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Critical configurations of planar robot arms

Research Article

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Abstract: It is known that a closed polygon P is a critical point of the oriented area function if and only if P is a cyclic polygon, that is, P can be inscribed in a circle. Moreover, there is a short formula for the Morse index. Going further in this direction, we extend these results to the case of open polygonal chains, or robot arms. We introduce the notion of the oriented area for an open polygonal chain, prove that critical points are exactly the cyclic configurations with antipodal endpoints and derive a formula for the Morse index of a critical configuration.

MSC: 52Cxx, 58E05

Keywords: Mechanical linkage • Robot arm • Configuration space • Moduli space • Oriented area • Morse function • Morse index • Cyclic polygon

1. Introduction

Geometry of various special configurations of robot arms modeled by open polygonal chains appears essential in many problems of mechanics, robot engineering and control theory. The present paper is concerned with certain planar configurations of robot arms appearing as critical points of the oriented area considered as a function on the moduli

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space of the arm in question. This setting naturally arose in the framework of a general approach to extremal problems on configuration spaces of mechanical linkages developed in [2, 4, 5], which has led to a number of new results on the geometry of cyclic polygons [3, 6] and suggested a variety of open problems. The approach and results of [2, 5] provided a paradigm and basis for the developments presented in this paper.

Let us now outline the structure and main results of the paper. We begin with recalling necessary definitions and basic results concerned with moduli spaces and cyclic configurations. In Section 3 we prove that critical configurations of a planar robot arm are given by the cyclic configurations with diametrical endpoints called diacyclic, Theorem 3.1, and describe the structure of all cyclic configurations of a robot arm, Theorem 3.3. Next, we establish that, for a generic collection of lengths of the links, the oriented area is a Morse function on the moduli space, Theorem 3.6, and provide some explications in the case of a 3-arm. In Section 4 we prove an explicit formula for the Morse index of a diacyclic configuration, Theorem 4.6, and illustrate it by a few visual examples. In conclusion we briefly discuss several open problems and related topics.

2. Oriented area function for planar robot arm

Let $L = (l_1, \dots, l_n)$, $L \in \mathbb{R}^n_+$. Informally, a *robot arm*, or an *open polygonal chain* is defined as a linkage built up from rigid bars (edges) of lengths l_i consecutively joined at the vertices by revolving joints. It lies in the plane, its vertices may move, and the edges may freely rotate around endpoints and intersect each other. This makes various planar configurations of the robot arm. In the engineering context one can think of this as a closed chain with one link being "telescopic".

Let us make this precise. A *configuration* of a robot arm is defined as an $(n+1)$-tuple of points $R = (r_0, \dots, r_n)$ in the Euclidean plane $\mathbb{R}^2$ such that $|r_{i-1}r_i| = l_i$, $i = 1, \dots, n$. Each configuration carries a natural orientation given by vertices' order. To factor out the action of orientation-preserving isometries of the plane $\mathbb{R}^2$, we consider only configurations with two first vertices fixed: $r_0 = (0, 0)$, $r_1 = (l_1, 0)$. The set of all such planar configurations of a robot arm is called the *moduli space* of a robot arm. We denote it by $M^0(L)$. It is a subset of Euclidean space $\mathbb{R}^{2n-2}$ and inherits its topology and differentiable space structure so that one can speak of smooth mappings and diffeomorphisms in this context. After these preparations it is obvious that the moduli space of any planar robot arm is diffeomorphic to the torus $(S^1)^{n-1}$. We will use its parametrization by angle-coordinates β_i (that is, by angles between r_0r_1 and r_kr_{k+1}, $k = 1, \dots, n-1$).

In this paper we consider the oriented (signed) area as a function on $M^0(L)$.

Definition 2.1.
For any configuration R of L with vertices $r_i = (x_i, y_i)$, $i = 0, \dots, n$, its (*doubled*) *oriented area* $A(R)$ is defined by

$$2A(R) = (x_0y_1 - x_1y_0) + \dots + (x_ny_0 - x_0y_n).$$

In other words, we add the connecting side r_nr_0 turning a given configuration R into an $(n+1)$-gon and compute the oriented area of the latter. Obviously, $A(R)$ is a smooth function on the moduli space $M^0(L)$ of any robot arm L.

3. Critical configurations. 3-arms

A configuration $R = (r_0, \dots, r_n)$ of a robot arm $L = (l_1, \dots, l_n)$ is *cyclic* if all its vertices lie on a circle. A configuration is *quasicyclic* (a QC-configuration for short) if all its vertices lie either on a circle or on a (straight) line. A configuration is *closed cyclic* if the last and the first vertices coincide: $r_0 = r_n$. A configuration is *diacyclic* if it is cyclic and the "connecting side" r_nr_0 is a diameter of the circumscribed circle ("diacyclic" is a sort of shorthand for "diametrally cyclic"). In other words, the connecting side r_nr_0 passes through the center of the circumscribed circle or, equivalently, each interval r_0r_k is orthogonal to the interval r_kr_n for $k = 1, \dots, n-1$.

Theorem 3.1.

For any robot arm $L \in \mathbb{R}^n_+$, critical points of A on the moduli space $M^0(L)$ are exactly the diacyclic configurations of L.

Proof. As above, we assume that $r_0 = (0, 0)$, $r_1 = (l_1, 0)$. For a configuration $R = (r_0, \dots, r_n)$ we put $e_i = r_i - r_{i+1}$, $i = 1, \dots, n$. Obviously, $r_i = e_1 + \dots + e_i$ and $e_i = l_i(\cos\beta_i, \sin\beta_i)$. Denote by $a \times b$ the oriented area of the parallelogram spanned by vectors a and b (i.e., we take the third coordinate of their vector product). The differentiation of vectors e_i with respect to angular coordinates β_j will be denoted by upper dots (i.e., there will appear terms of the form $\dot{e}_i$). With these assumptions and notations we can write

$$A = \sum_{j=1}^{n} r_{j-1} \times r_j = \sum_{j=2}^{n} (e_1 + \dots + e_{j-1}) \times e_j = \sum_{1 \le i < j \le n} e_i \times e_j.$$

Taking partial derivatives with respect to β_k, $k = 2, \dots, n$, we get

$$\frac{\partial A}{\partial \beta_k} = -\sum_{i=1}^{k-1} e_i \times \dot{e}_k + \sum_{i=1}^{k-1} e_k \times \dot{e}_i.$$

Notice now the identities

$$\dot{e}_i \times e_j = e_i \cdot e_j = -e_i \times \dot{e}_j.$$

Eventually we get

$$\frac{\partial A}{\partial \beta_k} = -\sum_{i=1}^{k-1} e_k \cdot e_i + \sum_{i=k+1}^{n} e_k \cdot e_i = \left(-\sum_{i=1}^{k-1} e_i + \sum_{i=k+1}^{n} e_i \right) \cdot e_k.$$

Consider now the equations $\partial A / \partial \beta_k = 0$, $k = 2, \dots, n$, defining the critical set of A. By taking appropriate linear combinations of equations, this system of $n - 1$ equations is easily seen to be equivalent to the system of equations

$$\left(\sum_{i=1}^{k-1} e_i \right) \cdot \left(\sum_{i=k}^{n} e_j \right) = 0, \qquad k = 2, \dots, n.$$

In geometric terms this means that the intervals $r_0 r_{k-1}$ and $r_{k-1} r_n$ are orthogonal for $k = 2, \dots, n$. It remains to refer to Thales' theorem to conclude that the points $r_0, \dots, r_n$ lie on a circle with diameter $r_0 r_n$. $\square$

Lemma 3.2.

The order of the lengths $l_1, \dots, l_n$ does not matter: for any permutation σ, there exists a diffeomorphism taking $M^0(L)$ to $M^0(\sigma L)$ which preserves the function A, and therefore, all the critical points together with their Morse indices.

The proof (which repeats the proof of the similar lemma for closed polygons from [6]) is as follows. Two consecutive edges of a configuration can be (geometrically) permuted in such a way that the oriented area remains unchanged. Such a geometrical permutation yields a diffeomorphism from one configuration space to another.

Theorem 3.3.

Assume that $l_1 > l_i$ for all $i = 2, \dots, n$. Then we have the following:

(a) *The set of all quasicyclic configurations is a disjoint collection of 2^{n-2} embedded (topological) circles (QC-components for short).*

(b) *Each of the circles contains at least two critical points of A.*

(c) *Assuming that all critical points are Morse non-degenerate, A is a perfect Morse function if and only if each circle has exactly two critical points of A.*

(d) *Each of the circles contains exactly two aligned configurations.*

Proof. We shall use the following notation: For a quasicyclic configuration, we define $\varepsilon_i = 1$ if the center of the circle lies to the left with respect to $r_{i-1}r_i$. Otherwise we put $\varepsilon_i = -1$. If the configuration is aligned, it lies on the axis OX. We define ε_i according to the orientation of the axis.

We show that each collection of signs $\varepsilon_i = \pm 1$, $i = 3, \dots, n$, yields a (topological) circle of quasicyclic configurations. Indeed, fix $\varepsilon_3, \dots, \varepsilon_n$. Take a (metric) circle $S(\rho)$ whose radius ρ varies from l_1 to infinity. A differentiable coordinate for a QC-component is e.g. the angle between the first and the second arm (mod 2π). The change of this angle induces a differentiable change of the radius ρ and each vertex moves around the intersection of a circle with center r_i and radius l_i, which intersects the circumscribed circle (with radius ρ) transversally (due to the condition $l_1 > l_i$).

If $l_1 < \rho < \infty$, the circle $S(\rho)$ has exactly one (up to a rigid motion) inscribed configuration with $E = (\pm 1, 1, \varepsilon_3, \varepsilon_4, \dots, \varepsilon_n)$ and exactly one inscribed configuration with $E(R) = (\pm 1, -1, -\varepsilon_3, -\varepsilon_4, \dots, -\varepsilon_n)$. The QC-component becomes in this way divided into four arcs, each with prescribed type of E, parameterized by the radius ρ. At the endpoints (that is, if $\rho = l_1$ or $\rho = \infty$) the four arcs join. More precisely, the arc that corresponds to $1, 1, \varepsilon_3, \dots, \varepsilon_n$ is followed by the arc that corresponds to $-1, 1, \varepsilon_3, \dots, \varepsilon_n$, then the next one with $+1, -1, -\varepsilon_3, \dots, -\varepsilon_n$, then to the one with $-1, -1, -\varepsilon_3, \dots, -\varepsilon_n$, and then to $1, 1, \varepsilon_3, \dots, \varepsilon_n$, see Figure 1. Continuity reasons imply that each such a circle of quasicyclic configurations has at least two diacyclic ones.

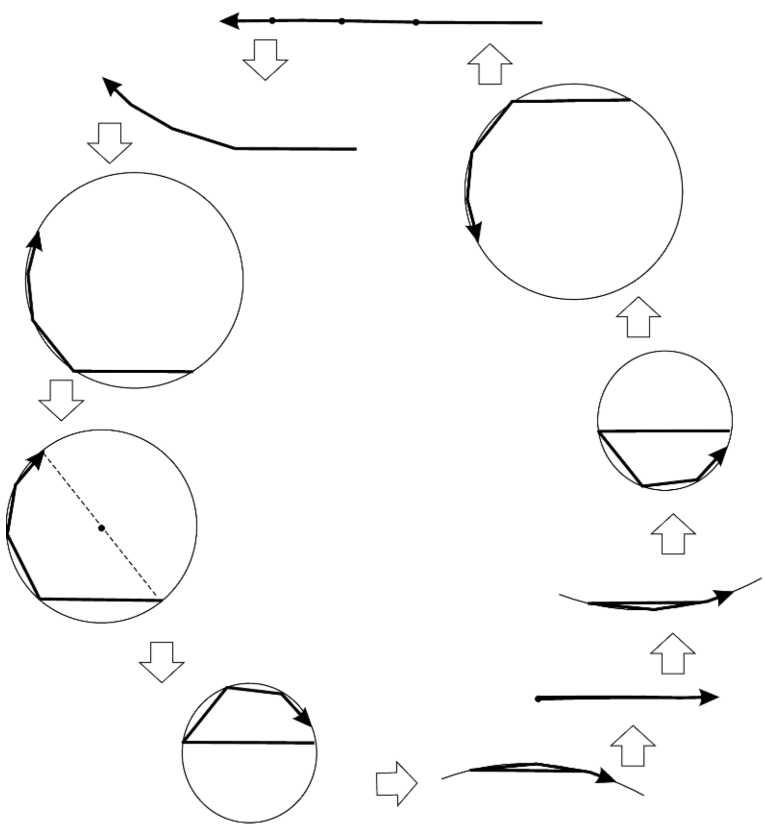

Figure 1. A circle of quasicyclic configurations

Remark 3.4.

The condition $l_1 > l_i$ is important: if there are several longest edges, the QC-components acquire common points. For instance, for an equilateral arm, they form a connected set.

Remark 3.5.

A QC-component can contain besides the diacyclic and aligned arms also closed cyclic arms (polygons). These special configurations are related to critical points of functions on configuration spaces (respectively, oriented area of an arm, squared length of the closing interval, see [1], and oriented area of a polygon, see [5]). Note that existence of a closed polygon on a QC-component (as well as the number of diacyclic configurations) depends on $l_1, \dots, l_n$. □

Theorem 3.6.

For a generic sidelength vector $L \in \mathbb{R}^n_+$, the function A has only non-degenerate critical points on $M^0(L)$.

Proof. The proof from [6] is applicable with some evident modifications. Namely, after introducing a local coordinate system with diagonals as coordinates, the Hessian matrix becomes tridiagonal with analytic entries. Deformation arguments show that a perturbation of just two of the edge lengths l_i makes the Hessian non-zero. □

For a 2-arm we obviously have two points: one maximum and one minimum.

Proposition 3.7.

Generically, for a 3-arm A has exactly four critical points on $M^0(P)$. If A is a Morse function (that is, if the Hessian is non-degenerate), these are two extrema and two saddles, see Figure 2. Extrema are given by the convex diacyclic configurations.

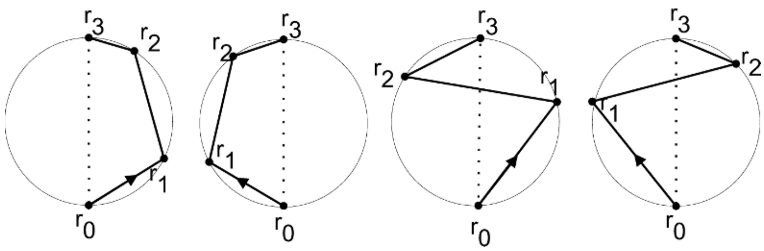

Figure 2. Diacyclic configurations for a generic 3-arm

Proof. Partial derivatives give the conditions for critical point:

$$\frac{\partial A}{\partial \beta_1} = l_1 l_2 \cos\beta_1 - l_2 l_3 \cos(\beta_2 - \beta_1) = 0, \qquad \frac{\partial A}{\partial \beta_2} = l_1 l_3 \cos\beta_2 + l_2 l_3 \cos(\beta_2 - \beta_1) = 0.$$

The orthogonality conditions are simply $r_0 r_1 \perp (r_1 r_2 + r_2 r_3)$, $(r_0 r_1 + r_1 r_2) \perp r_2 r_3$. The next step is to show that there are exactly four critical points. This can be done as follows. One uses elementary geometry to obtain a cubic equation for the length d of the connecting edge,

$$d^3 = (l_1^2 + l_2^2 + l_3^2)\, d \pm 2 l_1 l_2 l_3.$$

One has to solve these equations in d, taking into account $d \geq l_1$ and $d \geq l_3$. Elementary calculation shows that both the $+$ equation and the $-$ equation have a unique solution satisfying these conditions. From $\cos\beta_1 = \pm l_3/d$, $\cos(\beta_2 - \beta_1) = \pm l_1/d$ it follows that there are exactly two solutions in each case. They occur in pairs (β_1, β_2), $(-\beta_1, -\beta_2)$, which gives the result. □

Notice that this reasoning shows in all cases (except for $l_1 = l_2 = l_3$), that there are four critical points; in the generic case they are all Morse.

Example 3.8.

In the case $l_1 = l_2 = l_3$, there are three critical points on the torus: one maximum of A, one minimum, and a monkey saddle point. Figure 3, left depicts the level sets of A on the torus, whereas generically we have Figure 3, right. Note that in this case we obtain the minimal number of critical points of a differentiable function on a torus. It is equal to the Lusternik–Schnirelmann category of the torus, see [7].

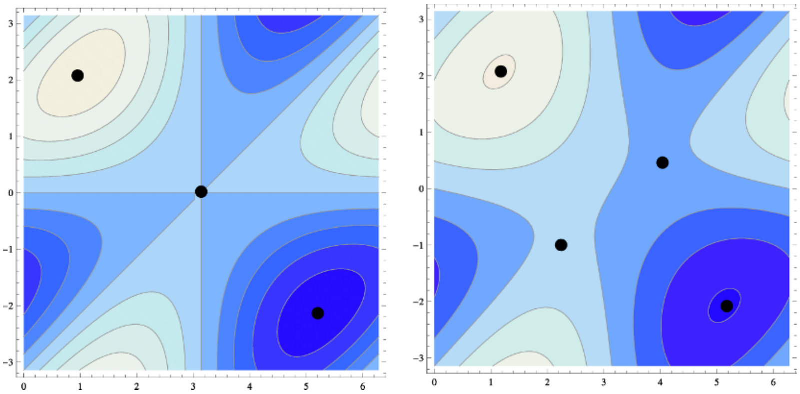

Figure 3. Level sets of the function A for $l_1 = l_2 = l_3$ (left) and generic (right)

4. On Morse index of a diacyclic configuration

We start with some examples. For arbitrary $n > 3$, the oriented area function A may or may not be a perfect Morse function.

Example 4.1.

Let $n = 4$ and $L = (10, 3, 2, 1)$. To be more precise, we take the lengths generically perturbed in order to guarantee non-degenerate critical points. Then configuration space is $M^0(L) = (S^1)^3$. Its Betti numbers are $\beta_0 = 1$, $\beta_1 = 3$, $\beta_2 = 3$, $\beta_3 = 1$. Direct computations show, that there are exactly eight critical points on $M^0(L)$ (the four configurations depicted in Figure 4 and their symmetric images). Therefore for this particular linkage A is a perfect Morse function.

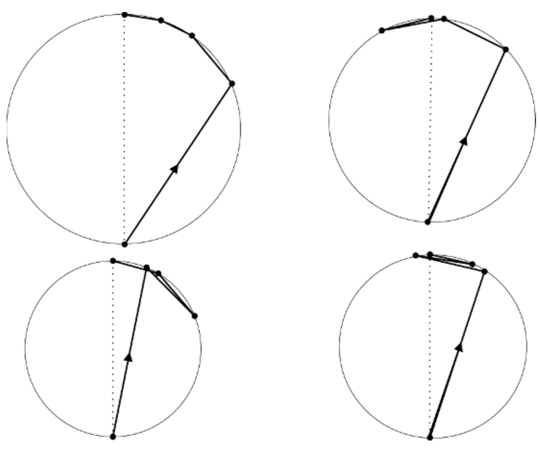

Figure 4. A is a perfect Morse function for $L = (10, 3, 2, 1)$

Example 4.2.

Let now $L = (22, 17, 21.9, 19)$. Again, $M^0(L) = (S^1)^3$. However, direct computations show that there are more than eight critical points on $M^0(L)$ (the six configurations depicted in Figure 5 and their symmetric images). Therefore, in this case A is not a perfect Morse function. There are two QC-components with three diacyclic configurations, whereas all others have only one.

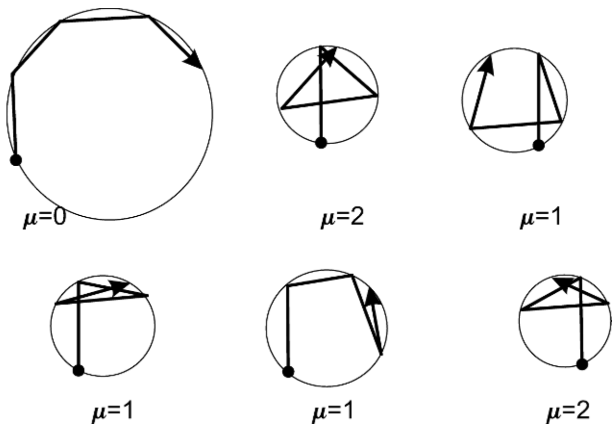

Figure 5. A is not a perfect Morse function for $L = (22, 17, 21.9, 19)$

Now we are going to find the Morse index of a diacyclic configuration of a robot arm by reducing the problem to the Morse index of a critical configuration of some closed linkage. First, we remind the reader the details about closed linkages. A *closed linkage* can be described as a flexible polygon on a plane. It is defined by its string of edges $L = (l_1, \dots, l_n)$, $L \in \mathbb{R}^n_+$. A *configuration of a closed linkage* is defined as an n-tuple of points $P = (p_1, \dots, p_n)$ in the Euclidean plane $\mathbb{R}^2$ such that $|p_i p_{i+1}| = l_i$, $i = 1, \dots, n$. Here the numeration is cyclic, i.e. $p_{n+1} = p_1$.

Definition 4.3.

For any configuration P of L with vertices $p_i = (x_i, y_i)$, $i = 1, \dots, n$, its (*doubled*) *oriented area* $A(R)$ is defined as

$$2A(P) = (x_1 y_2 - x_2 y_1) + \dots + (x_n y_1 - x_1 y_n).$$

Generically, the oriented area function is a Morse function on moduli space of a closed linkage.

Theorem 4.4 ([5]).

Generically, a polygon P is a critical point of the oriented area function A iff P is a cyclic configuration.

We will use the following notations for cyclic configurations, both open and closed:

- O is the center of the circumscribed circle.
- α_i is the half of the angle between the vectors $\overrightarrow{Op_i}$ and $\overrightarrow{Op}_{i+1}$. The angle is defined to be positive, orientation is not involved.
- Each edge has an orientation ε_i with respect to the circumscribed circle:

$$\varepsilon_i = \begin{cases} 1 & \text{if the center } O \text{ lies to the left of } p_i p_{i+1}, \\ -1 & \text{if the center } O \text{ lies to the right of } p_i p_{i+1}. \end{cases}$$

- $E(P) = (\varepsilon_1, \dots, \varepsilon_n)$ is the string of orientations of all the edges.
- $e(P)$ is the number of positive entries in $E(P)$.
- $\mu_P = \mu_P(A)$ is the Morse index of the function A at the point P.
- For cyclic configuration P of a closed linkage, ω_P is the winding number of P with respect to the center O.

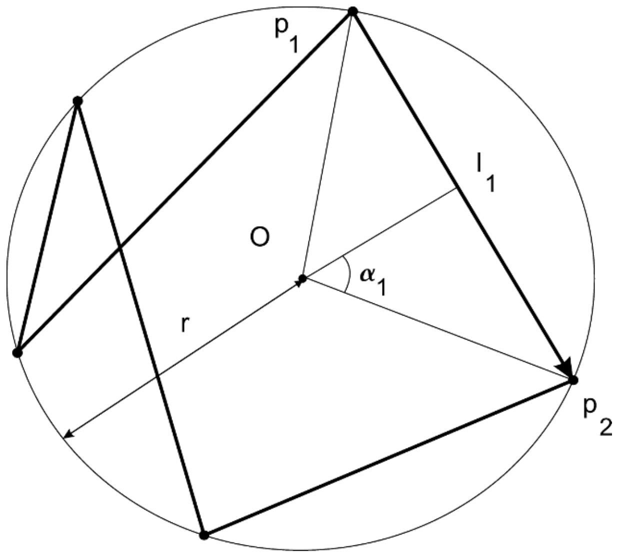

Figure 6. Notation for a pentagonal cyclic configuration with $E = (-1, -1, -1, 1, -1)$

Theorem 4.5 ([3]).
For a generic cyclic configuration P of a closed linkage L,

$$\mu_P(A) = \begin{cases} e(P) - 1 - 2\omega_P & \text{if} \quad \delta(P) > 0, \\ e(P) - 2 - 2\omega_P & \text{otherwise.} \end{cases}$$

Here $\delta P = \sum_{i=1}^{n} \varepsilon_i \tan \alpha_i$.

Returning to open chains, let R be a diacyclic configuration. Define its *closure* R^{Cl} as a closed cyclic polygon obtained from R by adding two positively oriented edges, see Figure 7, and denote by ω_R the winding number of the polygon R^{Cl} with respect to the center O. After this preparation we can present the below formula for the Morse index.

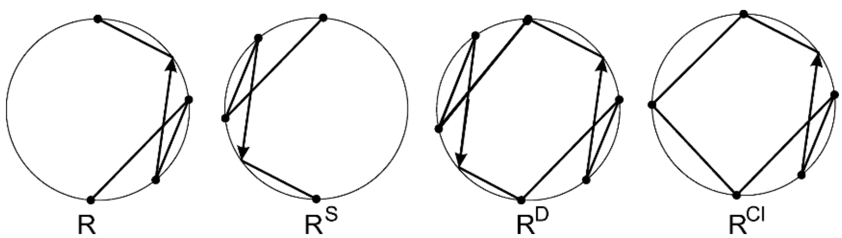

Figure 7. An open chain, its symmetry image, duplication and closure

Theorem 4.6.

Let $L = (l_1, \dots, l_n)$ be a generic open linkage, and let R be one of its critical configuration. For the Morse index $\mu_R(A)$ of the oriented area function A at the point R, we have

$$\mu_R(A) = \begin{cases} e(R) - 2\omega_R + 1 & \text{if} \quad \delta(R) > 0, \\ e(R) - 2\omega_R & \text{otherwise.} \end{cases}$$

Here $\delta R = \sum_{i=1}^{n} \varepsilon_i \tan \alpha_i$.

Proof. Consider the manifold $M_2^\circ(L) \times M_2^\circ(L) = \{R_1 \times R_2 : R_1, R_2 \in M_2^\circ(L)\}$. Generically, the function $A(R_1 \times R_2) = A(R_1) + A(R_2)$ is a Morse function on $M_2^\circ(L) \times M_2^\circ(L)$. Next, define the *duplication* of L as the closed linkage $L^D = (l_1, l_2, \dots, l_n, l_1, l_2, \dots, l_n)$. Consider a mapping ϕ which splits a polygon $P \in L^D$ into two open chains, R_1 and R_2. The mapping ϕ embeds $M_2(L^D)$ as a codimension one submanifold of $M_2^\circ(L) \times M_2^\circ(L)$.

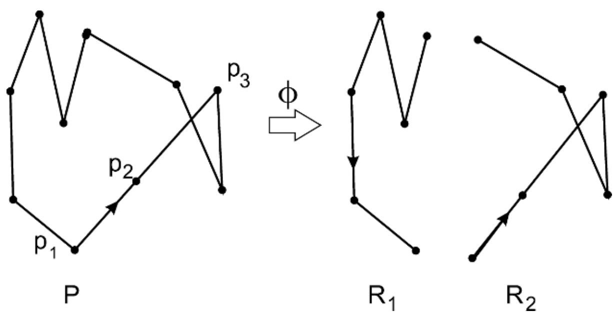

Figure 8. The mapping ϕ splits a closed chain into two open chains

For a cyclic open chain R, define R^S as the symmetric image of R with respect to the center O. Define also $R^D \in M_2(L^D)$ as a cyclic closed polygon obtained by patching together R and R^S. By Theorem 4.4, R^D is a critical point of the oriented area. On the one hand, the Morse index of its image $\phi(R^D) = R \times R^S$ on the manifold $M_2^\circ(L) \times M_2^\circ(L)$ equals $2\mu_R$. On the other hand, the Morse index of R^D on the manifold $M_2(L^D)$ is known by Theorem 4.5.

Since $M_2(L^D)$ embeds as a codimension one submanifold of $M_2^\circ(L) \times M_2^\circ(L)$, the Morse indices differ at most by one. More precisely, we have the following lemma.

Lemma 4.7.

Either $\mu_{R^D} = 2\mu_R$, or $\mu_{R^D} = 2\mu_R - 1$.

By Theorem 4.5,

$$\mu_{R^D} = \begin{cases} e(R^D) - 2\omega(R^D) - 1 & \text{if} \quad \delta(R^D) > 0, \\ e(R^D) - 2\omega(R^D) - 2 & \text{otherwise.} \end{cases}$$

Clearly, we have $e(R^D) = 2e(R)$, $\delta(R^D) = 2\delta(R)$, and $\omega(R^D) = 2\omega(R) - 1$. This gives us

$$\mu_{R^D} = \begin{cases} 2e(R) - 4\omega(R) + 1 & \text{if} \quad \delta(R) > 0, \\ 2e(R) - 4\omega(R) & \text{otherwise.} \end{cases}$$

Assume that $\delta(R) > 0$. Then $\mu_{R^D} = 2e(R) - 4\omega(R) + 1$ which is an odd number. The only possible choice in Lemma 4.7 is $2\mu_R = 2e(R) - 4\omega(R) + 2$. Analogously, if $\delta(R) < 0$ we conclude that $2\mu_R = 2e(R) - 4\omega(R)$. $\square$

Example 4.8.

Figure 7 depicts a number of diacyclic configurations for which we obviously have $\delta(R) > 0$. The Morse indices are calculated easily. The robot arm in question has four more diacyclic configurations symmetric to the depicted ones. For them, we easily have Morse indices $2, 2, 2$, and 0. The robot arm in Figure 5 presents more diacyclic configurations with their Morse indices.

5. Concluding remarks

We now wish to outline certain of the natural problems and perspectives suggested by the above results.

1. The most intriguing problem is to find an analog of the generalized Heron polynomial for n-arm, i.e., a univariate polynomial such that its roots give the critical values of area on the moduli space of an arm. Specifically, to find out what is the minimal algebraic degree of such a polynomial. Existence of such a polynomial follows from the general results of algebraic geometry using elimination theory but this does not give sufficient information about its algebraic degree.

2. Consider all n-arms with fixed n. What is the exact upper bound for the number of diacyclic configurations of such an n-arm? An estimate is provided by the degree of generalized Heron polynomial of the duplicate $2n$-gon but this upper bound is far from exact and the problem remains unsolved starting with $n = 4$. An exact upper bound could be obtained as the algebraic degree of a generalized Heron polynomial sought in the first problem.

3. As we have shown, the oriented area may or may not be a perfect Morse function on the configuration space of n-arm. For which collection of the lengths l_i is it perfect, i.e. has the minimal possible number of nondegenerate critical points equal to the sum of Betti numbers of moduli space? In other words, we seek for a criterion of perfectness of oriented area in terms of the lengths of the links. A related problem is to find out if the area can be a function with the minimal possible number of critical points given by the Lusternik–Schnirelmann category of the moduli space. As we have seen, this is the case for equilateral 3-arms. Does the same hold for equilateral 4-arms?

4. An interesting issue is suggested by our description of quasi-cyclic configurations. Namely, as we have seen, each component of quasi-cyclic configurations contains special points of three types: diacyclic, closed cyclic and critical points of the square of the connecting side. Are there any relations between the points of these three types?

5. Analogous problems may be considered for configurations of an arm in three-dimensional space.

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