_1$	$d=4$ $D_6 \oplus A_1$
$d=4$ $2A_3 \oplus A_1$	$d=4$ A_7	$d=5$ $J_{2,4}$
$d=3$ A_2	$d=4$ $A_5 \oplus A_2$	$d=4$ E_7

The list of plane curves is included in the total list of $\mathrm{pol}(V) = 2$ curves found by Fasarella and Medeiros (2012).

recall Huh's main lemma: Bound from below

Lemma

For a hypersurface V with only isolated singularities, $p \in \mathrm{Sing}(V)$ and H generic through p one has:

$$\mathrm{pol}(V) \geq \mu_p(V \cap H)$$

unless V is a cone with apex p.

As explained before: this follows from

$$\mathrm{pol}(V) = \alpha(V, H) + \beta(V, H)$$

after choosing a generic admissible hyperplane through p.

Corollary

For hypersurfaces with isolated singularities one has:

- *1. If* $\mathrm{pol}(V) = 1$ *then* V *has only* A_k*-singularities,*
- *2. If* $\mathrm{pol}(V) = 2$ *then* V *has only* A_k, D_k, E_*, J_* *singularities*

Proof.

If $\mu_p(V \cap H) = 1$ one can choose local coordinates such that the 2-jet in p becomes $x_0^2 + \cdots + x_{n-1}^2$.
By the classification of singularities (see the lecture A from the 22021-school) it follows that the local singularity at p is of type A_k.
If $\mu_p(V \cap H) = 2$: similar, but more advanced. See the paper [SST]. □

1. *Singularities of corank 2 with nonzero 3-jet.* Besides the simple singularities A, D, E_6, E_7, E_8 there is a further infinite series of classes:

Notation	Normal form	Restrictions	Multiplicity μ	Modality m
$J_{k,0}$	$x^3 + bx^2y^k + y^{3k} + cxy^{2k+1}$	$k > 1,\ 4b^3 + 27 \neq 0$	$6k - 2$	$k - 1$
$J_{k,i}$	$x^3 + x^2y^k + ax^{3k+i}$	$k > 1,\ i > 0,\ a_0 \neq 0$	$6k - 2 + i$	$k - 1$
E_{6k}	$x^3 + y^{3k+1} + axy^{2k+1}$	$k \geqq 1$	$6k$	$k - 1$
E_{6k+1}	$x^3 + xy^{2k+1} + ay^{3k+2}$	$k \geqq 1$	$6k + 1$	$k - 1$
E_{6k+2}	$x^3 + y^{3k+2} + axy^{2k+2}$	$k \geqq 1$	$6k + 2$	$k - 1$

Deformation to $f_{n,d}$

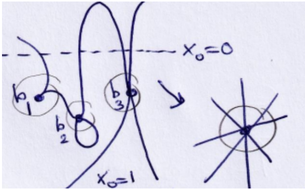
Let $H := \{x_0 = 0\} \subset \mathbb{P}^n$ be a generic hyperplane with respect to our hypersurface $V := \{f = 0\}$ and let $f = f_d + x_0 f_{d-1} + \cdots + x_0^d f_0$. We consider the family:

$$g_s(1, x_1, \cdots, x_n) := f_d + s f_{d-1} + \cdots + s^d f_0 \text{ on } \mathbb{C}^n = \mathbb{P}^n \setminus H.$$

where f_d is a general homogeneous polynomial, topologically equivalent to

$$f_{n,d} := x_1^d + \cdots + x_n^d.$$

The family g_s describes a deformation of $g_1 = f_{|\mathbb{C}^n}$ to $g_0 = f_d$, which deforms the hypersurface $V \setminus H$ to the hypersurface $\{f_d = 0\}$, with a single isolated singularity at 0, and which is topologically equivalent to the hypersurface $\{f_{n,d} = 0\}$.



We get a *multi-adjacency* of the local singularities at b_i to the singularity at O of $f_{n,d}$.

Intermezzo: Singularity invariants

One of the invariants of the singularities is its Milnor number. But there are also finer invariants. We first look at the intersectionform S on the homology of the Milnor fibre: This is the pairing given by the sequence of maps:

$$S : H_{n-1}(F) \otimes H_{n-1}(F) \to H_{n-1}(F, \partial F) \otimes H_{n-1}(F) \to \mathbb{Z}$$

The geometric meaning of S is the intersection of $(n-1)$cycles in the Milnor fibre, which has real dimension $2n-2$. The second map is a non-degenerate pairing (by Poincaré-duality).
In dimension $n = 3 mod\ 4$ the intersection form is symmetric and self-intersections are -2 . We consider this case below, other cases are almost similar.

The intersection form will give us three numbers: μ_-, μ_0, μ_+ which add up to μ and are resp. the dimensions of the negative, zero and positive eigenspaces. These numbers are studied in great detail in work of Brieskorn, Arnol'd, Gusein Zade, Ebeling. The numbers μ_-, μ_+ and the rank of S are semi-continuous in (multi)-adjacencies.
The simple singularities A_k, D_k, E_6, E_7, E_8 are the only negative definite forms. They are related to the famous Dynkin diagrams.

Continuation of proof

The multi-adjacency induces an inclusion of Milnor lattices:

$$\oplus_{b \in \mathrm{Sing}\, V} H_{n-1}(F_{b_i}) = \oplus \mathbb{Z}^{\mu_{b_i}} \longrightarrow \mathbb{Z}^{(d-1)^n} = H_{n-1}(x_1^d + \cdots + x_n^d = t)$$

with intersection matrices S_i resp S.
(See e.g. D. Siersma: Classification and Deformation of Singularities, page 74.)
Consider now the case pol$(V) = 1$ We know that the singularities at b_i are of type A_k, the S_i is negative definite and the rank of S_i is equal to μ_{b_i}. For $f_{n,d}$ holds:

$$\mu_- = (d-1)^3 - 2\binom{d}{3} \;\; ; \;\; \mu_0 = 2\binom{d-1}{2} = d^2 - 3d + 2 \; ; \;\; \mu_+ = 2\binom{d-1}{3}$$

and especially $\mu_0(x_1^d + \cdots + x_n^d) \geq 2$ if $n = 2, d \geq 4$ or $n > 2, d \geq 3$.
Since pol$(V) = 1$ we have $\sum rankS_i = (d-1)^n - 1$ but $\mathrm{rank}\ S \leq (d-1)^n - 2$.
The semi continuity of rank excludes all hypersurfaces, which are not in Dolgacev's list.

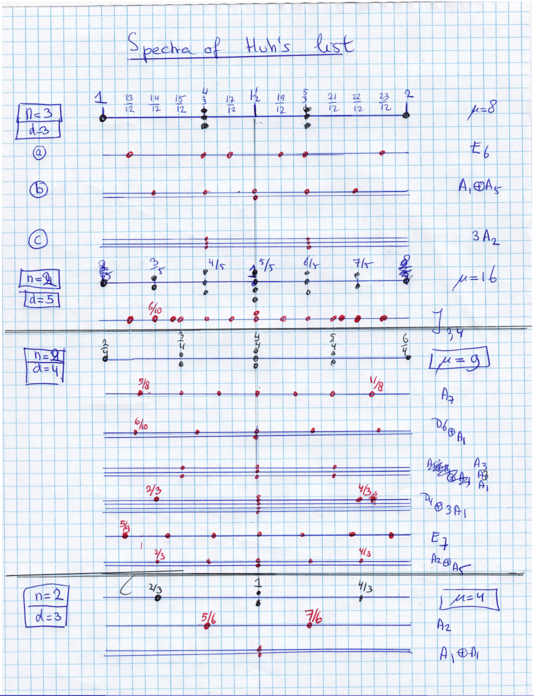
This finishes the proof in case pol$(V) = 1$.
The argument above is different form Huh's argument. We will also use this for the case that pol$(V) = 2$. The intersection form argument still works for $n = 3$, but for the general case we will use the semi-continuity of the spectrum.

Intermezzo: Spectrum numbers

The spectrum numbers of a singularity are related to the *eigenvalues of the monodromy*. They were well-studied by Steenbrink and Varchenko. Their definition uses the Hodge filtration of the homology of the Milnor fibre. The theory is rather complicated and we only sketch here how these number can be used in the proof of our theorem. For details see the paper [SST] and the references mentioned there. The *spectrum numbers* are defined by

$$s_i = log(\lambda_i) + n - \rho$$

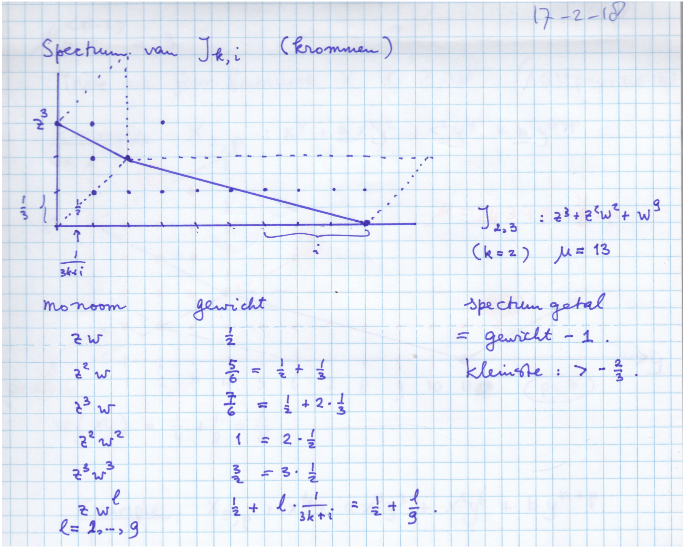
where λ_i is an eigenvalue of monodromy. The *log* is taken in the interval $(-1, 0]$ and ρ is the level in the Hodge filtration.
Spectrum numbers of quasi-homogeneous singularities or those with non-degenerate Newton diagram are computable. (Steenbrink-Varchenko). We list those, which appear in Huh's conjecture in the next slide. Each part of the list starts with the spectrum of $f_{n,d}$ and below them (with difference of Milnor numbers 2) one sees the spectra the combinations of singularities.


Spectra of Huh's list
n=3
d=3
μ=8
E6
A1⊕A5
3A2
d=5
μ=16
J3,4
d=4
μ=9
A7
D6⊕A1
D4⊕3A1
E7
A2⊕A5
n=2
d=3
μ=4
A2
A1⊕A1

End of the proof

The semi-continuity of the spectrum means that the spectrum numbers of the local singularities are well-devided between the spectrum numbers of $f_{n,d}$. Look e.g. at the list. We don't state this rule in detail.
The strategy of the proof is now to show that other combinations of singularities, where the Milnor numbers add up to $\mathrm{pol}(V) - 2$ don't satisfy this rule. It is easy to rule out several big classes of hypersurfaces; in a few remaining cases one has to do some detailed computations.

You find a sample computation (made by my friend Steenbrink on the next slide). It is not the idea that you will understand this here. But anyhow you could recognize a Newton diagram and the generators of the Milnor Algebra. (NB. The text is written in Ducth, the language of The Netherrlands)



This end this 'proof"

Finiteness for isolated singularities

We could also solve:

Conjecture (Huh, 2014)

There is no projective hypersurface $V \subset \mathbb{P}^n$ of polar degree k with only isolated singular points, for sufficiently large n and $d = \deg V$.

A more precise statement?

Theorem (Finiteness Theorem, SST)

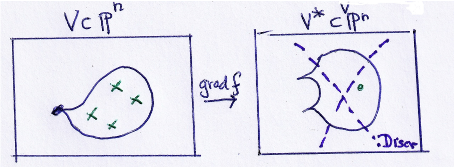
For any integer $k \geq 2$, let K_k denote the set of pairs of integers (n, d) with $n \geq 2$ and $d \geq 3$, such that there exists a projective hypersurface V in $\mathbb{P}^n$ of degree d with isolated singularities and $\mathrm{pol}(V) = k$.
Then K_k is finite for any $k \geq 2$.

The proof is also an application of spectrum numbers. We refer again to [SST].

Polar degree, history and examples

We recall

Let $V \subset \mathbb{P}^n$ be a complex projective hypersurface of degree $d \geq 2$, defined by a homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, $n \geq 2$, of degree d. The *polar degree* pol(V) is defined as the *topological degree* of the gradient mapping: (sometimes called Gauss map): $\mathrm{grad} f : \mathbb{P}^n \setminus \mathrm{Sing}\, V \to \mathbb{P}^n$



If the degree is non-zero then the image has full dimension. The target space contains a discriminant hypersuface where the gradient map is not covering map The image of $V \setminus \mathrm{Sing}\, V$ is V^*, the dual of V..

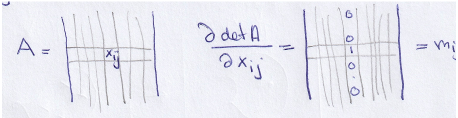
Historically the focus has been on the algebraic properties of the definition. The topological approach started around 2000.

Examples

By using the algebraic defintion one can check the following examples:

1. Smooth hypersurface $f = x_0^d + \cdots x_n^d$ has $\mathrm{pol}(V) = (d-1)^n$
2. Projective cone $f(x_0, x_1, \cdots, x_n) = h(x_1, \cdots, x_n)$ has $\mathrm{pol}(V) = 0$.
3. Hankel type determinant $f = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & y_3 & y_4 & y_5 \\ y_3 & y_4 & y_5 & 0 \\ y_4 & y_5 & 0 & 0 \end{vmatrix}$ has $\mathrm{pol}(V) = 1$.
4. The determinantal hypersurface $f = \det(A) = 0$ in $\mathbb{P}^{n^2-1}$ has $\mathrm{pol}(V) = 1$. Note that $\mathrm{grad}\det A$ is an invertible mapping due to Cramer's rule:

$$A \cdot \mathrm{grad}\det A = (detA) \cdot I$$



Exercise Determine the topology of the complex link of the determinantal hypersurface in $\mathbb{C}^{n^2}$.

pol$(V) = 0$; Gradient map has lower dimensional image

Otto Hesse (1851 and 1856) claimed that if the Hessian determinant $\det(\frac{\delta^2 f}{\delta x_i \delta x_j})$ is identically zero, (which equivalent to: *the polar degree is zero) if and only if the hypersurface V is a projective cone*; that is elimnation of one variable is possible.
NB. If the Hessian determinant is non zero, then the gradient map is a local diffeomorphism.
The proof was considered as not 'rigourous'.

In 1875 M. Pasch showed that Hesse's claim was true for ternairy and quartic cubics.

Gordan and Noether (1876) disproved it , but showed that it is true up to birational transformations.
In fact any relation $F(\frac{\delta f}{\delta x_0}, \cdots, \frac{\delta f}{\delta x_n}) = 0$ gives pol$(V) = 0$
NB. Hesse considered only linear relations with constant coefficients:

$$z_0 \frac{\delta f}{\delta x_0} + \cdots z_n \frac{\delta f}{\delta x_n} = 0$$

A typical example with a 2-dimensional singular set is:

$$f = x_3^{d-1} x_0 + x_3^{d-2} x_4 x_1 + x_4^{d-1} x_2 = 0 \ \ ; d \geq 3$$

Exercise:: Compute the singular set and the relation F in this case.

pol(V) = 1 ; Homaloidal polynomials

In this case the gradient map is a birational map. The name *Cremona transformation* is used for these maps. They form an important class of maps in algebraic geometry and are well studied.
We mention some examples:

1. The determinantal hypersuface
2. Determinants of generic sub-Hankel matrices
3. The smooth quadratic hypersurface
4. The (generic) arrangement of hyperplanes $z_0z_1z_2z_3 = 0$ in $\mathbb{P}^3$.
5. For hypersurfaces with isolated singularities: the list of Dolgacev (2000), see before.

For the algebraic approach we refer especially to [CRS]:
C. Cilberto, F. Russo, A. Simis, *Homaloidal hypersurfaces and hypersurfaces with vanishing Hessian*, Adv. Math. 218 (2008), no 6, 1759-1805.

General Impression: Homaloidal hypersurfaces with isolated singularities occur only in $\mathbb{P}^3$ (Dolgacev's list). For $\mathbb{P}^n$ they have severe singularties (increasing with the dimension).

Semi-continuity of pol(V) under deformations

The following result holds for any singular locus and any value of pol(V):

Proposition

The polar degree is lower semi-continuous in deformations of fixed degree d. More precisely, if f_s is a deformation of $f_0 := f$ of constant degree, then $\mathrm{pol}(V_s) \geq \mathrm{pol}(V)$ *for $s \in \mathbb{C}$ close enough to 0, where $V_s := \{f_s = 0\}$.*

proof: Let $b \in \mathbb{P}^n$ be a regular value for $\mathrm{grad} f : \mathbb{P}^n \setminus \mathrm{Sing}\, V \to \mathbb{P}^n$ and let $(\mathrm{grad} f)^{-1}(b) = \{a_1, \cdots, a_k\}$, $k \geq 0$. There exist disjoint compact neighborhoods U_i of a_i and U' of b such that $\mathrm{grad} f : (U_i, a_i) \to (U', b)$ is a diffeomorphism. Next take s so close to 0 such that $\mathrm{grad} f_s | U_i$ are still diffeomorphisms, and that $\mathrm{grad} f_s(U_i)$ still contains b in its interior. Let $W = \cap_{i=1}^{k} \mathrm{grad} f_s(U_i)$ and $Z_i = (\mathrm{grad} f_s)^{-1}(W) \cap U_i$. The restriction $\mathrm{grad} f_s : \bigsqcup_i Z_i \to W$ is a diffeomorphism on each component Z_i, and has topological degree pol(V). Moreover b is still a regular value for this restriction, but perhaps not anymore for the full map $\mathrm{grad} f_s : \mathbb{P}^n \setminus \mathrm{Sing}\, V_s \to \mathbb{P}^n$. Arbitrarily close to b there exist points b' which are regular values for $\mathrm{grad} f_s$. Then the number of counter-images $\#(\mathrm{grad} f_s)^{-1}(b')$ is pol(V_s) and $(\mathrm{grad} f_s)^{-1}(b')$ contains at least one point in each Z_i. This shows the inequality $\mathrm{pol}(V_s) \geq \mathrm{pol}(V)$.

Generalized Dimca-Papadima Formula

In the presense of 0- and 1-dimensional singularities we will prove the Dimca-Papadima formula and its generalization via the vanishing homolgy defined in the F2(B2)-lecture on nonisolated singularities. A general reference is:
D. Siersma, M. Tibăr, *Polar degree of hypersurfaces with 1-dimensional singularities* >> >>CLICK

Proposition

Let $V \in \mathbb{P}^n$ *with only isolated singularities with Milnor numbers* $\mu_p(V)$. *Then:*

$$\mathrm{pol}(V) = (d-1)^n - \sum_p \mu_p(V)$$

Let $V \subset \mathbb{P}^n$ *be a hypersurface of degree* d *with a 1-dimensional singular set. Then:*

$$\mathrm{pol}(V) = (d-1)^n - \sum_{p \in \Sigma^{isol}} \mu_p(V) - \sum_{i=1}^{r} c_i \mu_i^{\pitchfork} + (-1)^n \sum_{q \in Q} (\chi(\mathcal{A}_q) - 1)$$

where $c_i = 2g_i + \gamma_i + (d+1)\deg \Sigma_i^c - 2$, *where* g_i *is the genus of the normalization* $\tilde{\Sigma}_i^c$ *of* Σ_i^c, *and where* $\deg \Sigma_i^c$ *denotes the degree of* Σ_i^c *as a reduced curve.*

Proof: We show the second formula; the proof of the first is included as a special case. Recall $\mathrm{pol}(V) = b_{n-1}(V - H)$ for generic H. Moreover (since the homology is concentrated) :
$\chi(V - H) = 1 + (-1)^{n-1}\,\mathrm{pol}(V)$ and $\chi(V_\epsilon - H) = 1 + (-1)^{n-1}\,\mathrm{pol}(V_\epsilon)$ and

$$\mathrm{pol}(V) - \mathrm{pol}(V_\epsilon) = (-1)^{n-1}\{\chi(V - H) - \chi(V_\epsilon - H)\}$$

In the (notes of) lecture 2 of F2 we looked at the pair (V_Δ, V_ϵ) and showed that:

$$\chi(V) - \chi(V_\epsilon) = \chi^\curlyvee(V) = (-1)^{n+1}\sum_{i=1}^{\rho}\chi(\Sigma^*)\mu_i^\pitchfork - \sum_{q\in Q}\tilde{\chi}(\mathcal{A}_q) + (-1)^{n+1}\sum_{r\in R}\mu_r.$$

$$\chi(V \cap H) - \chi(V_\epsilon \cap H) = \chi^\curlyvee(V \cap H) = (-1)^{n-1}\sum_{a\in sing(V\cap H)}\mu_a(V \cap H)$$

Note $\chi(\Sigma^*) = 2g_i + \gamma_I + \nu_i - 2$, where ν_i is the number of axis points on Σ_i and $\mu_a(V \cap H) = \mu_i^\pitchfork$ if $a \in \Sigma_i$. Next apply

$$\chi(V - H) = \chi(V) - \chi(V \cap H),$$

also for V_ϵ and combine the formula's above.
Excercise: Give the details.

Polar degree and vanishing Euler characteristic

The above proof shows also that polar degree is related to the vanishing homology via the vanishing Euler-characteristic $\chi^{\curlyvee}(V) = \chi(V) - \chi(V_\epsilon) = \chi^{\curlyvee}(V)$:

Proposition

For a projective hypersurface V of arbitrary dimension of its singular set and a generic hyperplane H:

$$\mathrm{pol}(V) = (d-1)^n - (-1)^{n-1}\{\chi^{\curlyvee}(V) - \chi^{\curlyvee}(V \cap H)\}.$$

Examples and excercise: Cubic surfaces

The classification of reduced surfaces by Bruce and Wall is a good source of examples, where the above fomula can be used. We leave the compuations as an excercise.

For *isolated singularities*, we give the number of singularities and their types.
$\mathrm{pol}(V) = 8$: the smooth cubic
$\mathrm{pol}(V) = 7 : A_1$.
$\mathrm{pol}(V) = 6 : 2A_1$ or A_2.
$\mathrm{pol}(V) = 5 : 3A_1$ or $A_1 A_2$ or A_3.
$\mathrm{pol}(V) = 4 : 4A_1$ or $A_2 2A_1$ or $A_3 A_1$ or $2A_2$ or A_4 or D_4.
$\mathrm{pol}(V) = 3 : A_3 2A_1$ or $A_1 2A_2$ or $A_4 A_1$ or A_5 or D_5.
$\mathrm{pol}(V) = 2 : 3A_2$ or $A_5 A_1$ or E_6.
$\mathrm{pol}(V) = 1$: no homaloidal surfaces.
$\mathrm{pol}(V) = 0 : \tilde{E}_6$, which is a cone.

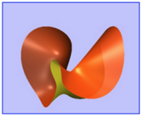
Pictures of cubic surfaces

It is interesting to look at pictures of cubic surfaces. You can draw the real parts by yourselves using software SURFER https://www.imaginary.org/program/surfer or find them on the web. Here are allready examples.

Surfaces with polar degree 2

Class XX: one E_6 singularity

In Cayley's notation: U8



$W X^2 + X Z^2 + Y^3$

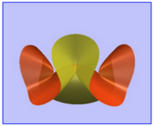
Class XIX: one A_1 and one A_5 singularity

In Cayley's notation: B6 + C2

$W X Z + Y^2 Z + X^3$

Class XXI: three A_2 singularities

In Cayley's notation: 3 B3



$W X Z + Y^3$

Next *irreducible* cubics with *nonisolated singularities* :
(CN) cone over a nodal curve,
(CC) cone over a cuspidal curve
both with $\mathrm{pol}(V) = 0$ because they are cones, and two other cases:
(E1) $x_0^2 x_2 + x_1^2 x_3$; $\mathrm{pol}(V) = 2$
(E2) $x_0^2 x_2 + x_0 x_1 x_3 + x_1^3$; $\mathrm{pol}(V) = 1$,
where the singular set is a projective line with two special points of type D_∞ in the first case, and a single special point of type $J_{2,\infty}$ in the second case
Among the *reducible* cubics there are only the following three cases with non-zero polar degree:

(QP) The union of a smooth quadratic with a general hyperplane: $\mathrm{pol}(V) = 2$,

(QT) The union of a smooth quadratic with a tangent hyperplane: $\mathrm{pol}(V) = 1$,

(CP) The union of a quadratic cone and a general hyperplane: $\mathrm{pol}(V) = 1$.

All the other reducible cubics are cones and thus have $\mathrm{pol}(V) = 0$.

From all these results:

Proposition

There are only three homaloidal cubic surfaces, all with nonisolated singularities.

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