Topology of polynomial functions and monodromy dynamics

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Abstract. Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function. We define global polar invariants associated to fibres of f and we describe a CW-complex model of a fibre. We show how to use affine polar curves in order to study the monodromy around atypical values of f, including the value infinity. We give a zeta-function formula for such a monodromy.

Topologie des fonctions polynomiales et dynamique monodromique

Résumé. Soit $f : \mathbb{C}^n \to \mathbb{C}$ une fonction polynomiale. On définit des invariants polaires globaux associés aux fibres de f à l'aide desquels on décrit un modéle d'une fibre de f comme CW-complexe. On montre comment utiliser les courbes polaires affines pour étudier la monodromie autour d'une valeur atypique, y compris la valeur infini. On donne une formule pour la fonction zêta d'une telle monodromie.

Version française abrégée

On étudie une fonction polynomiale $f : \mathbb{C}^n \to \mathbb{C}$, $n \geq 2$, ayant comme but de decrire la variation dans la topologie de la fibre de f due à la présence des fibres atypiques. Une valeur $t_0 \in \mathbb{C}$ est *typique* pour f si l'application f est une C^∞-fibration triviale en t_0. L'ensemble des valeurs atypiques est fini (cf. [8], [12]) et il inclut l'ensemble des valeurs *critiques* de f. Les valeurs atypiques non critiques sont dues au comportement asymptotique "mauvais" d'un nombre de fibres.

On définit d'abord des invariants polaires globaux γ^* (introduits en [11] pour le but de construire une théorie d'équisingularité globale des familles d'hypersurfaces affines) et on montre comment γ^* entrent dans la description d'un modele CW-complexe d'une fibre de f (Théorème 1.3).

On considere ensuite le problème de définir une monodromie géométrique globale,

c'est à dire une représentation $\rho : \pi_1(\mathbb{P}^1 \setminus \Lambda) \to \mathrm{Diff}(F)$, où Λ est l'ensemble des valeurs atypiques, F est la fibre générale de f et $\mathrm{Diff}(F)$ est le groupe de C^∞-difféomorphismes de F. Dans le cas d'une monodromie autour une valeur atypique (y compris la valeur ∞ !) on définit un champs de vecteurs controlé à l'infini et tangent à la courbe polaire $\Gamma(l, f) :=$ adhérence$\{\mathrm{Sing}\,(l, f) \setminus \mathrm{Sing}\, f\}$, pour une forme linéaire suffisamment générale. La construction est basée sur une décomposition en régions de la fibre F, chaque région ayant une dynamique spéciale. La *méthode du carrousel* de Lê D.T. [3], [4] joue un rôle important dans le tableau. On prouve une formule pour la fonction zêta de la monodromie autour d'une valeur atypique d'un polynôme quelconque et on finit par deux exemples qui montrent que la monodromie globale a un comportement bien différent d'une monodromie locale.

We study a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ aiming to describe the variation of topology in the fibration induced by f, at atypical fibres (i.e. fibres with special asymptotic behavior at infinity). First we define numerical invariants which enter as data in a CW-complex model of any fibre of f. These are global polar invariants introduced and used by the second author for constructing a theory of global equisingularity of families of affine hypersurfaces [11].

Let us define the space $\mathbb{X} = \{[x_0 : x_1 : \cdots x_n] \in \mathbb{P}^n, [s : t] \in \mathbb{P}^1 \mid s\tilde{f} - tx_0^d = 0\} \subset \mathbb{P}^n \times \mathbb{P}^1$, where $\tilde{f}$ is the homogenized of the polynomial f which is of degree d. Denote by $p : \mathbb{X} \to \mathbb{P}^1$ the second projection and by $\mathbb{X}^\infty$ the part at infinity $\mathbb{X} \cap \{x_0 = 0\}$ of $\mathbb{X}$. There is a finite set $\Lambda \subset \mathbb{P}^1$ such that both $p : \mathbb{X} \setminus p^{-1}(\Lambda) \to \mathbb{P}^1 \setminus \Lambda$ and its restriction $p_| : \mathbb{X} \setminus (p^{-1}(\Lambda) \cup \mathbb{X}^\infty) \to \mathbb{P}^1 \setminus \Lambda$ are locally trivial C^∞-fibrations [8], [12]. We take by definition $[0 : 1] = \infty \in \Lambda$. We consider the polar locus $\Gamma(l, f) :=$ closure$\{\mathrm{Sing}\,(l, f) \setminus \mathrm{Sing}\, f\}$ of f with respect to a general linear form $l : \mathbb{C}^n \to \mathbb{C}$ and denote by l_H the linear form associated to a projective hyperplane $H \in \check{\mathbb{P}}^{n-1}$.

We first need the following global result, an improved version of the Polar Curve Theorem, see [10, Lemma 2.4]. We denote by $\Delta := \Delta(l, f)$ the discriminant in $\mathbb{C}^2$ of the map (l, f).

1.1 Lemma *There is a Zariski-open set $\Omega \subset \check{\mathbb{P}}^{n-1}$ such that, for any $H \in \Omega$, the polar locus $\Gamma(l_H, t)$ is a reduced curve or it is empty, that the map $(l, f) : \mathbb{C}^n \to \mathbb{C} \times \mathbb{C}$ is a C^∞-trivial fibration over $(\mathbb{C} \times (\mathbb{C} \setminus \Lambda')) \setminus \Delta$, where Λ' is a finite set in $\mathbb{C}$ and that no component of $\Gamma(l_H, t)$ is contained into a fibre of f.* $\diamond$

This can be proved by the methods of [10, §5.]; we shall not do this here. However, it will be important, for later use, to mention that the excluded set $\mathbb{C} \times \Lambda'$ is a one-to-one projection by $(\bar{l}, \bar{f})$ of the polar locus within $\mathbb{X}^\infty$ of the projective compactification $(\bar{l}, \bar{f})$ of (l, f).

1.2 Definition [11] For any $c \in \mathbb{C}$, the *set of generic polar intersection multiplicities*

$$\gamma_c^* := \{\gamma_c^{n-1}, \ldots, \gamma_c^1, \gamma_c^0\}$$

is defined as follows. The number $\gamma_c^{n-1} = \gamma^{n-1}(f^{-1}(c)) = \mathrm{int}(\Gamma(l_H, f), f^{-1}(c))$, is the sum of the local intersection multiplicities at each point of the finite set $\Gamma(l_H, f) \cap f^{-1}(c)$, for a generic $H \in \Omega$.

Next, γ_c^{n-2} is the generic polar intersection multiplicity of $f_{|\mathcal{H}}^{-1}(c)$ with $\Gamma(l_1, f_{|\mathcal{H}})$, for a general hyperplane $\mathcal{H}$ identified with $\mathbb{C}^{n-1}$ and general linear form $l_1 : \mathbb{C}^{n-1} \to \mathbb{C}$. We define in this way, by induction, γ_c^{n-i}, for $1 \leq i \leq n-1$. We put $\gamma_c^0 := \deg f$, by definition.

By a standard connectivity argument, the set of polar intersection multiplicities is well defined, i.e. it does not depend on the choices of generic hyperplanes, and it is constant on the complement of a finite set in $\mathbb{C}$. The "jump" of γ_c^i is due to the loss of intersection points to infinity.

The invariants γ_c^* enter in the description of a fibre $F_c := f^{-1}(c)$ as CW-complex. For simplicity, we assume that F_c has isolated singularities. Let $\mu(F_c)$ denote the sum of the Milnor numbers of the isolated singularities of F_c.

1.3 Theorem [11] *Let $c \in \mathbb{C}$ such that F_c has isolated singularities. Then F_c is homotopy equivalent to a generic hyperplane section $F_c \cap \mathcal{H}$ to which one attaches $\gamma_c^{n-1} - \mu(F_c)$ cells of dimension $n-1$. Furthermore, F_c is homotopy equivalent to the CW-complex obtained by attaching to $\deg f$ points a number of γ_c^1 cells of dimension 1, then attaching to this γ_c^2 cells of dimension 2, ..., γ_c^{n-2} cells of dimension $n-2$ and finally $\gamma_c^{n-1} - \mu(F_c)$ cells of dimension $n-1$. In particular,*

$$\chi(F_c) = (-1)^n \mu(F_c) + \sum_{i=0}^{n-1} (-1)^i \gamma_c^i$$

and

$$\chi(F_{\text{gen}}) - \chi(F_c) = (-1)^{n-1} \mu(F_c) + \sum_{i=0}^{n-1} (-1)^i (\gamma_{\text{gen}}^i - \gamma_c^i),$$

where F_{gen} is a general fibre of f and γ_{gen}^* is its generic polar intersection multiplicities. ◇

It follows that the invariants γ_c^* are more refined than the number λ_c defined as the total jump at infinity in case of a polynomial with isolated $\mathcal{W}$-singularities at infinity [5, Corollary 3.5]. Further results on γ^* are contained in [11].

1.4 Corollary *Let $l \in \Omega$.*

(a) *If F_a has an isolated singularity at $\alpha \in F_a$ then $\alpha \in \Gamma(l, f)$.*

(b) *If $\Gamma(l, f) = \emptyset$ then $H_j(F_{\text{gen}}, \mathbb{Z}) = 0$, for $j \geq n-1$.* ◇

Monodromy

We show how to use affine polar curves, as defined in [10], in order to study the monodromy around an atypical value or ∞. The polar curve $\Gamma(l, f)$ is needed for defining a controlled vector field lifting the unitary tangent vector field of a path within $\mathbb{P}^1 \setminus \Lambda$. In this way one defines a *global geometric monodromy* along any simple loop within $\mathbb{P}^1 \setminus \Lambda$, thus getting a geometric monodromy representation

$$\rho : \pi_1(\mathbb{P}^1 \setminus \Lambda) \to \text{Diff}(F),$$

where F is a general fibre of f and $\mathrm{Diff}(F)$ is the group of C^∞-diffeomorphisms of F, which induces an algebraic monodromy representation

$$\rho_{\mathrm{alg}} : \pi_1(\mathbb{P}^1 \setminus \Lambda) \to H_*(F, \mathbb{Z}).$$

We keep control over the vector field "at infinity" by stratifying the space $\mathbb{X}$ (as in [5], [10]) and by imposing tangency conditions along the strata at infinity.

We describe in the following a geometric monodromy of a general fibre of f along a small circle in the base space $\mathbb{P}^1 \setminus \tilde{\Lambda}$, where $\tilde{\Lambda}$ is a finite set including Λ', the critical values of f and the values $a \in \mathbb{C}$ such that $\Gamma(l, f)$ tends asymptotically to $f^{-1}(a)$.

Take a small closed disc D_a at $a \in \tilde{\Lambda}$ such that $\tilde{\Lambda} \cap D_a = \{a\}$. By Lemma 1.1 and the argument following it, one can lift the unitary vector field $\mathbf{u}$ on the circle ∂D_a to a vector field $\mathbf{w}$ in the tube $f^{-1}(\partial D_a)$ such that $\mathbf{w}$ is tangent to $p^{-1}(\partial D_a) \cap \mathbb{X}^\infty$ in a stratified sense and tangent to $\Gamma(l, f) \cap f^{-1}(\partial D_a)$, for some general linear l. Note that the set $\Gamma(l, f) \cap f^{-1}(\partial D_a)$ is a finite union of circles.

Moreover, one can construct a vector field $\mathbf{w}$ by lifting $\mathbf{u}$ in two steps:

$$f^{-1}(\partial D_a) \xrightarrow{(l,f)} \mathbb{C} \times \partial D_a \xrightarrow{\mathrm{pr}_2} \partial D_a.$$

This idea was used in the local case by Lê D.T. [3], [4]. In the global setting, we may decompose the monodromy flow in regions where the local *carrousel construction* of Lê can be used. There is a carrousel construction associated to each point $q \in \overline{\Gamma(l, f)} \cap p^{-1}(a)$, including the case where $q \in \mathbb{X}^\infty \cap p^{-1}(a)$. Namely, there is a small closed disc δ at each point $\bar{l}(q)$ of $\mathbb{P}^1$, $q \in \overline{\Gamma(l, f)} \cap p^{-1}(a)$, such that $\mathbf{v}$ is the carrousel vector field on $\delta \times \partial D_a$, for a small enough D_a. In particular, the lift $\mathbf{v}$ of $\mathbf{u}$ to $\mathbb{C} \times \partial D_a$ is tangent to the discriminant $\Delta(l, f)$ and the vector field $\mathbf{v}$ is the identical lift by the projection $\mathrm{pr}_2 : \{b\} \times \partial D_a \to \partial D_a$, for any point $b \in \partial\delta$. Moreover, this is the case for any point $b \in \mathbb{P}^1 \setminus \cup_{i=1}^k \delta_i$, where δ_i is a small enough disc centered at d_i and the set $\{d_1, \ldots, d_k\} \in \mathbb{P}^1$ is the image by $\bar{l}$ of the set $\overline{\Gamma(l, f)} \cap p^{-1}(a)$. By convention, d_1 denotes the point $\infty \in \mathbb{P}^1$.

This special vector field $\mathbf{v}$ is now lifted to $f^{-1}(\partial D_a)$ giving rise, by integration, to a geometric monodromy, denoted by h_a. Note that, for some point $b \in \mathbb{P}^1 \setminus \cup_{i=1}^k \delta_i$, this monodromy restricts to a monodromy of the slice fibration:

$$f^{-1}(\partial D_a) \cap l^{-1}(b) \to \partial D_a.$$

Using this construction, one can prove the following result for the zeta function of the monodromy h_a.

1.5 Theorem $\zeta_{h_a}(t) = \zeta_{h_{a|F'}}(t) \cdot \zeta_{\mathrm{rel}}(t)$,
where F' is a general fibre of the map (l, f) and $\zeta_{\mathrm{rel}}(t) = \prod_{i=1}^k \zeta_{rel,i}(t)$. By $\zeta_{rel,i}(t)$ we denote the zeta-function of the relative monodromy of the pair $(F_c \cap l^{-1}(\check{\delta}_i), F_c \cap l^{-1}(s_i))$, for $c \in \partial D_a$ and s_i some fixed point on the circle $\partial\delta_i$. We denote $\check{\delta}_i = \delta_i$, for $i \geq 2$ and $\check{\delta}_1 = \delta_1 \setminus \alpha_1$, where α_1 is the radius from d_1 to a point on the circle $\partial\delta_1$ different from s_1.

Proof Note first that $\delta_i \times \{q\}$ contains by definition all the points of $\bar{\Delta}_i(l,f) \cap \mathbb{P}^1 \times \{q\}$, where $q \in \partial D_a$ and $\bar{\Delta}_i(l,f)$ denotes the germ of $\bar{\Delta}(l,f)$ at $(d_i, a) \in \mathbb{P}^1 \times \mathbb{C}$. By the above description, the relative homology $H_*(F_c, F')$ is concentrated in dimension $n-1$. From the exact sequence of the pair (F_c, F') we get the 4-term exact sequence:

$$0 \to H_{n-1}(F_c) \to H_{n-1}(F_c, F') \to H_{n-2}(F') \to H_{n-2}(F_c) \to 0$$

and, for $j \geq 3$, the isomorphisms:

$$0 \to H_{n-j}(F') \to H_{n-j}(F_c) \to 0.$$

The geometric monodromy constructed above acts on the exact sequence of the pair (F_c, F'). Now, the relative homology splits into a direct sum and the same holds for the action of the monodromy:

$$H_{n-1}(F_c, F') = \oplus_{i=1}^k H_{n-1}(F_c \cap l^{-1}(\check{\delta}_i), F_c \cap l^{-1}(s_i)).$$

$\diamond$

Moreover, the relative homology $H_{n-1}(F_c \cap l^{-1}(\check{\delta}_i), F_c \cap l^{-1}(s_i))$ is localizable at the points $\Gamma(l,f) \cap F_c \cap l^{-1}(s_i)$. This can be seen in the next examples. For further developments, see [7]. The second author used polar curves to find a local zeta function result in [9].

1.6 Corollary (a) *If f has isolated $\mathcal{W}$-singularities at infinity and isolated singularities in $\mathbb{C}^n$ then $h_{a|F'}$ is the identity.*

(b) *Let $c_i \in \mathbb{C}$. If $l^{-1}(c_i) \cap \operatorname{Sing} F_a = \emptyset$ then the relative monodromy of the pair $(F_c \cap l^{-1}(\delta_i), F_c \cap l^{-1}(s_i))$ is the identity.*

Proof (a) follows from [5, Cor. 3.6] and the constructions in [10, §5].
(b) follows from the fact that f is a stratified submersion in the neighborhood of $q \in F_a \cap l^{-1}(c_i)$, relative to the stratification $\{\mathbb{C}^n \setminus \Gamma(l,f), \Gamma(l,f)\}$. $\diamond$

Finally, we consider two examples in case $n = 2$.

1.7 Example $f : \mathbb{C}^2 \to \mathbb{C}$, $f(x,y) = x + x^2 y$. This was brought into attention by Broughton [1] as the simplest polynomial with a noncritical atypical fibre, see also [5], [10] for further comments on it. The point $[0:1] \in \mathbb{P}^1$ is critical at infinity for the fibre F_0, in the sense of [5]. The jump at this point, as defined by Lê D.T. and Hà H.V. [2], is $\lambda = 1$.

For a general l, say $l = x + y$, the polar curve $\Gamma(l,f)$ intersects a general fibre F_{gen} in 3 points and the fibre F_0 in 2 points. This gives $\gamma^1_{\text{gen}} = 3$, $\gamma^0_{\text{gen}} = \gamma^0_0 = 3$ and $\gamma^1_0 = 2$, therefore $\chi(F_{\text{gen}}) = 0$, $\chi(F_0) = 1$. However, it is easy to see by direct computations that $F_{\text{gen}} \overset{\text{ht}}{\simeq} S^1$ and $F_0 \overset{\text{ht}}{\simeq} \mathbb{C} \coprod S^1$. The zeta-function of the monodromy h_0 around the value 0 is equal, by Corollary 1.6 (a) and Theorem 1.5, to $\zeta_{F'}(t) \cdot \prod_{i=1}^3 \zeta_{\text{rel},i}(t)$. The pair $(F_t \cap l^{-1}(\check{\delta}_1), F_t \cap l^{-1}(s_1))$, where δ_1 is a small disc around $\infty \in \mathbb{P}^1$, is homotopy equivalent to $(I, \partial I)$ (since $\Gamma(l,f) \cap F_t \cap l^{-1}(\delta_1)$ is just one point) and it appears that the monodromy acts on the two points ∂I as the identity (I denotes the interval $[0,1]$). By Corollary 1.6 (b), $\zeta_{\text{rel},2}(t) = \zeta_{\text{rel},3}(t) = (1-t)^{-1}$ and by the above, $\zeta_{\text{rel},1}(t) = (1-t)^{-1}$ also. We get $\zeta_{h_0}(t) = 1$, since $\zeta_{h_{0|F'}}(t) = (1-t)^3$.

1.8 Example $f: \mathbb{C}^2 \to \mathbb{C}$, $f(x,y) = x^2y^2 + xy + x$. This is contained in the classification list of polynomials of small degrees, with respect to their singularities including those at infinity, of the first author with Smeltink [6]. There is a Morse singularity at $(0,-1)$, on the fibre F_0 and a singularity at infinity at $[0:1] \in \mathbb{P}^1$ for the fibre $F_{-\frac{1}{4}}$. Hence the total Milnor number is $\mu = 1$ and the total jump is $\lambda = 1$. The general fibre is homotopy equivalent to $S^1 \vee S^1$. We may take as general linear form $l = x + y$. Then $\Gamma(l,f) = \{2xy^2 + y + 1 - 2x^2y - x = 0\}$ and its intersection with F_t is 4 points, if $t = 0$ or $t = -\frac{1}{4}$ and 5 points for the other values of t. The fibre $F' = F_{\rm gen} \cap \{x + y = s\}$ is 4 points, for generic s. We get $\gamma^1_{\rm gen} = \gamma^1_0 = 5$, by Theorem 1.5 and $\gamma^1_{-\frac{1}{4}} = 4$. We compute the zeta-function of the monodromy $h_{-\frac{1}{4}}$. We have 4 points $\{q_2, q_3, q_4, q_5\} = \Gamma(l,f) \cap F_{-\frac{1}{4}}$ and to each such point there corresponds a monodromy on the relative homology $H_1(I, \partial I)$ which is the identity. The monodromy corresponding to the point $q_1 = \overline{\Gamma(l,f)} \cap \overline{F_{-\frac{1}{4}}}$ *is not the identity*, since it switches the points ∂I. This relative monodromy cannot occur as monodromy of an ordinary Morse point, therefore the singularities at infinity represent a new type of singularities. Their behavior, from the point of view of monodromy, is investigated in [7].

By Theorem 1.5, we get:

$$\zeta_{h_{-\frac{1}{4}}}(t) = (1-t)^4(1-t)^{-4}(1+t) = 1 + t.$$

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