

# EQUILIBRIA OF THREE CONSTRAINED POINT CHARGES

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ABSTRACT. We study the critical points of Coulomb energy considered as a function on configuration spaces associated with certain geometric constraints. Two settings of such kind are discussed in some detail. The first setting arises by considering polygons of fixed perimeter with freely sliding positively charged vertices. The second one is concerned with triples of positive charges constrained to three concentric circles. In each of these cases the Coulomb energy is generically a Morse function. We describe the minima and other stationary points of Coulomb energy and show that, for three charges, a pitchfork bifurcation takes place accompanied by an effect of the Euler's Buckling Beam type.

## 1. INTRODUCTION

We deal with equilibrium configurations of point charges with Coulomb interaction satisfying certain geometric constraints. Our approach to this topic is similar to the paradigms used in [4], [5]. Namely, we consider the Coulomb energy as a function on a certain configuration space naturally associated with the constraints in question, and investigate its critical points. The main attention in this paper is given to two specific problems naturally arising in this setting. The first one deals with identification and calculation of equilibrium configurations of charges satisfying the given constraints. The second one, called the *inverse problem*, is concerned with characterizing those configurations of points for which there exists a collection of non-zero charges such that the given configuration is a critical point of Coulomb energy on the corresponding configuration space. Such configurations are called *Coulomb equilibria*.

The geometric constraints considered below come from two sources: (i) Coulomb energy of point charges freely sliding along a flexible planar contour of fixed length, and (ii) Coulomb energy of concentric orbitally constrained triples of charges. The first setting was inspired by the concept of "necklace with interacting beads" introduced and investigated

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by P. Exner[1]. This setting has been considered in a similar situation in our previous paper [6].

It turns out that interesting results exist even in the case of three charges. We focus on the minimum energy and the other stationary points and values. While for almost equal charges the minimum is achieved on a triangle configuration, it turns out that in both settings the global minimum is achieved in an aligned situation if one of the charges is much smaller (or bigger, depending on the setting) than the others. The transition between these two states exhibits the well-known *supercritical pitchfork bifurcation* accompanied by a *fixing* effect, similar to the Euler Buckling Beam phenomenon [3, 8].

It seems worthy of adding that in most of our considerations the Coulomb forces can be replaced by various other central forces. The qualitative behaviour will be the same with modifications on the quantitative side.

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## 2. CONFIGURATION SPACES AND COULOMB ENERGY

We are basically interested in studying the equilibrium configurations of a system of repelling charges. As usual equilibria are defined as the critical (stationary) points of Coulomb energy of an  $n$ -tuple of points  $\{p_i\}$  defined by the formula

$$E = \sum_{i < j} \frac{q_i q_j}{d_{ij}},$$

where  $q_i$  are some positive numbers (charges) assigned to the points  $p_i$ , and  $d_{ij}$  are the distances  $|p_i p_j|$ . To this end the Coulomb energy is considered as a function on a certain configuration space naturally associated with the given geometric constraints.

Our first configuration space is the space of all labeled  $n$ -tuples of points<sup>1</sup> with the constraint that the perimeter is not bigger than 1:

$$\overline{M}(n) = \{(p_1, \dots, p_n) \mid p_i \in \mathbb{R}^2, p_1 = 0, \sum_{i=1}^n |p_i p_{i+1}| \leq 1\} / SO(2).$$

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<sup>1</sup>Indices are modulo  $n$ , so in the summation we assume  $p_{n+1} = p_1$ .

Informally, one can think of a closed rope with freely sliding positively charged points on it. Factorization by  $SO(2)$  means that we are only interested in the shape of a configuration and ignore orientation preserving motions. However, we do not identify symmetric  $n$ -tuples.

Let us also introduce the space

$$M(n) = \{(p_1, \dots, p_n) \mid p_i \in \mathbb{R}^2, p_1 = 0, \sum_{i=1}^n |p_i p_{i+1}| = 1\} / SO(2).$$

It is naturally identified with the space of all planar polygons with fixed perimeter (vertices are allowed to coincide) factorized by orientation preserving motions. The elements of the space  $M(n)$  are called either *polygons* or *configurations*.

For our purposes, it is important to know the topological type of the configuration space.

**Theorem 1.** *The space  $M(n)$  is diffeomorphic to the complex projective space  $\mathbb{C}P^{n-2}$ .*

Proof. By definition,

$$\mathbb{C}P^{n-2} = \{(u_1 : \dots : u_{n-1}) \mid u_i \in \mathbb{C}, \text{ not all } u_i = 0\},$$

assuming that two proportional  $(n-1)$ -tuples are identified. We add one more term and write

$$\mathbb{C}P^{n-2} = \{(u_1 : \dots : u_{n-1} : -\sum_{i=1}^{n-1} u_i) \mid u_i \in \mathbb{C}, \text{ not all } u_i = 0\},$$

with identification

$$(u_1 : \dots : u_{n-1} : -\sum_{i=1}^{n-1} u_i) = (\lambda \cdot u_1 : \dots : \lambda \cdot u_{n-1} : -\lambda \cdot \sum_{i=1}^{n-1} u_i).$$

This can be interpreted as the space of all  $n$ -gons with non-zero perimeter. Indeed, complex numbers  $u_i$  yield vectors in the plane. The factorization by multiplication by complex numbers amounts to factorization of the space of polygons by all possible rotations and scalings.  $\square$

The space  $\overline{M}(n)$  is a cone over  $M(n)$ , and we have a natural inclusion  $\overline{M}(n) \supset M(n)$ . All the polygons with non-zero perimeter strictly smaller than 1 form a manifold diffeomorphic to  $\mathbb{C}P^{n-2} \times \mathbb{R}$ , so it makes sense to speak of critical points of the Coulomb energy  $E$ . The informal message of the following proposition is that the "sliding charges on a closed rope" problem reduces to "fixed perimeter" problem.

**Proposition 1.** *The Coulomb energy has no critical points in  $\overline{M}(n)$  outside  $M(n)$ .*

Proof. Assume  $P$  is a critical polygon whose perimeter is strictly smaller than 1. Its dilation gives a tangent vector with a non-zero derivative of the Coulomb energy since the dilation strictly increases all pairwise distances between the points.  $\square$

The second constrained system is associated with a system of three concentric circles. It is defined by a triple of positive numbers  $r_1, r_2, r_3$ . The corresponding configuration space can be represented as

$$T(r_1, r_2, r_3) = \{(p_1, p_2, p_3) | p_i \in \mathbb{R}^2, |p_1| = r_1, |p_2| = r_2, |p_3| = r_3\} / SO(2).$$

In other words, it is the configuration space of triples of points lying on three concentric circles. Clearly,  $T$  is diffeomorphic to the torus  $S^1 \times S^1$ . The elements of  $T(r_1, r_2, r_3)$  are also called *configurations*.

For both of the above spaces, a configuration is called *aligned* if all its vertices lie on a line.

**Remark.** On each of the above spaces there is a natural involution  $\nu$  which takes a configuration to its image under reflection (with respect to a line). For a configuration  $P = \{p_i\}$ , we have  $\nu(P) = P$  if and only if all the points  $p_i$  lie on a line. Since the Coulomb energy is symmetric with respect the involution, that is,  $E(P) = E(\nu(P))$ , each non-aligned critical configuration comes together with its symmetric image, which is also critical, and has the same Morse index.

### 3. THREE CHARGES ON A CONTOUR WITH GIVEN PERIMETER

In this section we study Coulomb energy on the space  $M(3)$  which we also call the space of triangles. Theorem 1 implies that  $M(3)$  is homeomorphic to the two-sphere  $S^2$ .

Let us first analyze Coulomb energy of three constrained charges in the case when the ambient space has dimension one. That is, we consider the configuration space of all aligned triangles with fixed perimeter. Topologically this configuration space is a circle. Each such aligned triangle is a segment of length  $\frac{1}{2}$  whose endpoints are two of the points  $p_i$  and the third point lies between these two (or equals one of them). Two such aligned triangles are identified if they differ by reflection.

The Coulomb energy has three poles on the circle (which correspond to maxima), and three minima. Each of the minima corresponds to the choice of the point that lies between two others. The following observation is well known and easy to prove by direct computation.

**Lemma 1.** *Assume that charges  $(q_1, q_2, q_3)$  are positioned on a line at points  $p_1, p_2, p_3$  satisfying the fixed perimeter condition. Then there exists a unique critical point (which is the local minimum) of the Coulomb energy with the point  $p_2$  lying between  $p_1$  and  $p_3$ . The point  $p_2$  does not depend on  $q_2$  and is given by the proportion*

$$\frac{d_{12}}{d_{23}} = \frac{\sqrt{q_1}}{\sqrt{q_3}}.$$

*The other local minima (for  $p_1$  and for  $p_3$  as intermediate charges) come analogously. The global minimum is the one with the smallest intermediate charge. All these critical points are non-degenerate Morse points.*  $\square$

Now let us return to the space  $M(3)$ . We use a shorter notation  $l_1 = d_{23}, l_2 = d_{31}, l_3 = d_{12}$ . In this notation:

$$E = \frac{q_1 q_2}{l_3} + \frac{q_2 q_3}{l_1} + \frac{q_3 q_1}{l_2}.$$

**Theorem 2.** *For any three positive charges, the Coulomb energy  $E$  has three obvious poles:  $p_1 = p_2$ ,  $p_2 = p_3$ , and  $p_1 = p_3$ . The minima points and the saddle points depend on the charges. More precisely,*

- (1) *If the triple  $(\frac{1}{\sqrt{q_1}} : \frac{1}{\sqrt{q_2}} : \frac{1}{\sqrt{q_3}})$  satisfies the strict triangle inequality then we have:*
  - (a)  *$E$  has exactly two minima points. They correspond to mutually symmetric triangles whose sidelengths come from the proportion  $(l_1 : l_2 : l_3) = (\frac{1}{\sqrt{q_1}} : \frac{1}{\sqrt{q_2}} : \frac{1}{\sqrt{q_3}})$ .*
  - (b)  *$E$  has exactly three saddle Morse points which correspond to aligned configurations from Lemma 1.*
- (2) *Assume that the triple  $(\frac{1}{\sqrt{q_1}} : \frac{1}{\sqrt{q_2}} : \frac{1}{\sqrt{q_3}})$  does not satisfy the triangle inequality, namely  $\frac{1}{\sqrt{q_1}} > \frac{1}{\sqrt{q_2}} + \frac{1}{\sqrt{q_3}}$ . Then we have:*
  - (a)  *$E$  has exactly one minimum point. It corresponds to the aligned configuration described in Lemma 1 with  $p_1$  lying between  $p_2$  and  $p_3$ .*
  - (b)  *$E$  has exactly two saddle Morse points. They correspond to remaining aligned configurations described in Lemma 1 with either  $q_2$  or  $q_3$  lying between the two other charges.*
- (3) *Assume that the triangle inequality becomes an equality:  $\frac{1}{\sqrt{q_1}} = \frac{1}{\sqrt{q_2}} + \frac{1}{\sqrt{q_3}}$ . Then we have:*
  - (a)  *$E$  has exactly one degenerate minimum point. It corresponds to the aligned configuration described in Lemma 1*

with  $p_1$  lying between  $p_2$  and  $p_3$ . It can be viewed as a meeting point of two minimum points and one saddle point.

- (b)  $E$  has exactly two non-degenerate saddle Morse points, as in the previous case.

Proof. First assume that a non-aligned triangle is a critical point of Coulomb energy. Observe that with respect to coordinates  $(l_1, l_2, l_3)$  we have:  $\nabla(E) = (-\frac{q_1 q_2}{l_3^2}, -\frac{q_2 q_3}{l_1^2}, -\frac{q_3 q_1}{l_2^2})$ . The Langrange multiplier method with constraint  $l_1 + l_2 + l_3 = 1$  gives:

$$l_1^2 q_1 = l_2^2 q_2 = l_3^2 q_3,$$

which has a unique solution for given  $(q_1, q_2, q_3)$ . As soon as  $(l_1, l_2, l_3)$  satisfy the strict triangle inequality this is also a solution of our problem (case 1). Otherwise all critical configurations are aligned.

Next we consider an aligned configuration. Let us prove that it is critical in the plane if and only if it is critical in dimension one. Assume that the intermediate vertex is  $p_2$ , and denote the configuration by  $P$ . Since we can no longer use  $l_i$ 's as local coordinates (as we did in the proof of Lemma 1), we introduce other local coordinates on the space  $M(3)$  in a neighborhood of  $P$ . Namely, for a configuration close to  $P$ , we may assume that the points  $p_1$  and  $p_3$  lie on the  $x$ -axis. The position of  $p_2$  uniquely defines the triangle, so we can take the coordinates of  $p_2$  as local coordinates. Since  $E(x, y) = E(x, -y)$ , we have a critical point as soon as the restriction to the  $x$ -axis is critical. Moreover, the Hessian matrix is diagonal in  $(x, y)$  coordinates.

Now we know that there are exactly three critical points that are aligned configurations. To complete the proof one has to use the relation  $\sharp(\text{maxima}) + \sharp(\text{minima}) - \sharp(\text{saddles}) = 2$ . That is,

(1) If the triangle inequality holds, there exist two symmetric non-aligned critical points. Elementary calculations show that these are minimum points. In this case the three aligned configurations are saddles.

(2) Otherwise, the minimum is one of the aligned configurations. Clearly it is the one with the smallest charge as the intermediate point. The other two are saddles.

It only remains to check that the critical points are generically non-degenerate. Besides, the below detailed analysis of the determinant of the Hessian gives a better understanding of what's going on. A similar approach is used in the subsequent section.

Assume that we have an aligned configuration with sidelengths<sup>2</sup>  $a$  and  $b$  as is depicted in Figure 3, left. We know from Lemma 1 that  $\frac{\partial^2 E}{\partial x^2} > 0$ , so we are interested in the sign of  $\frac{\partial^2 E}{\partial y^2}$ . We show that the latter depends on the charges.

So we take the one-parametric family of configuration that depends symmetrically on  $y$  (see Figure 3, right) and write:

$$l_3 = \sqrt{(a-t)^2 + y^2} = a - t + \frac{y^2}{2a} + o(y^2),$$

$$l_1 = \sqrt{(b-s)^2 + y^2} = b - s + \frac{y^2}{2b} + o(y^2),$$

$$l_2 = a + b - s - t = 1/2 - s - t.$$

The perimeter stays the same, so

$$2s + 2t = \frac{y^2}{2a} + \frac{y^2}{2b} + o(y^2).$$

We may choose any symmetric family, so let us assume that  $s = t$ , which implies  $t = s = y^2(\frac{1}{8a} + \frac{1}{8b})$ . Substituting this in the Coulomb energy formula we get

$$E = \frac{q_1 q_3}{\frac{1}{2} - (\frac{1}{4a} + \frac{1}{4b})y^2} + \frac{q_1 q_2}{a - (\frac{1}{8a} + \frac{1}{8b})y^2 + \frac{y^2}{2a}} + \frac{q_3 q_2}{b - (\frac{1}{8a} + \frac{1}{8b})y^2 + \frac{y^2}{2b}} + o(y^2). \quad (*)$$

It suffices to make qualitative analysis of the formula. First, we conclude that generically,  $\frac{\partial^2 E}{\partial y^2}$  is non-zero, and therefore, we have a Morse point. We also see that the second derivative  $\frac{\partial^2 E}{\partial y^2}$  is positive if  $q_2$  is relatively small, since in this case the derivative of the first term is positive and majorates the other terms.  $\square$



FIGURE 1.

Now let us analyze the meaning of the above theorem from the bifurcation theory viewpoint. By homogeneity we may assume that  $q_1 + q_2 + q_3 = 1$ . Since we only consider positive charges the control

<sup>2</sup>We change notation for brevity.

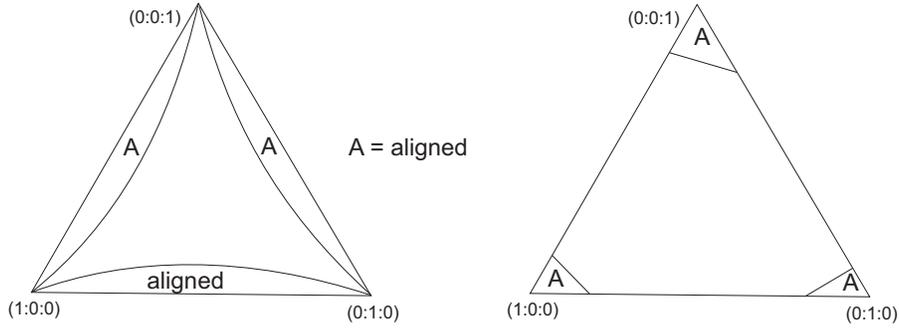


FIGURE 2. The control triangle and the bifurcation curves for the pitchfork bifurcation; left: Section 3: perimeter constraint, right: Section 5: concentric circles

space of our system is a triangle  $\Delta$ . In this triangle the bifurcation set is given by the equalities:

$$\frac{1}{\sqrt{q_1}} = \frac{1}{\sqrt{q_2}} + \frac{1}{\sqrt{q_3}}, \quad \frac{1}{\sqrt{q_2}} = \frac{1}{\sqrt{q_3}} + \frac{1}{\sqrt{q_1}}, \quad \frac{1}{\sqrt{q_3}} = \frac{1}{\sqrt{q_1}} + \frac{1}{\sqrt{q_2}}$$

The middle part of  $\Delta$  (see Fig. 2, left) corresponds to the case (1) of the above theorem: the minimum is achieved at two mutually symmetric non-degenerate triangles. For the other points of the triangle  $\Delta$  all critical configurations are aligned.

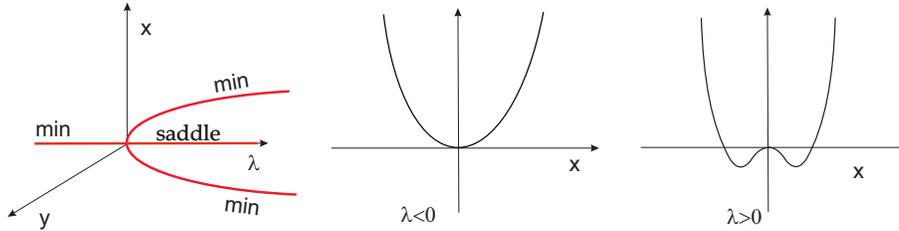


FIGURE 3. The pitchfork bifurcation

The situation we encounter here is a typical example of a (super-critical) pitchfork bifurcation. In the single variable case this is locally described by the potential  $f(x, y, \lambda) = \frac{x^4}{4} - \lambda \frac{x^2}{2} + y^2$ . Its critical set is given by  $x^3 - \lambda x = 0$ ,  $y = 0$ . So it consists of a parabola and a line in the  $(x, \lambda)$ -plane (see Fig. 3). The points on the  $\lambda$ -axis are critical with value 0 for all  $\lambda$ : a minimum for  $\lambda < 0$  and a saddle point for  $\lambda > 0$ . The points on the parabola are minima with value  $-\frac{1}{2}\lambda^2$ . This is the typical situation for a point on the bifurcation set [3].

We can conclude the following. Assume that we have the charge  $q_2 = 0$  whereas the other two charges are non-zero. Then the configuration  $(p_1, p_2, p_3)$  considered as a physical system becomes aligned with  $p_1$  and  $p_3$  at the endpoints, and the position of  $p_2$  does not matter. We begin to gradually increase the value of  $q_2$ . As soon as it is non-zero,  $p_2$  takes some specific position on the aligned configuration (see Lemma 1) (which does not depend on the value of  $q_2$  while it is small), stays at the same place (we call this *fixing effect*) until we cross over the bifurcation set, where it moves away from the line. We get two triangles as minima and the aligned position persists as a saddle point in the same position.

**Remark.** In view of the above, the inverse problem is now solved straightforwardly. Namely,

- (1) If three points in the plane are not collinear, their stabilizing charges are defined uniquely up to a scale.
- (2) If the three points are aligned, the stabilizing charges are defined not uniquely: the charges at the endpoints come from Lemma 1, whereas the charge of the intermediate point has to be small enough.

#### 4. GENERAL FIXING EFFECT AT ALIGNED POSITIONS

The direct computation for  $n$ -gons (even with  $n = 4$ ) leads to complicated equations that do not give a transparent solution. However some important qualitative observations can be done. An informal message of the below theorem is that for aligned configurations, there exists fixing effect of the Euler's Buckling Beam type.

**Theorem 3.** (1) *No convex critical configuration  $P$  has three charges lying on a line unless  $P$  is aligned. In other words, there is no fixing effect for non-aligned convex polygons.*

- (2) *If  $q_2, q_3, \dots, q_{n-1}$  are relatively small<sup>3</sup> with respect to the values of  $q_1$  and  $q_n$ , then the Morse index of an aligned configuration equals the Morse index of the same configuration in the case where the ambient space is  $\mathbb{R}^1$ . In particular, this means that minima points for the ambient space  $\mathbb{R}^1$  remain minima points for the ambient space  $\mathbb{R}^2$ , and we have the fixing effect for them.*

*Proof.* (1) Assume the contrary. In this case there exists an infinitesimal motion (that is, an element of the tangent space  $T_P M(n)$ ) which increases some of diagonals with non-zero derivative, whereas derivatives of all the rest of pairwise distances is zero, see Fig. 4.

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<sup>3</sup>This means that, for each pair  $q_1, q_n$ , there exists a positive number  $\delta$  such that the statement of the theorem is true for all  $q_2, \dots, q_{n-1}$  smaller than  $\delta$ .

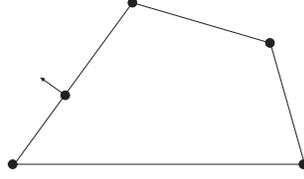


FIGURE 4. This infinitesimal motion gives non-zero derivative of  $E$ .

(2) Let us take an aligned configuration which is a critical point. We can assume that it lies on the  $x$ -axis. Assume that its endpoints are  $p_1$  and  $p_n$ . In its neighborhood in the space  $M(n)$  we choose  $(n-2)$  coordinates that correspond to  $x$ -coordinates of  $p_2, \dots, p_{n-1}$ . The other  $(n-2)$  coordinates correspond to  $y$ -coordinates of the same points. Symmetry arguments  $E(\vec{x}, \vec{y}) = E(\vec{x}, -\vec{y})$  imply that the Hessian matrix is a block matrix

$$\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix},$$

where  $H_1$  is the Hessian with respect to the  $x$ -coordinates, that is, the Hessian associated with the one-dimensional ambient space. We are going to show that the Hessian matrix  $H_2$  related to the coordinates  $y_i$  is positively definite provided that  $q_2, q_3, \dots, q_{n-1}$  are relatively small. Let us assume that  $n = 4$ ; the same approach works for  $n > 4$ .

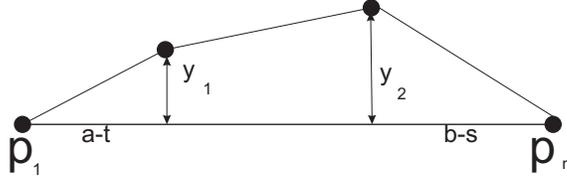


FIGURE 5. Notation for the proof of Theorem 3

We mimic the proof of formula (\*). Namely, we use the symmetry property of  $E$  and choose two shifts  $y_1$  and  $y_2$ . We can choose the coordinates in any symmetric way with respect to  $(y_1, y_2)$ , so let us take  $s = t$ . Using notations of Fig. 5, from the fixed perimeter condition we conclude that

$$s = t = \frac{1}{2} \left( \frac{y_1^2}{2a} + \frac{y_2^2}{2b} + \frac{(y_1 - y_2)^2}{2b} \right).$$

Therefore

$$E = \frac{q_1 q_4}{1 - \left( \frac{y_1^2}{2a} + \frac{y_2^2}{2b} + \frac{(y_1 - y_2)^2}{2b} \right)} + R.$$

Here  $R$  is a sum of fractions with smaller numerator and a denominator that quadratically depend on  $y_i$ . The first summand has an obviously positively definite Hessian and majorates the rest of the summands  $R$ .  $\square$

### 5. CONCENTRIC ORBITALLY CONSTRAINED TRIPLES OF CHARGES

We consider three concentric circles with radii  $r_1, r_2, r_3$  and three charges  $q_1, q_2, q_3$ . The corresponding configuration space  $T(r_1, r_2, r_3)$  can be identified with a 2-torus. Let us fix some notation (see Fig. 6):

$$d_1 = |p_2 p_3|, \quad d_2 = |p_3 p_1|, \quad d_3 = |p_1 p_2|,$$

$$\alpha_1 = \angle(p_2 p_0 p_3), \quad \alpha_2 = \angle(p_3 p_0 p_1), \quad \alpha_3 = \angle(p_1 p_0 p_2).$$

Note that  $d_i$  depends on  $\alpha_i$  via the cosine rule, e.g.  $d_3^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos \alpha_3$ .

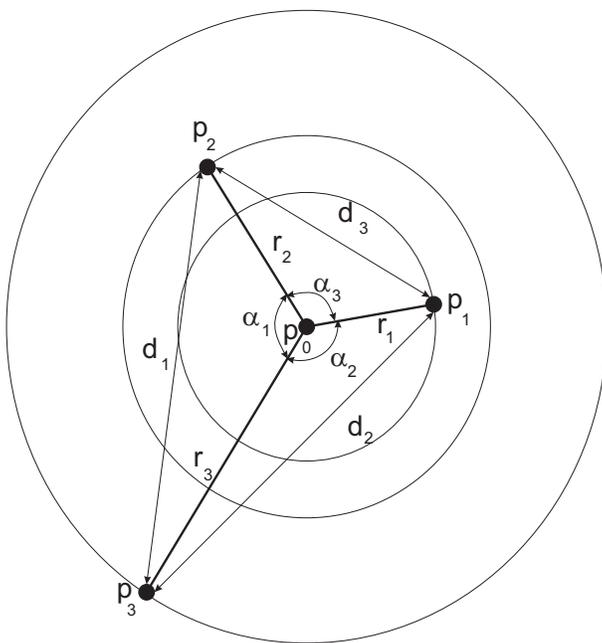


FIGURE 6. A configuration of concentrically constrained triple

We suppose that all charges are positive (but similar analysis works also if some of the charges are negative) and that all  $r_i$  are different.

The Coulomb energy

$$E = \frac{q_1 q_2}{d_3} + \frac{q_2 q_3}{d_1} + \frac{q_3 q_1}{d_2}$$

is thus a bounded function on the 2-torus for each  $q_1, q_2, q_3$ .

**Proposition 2.** *The critical points of  $E$  are given by:*

$$\frac{\sin \alpha_1}{d_1^3 r_1 q_1} = \frac{\sin \alpha_2}{d_2^3 r_2 q_2} = \frac{\sin \alpha_3}{d_3^3 r_3 q_3}$$

Proof. This follows by applying Lagrange multiplier method to  $E$  with the constraint  $\alpha_1 + \alpha_2 + \alpha_3 = 2\pi$ .  $\square$

An immediate corollary is that the aligned positions (where  $\alpha_i \equiv 0$  modulo  $\pi$ ) are among the critical points. There are four aligned configurations (see Fig. 7); let us study them first.

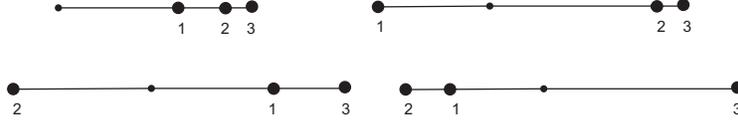


FIGURE 7. Aligned configurations

Using a parametrization via  $\alpha_1$  and  $\alpha_2$  we can compute the first and second derivatives of  $E$ , and evaluate them at the corresponding critical point. The matrix of second derivatives is (in the aligned case):

$$\begin{pmatrix} -q_1 q_2 \frac{r_1 r_2}{d_3^3} \cos \alpha_3 & q_2 q_3 \frac{r_2 r_3}{d_1^3} \cos \alpha_1 & -q_1 q_2 \frac{r_1 r_2}{d_3^3} \cos \alpha_3 \\ q_1 q_2 \frac{r_1 r_2}{d_3^3} \cos \alpha_3 & -q_1 q_2 \frac{r_1 r_2}{d_3^3} \cos \alpha_3 - q_3 q_3 \frac{r_3 r_1}{d_2^3} \cos \alpha_2 & \end{pmatrix}$$

The Hessian determinant (up to a positive multiple) becomes:

$$H(\alpha_1, \alpha_2, \alpha_3) = \frac{r_1}{d_2^3 d_3^3} q_1 \cos \alpha_2 \cos \alpha_3 + \frac{r_2}{d_3^3 d_1^3} q_2 \cos \alpha_3 \cos \alpha_1 + \frac{r_3}{d_1^3 d_2^3} q_3 \cos \alpha_1 \cos \alpha_2$$

Assume that  $\alpha_1 = \alpha_2 = \pi$ ,  $\alpha_3 = 0$ . In this case we have

$$d_1 = r_2 + r_3, \quad d_2 = r_1 - r_3, \quad d_3 = r_2 - r_1.$$

Therefore we have a linear form

$$H(\pi, \pi, 0) = -A_1 q_1 - A_2 q_2 + A_3 q_3 \quad (A_i > 0)$$

It follows that, for  $q_3$  big and  $q_1, q_2$  small, the Hessian is positive and gets negative on the other side of the line  $H = 0$  in the affine triangle  $\Delta$ . The first case corresponds to a local minimum; the second to a saddle point. Again we meet an example of the *pitchfork bifurcation*: fix  $q_1, q_2$  small and let  $q_3$  start big and decrease next. The system stays

at equilibrium minimum in the aligned position until it meets  $H = 0$ ; after that the aligned position become a saddle and two minima are born as symmetric triangles.

We give now the final result for all aligned situations.

**Theorem 4.** *All aligned configurations are critical points of the Coulomb energy. More precisely,*

- (1) *If  $\alpha_1 = \alpha_2 = \alpha_3 = 0$  then we have a non-degenerate maximum (which is the absolute maximum) for all  $(q_1, q_2, q_3)$ .*
- (2) *The other three cases depend each on  $(q_1, q_2, q_3)$ : we have a pitchfork bifurcation, which transforms a minimum into a saddle point.*

*Proof.*  $H$  has following form for each of the four cases (all constants below depend on the  $r_i$ 's and are positive)

- (1)  $(\alpha_1, \alpha_2, \alpha_3) = (\pi, \pi, 0) : H = -A_1q_1 - A_2q_2 + A_3q_3$
- (2)  $(\alpha_1, \alpha_2, \alpha_3) = (0, \pi, 0) : H = -B_1q_1 + B_2q_2 - B_3q_3$
- (3)  $(\alpha_1, \alpha_2, \alpha_3) = (\pi, 0, 0) : H = C_1q_1 - C_2q_2 - C_3q_3$
- (4)  $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0) : H = D_1q_1 + D_2q_2 + D_3q_3$

The sign of the Hessian determines the type of equilibrium. It can be computed from the  $q_i$ 's and  $r_j$ 's. In the first case the linear function  $H$  will change sign for positive  $q_i$ 's. In the last case  $H > 0$  for all  $q_i$ 's.  $\square$

Also here the control space (with normalized charges) is a triangle. The zero sets of the linear functions from Theorem 4 cut out three small triangles around vertices of this triangle (see Fig. 2, right). They don't intersect. Each of the triangles corresponds to a minimum state.

Unlike the constrained problem discussed in Section 3, the number of the non-aligned critical configurations is difficult to analyze. They are solutions of the equation in Proposition 2, which seems difficult to solve by hand. We proceed with discussing a few qualitative aspects and mention the results of some tests with Mathematica.

First, the non-degeneracy of  $E$  and Betti numbers of the torus imply that there are at least four critical points with sum of their Morse indices is equal to zero. In case of one minimum, two saddles and one maximum we have an exact Morse function. From Theorem 4 we have already four critical points, which includes a single maximum. If the three other points all are saddle points (where this happens can be computed from the above Hessians) then there must be at least two minima which correspond to (non-aligned) triangles. They come in couples via symmetry with the same critical value. Computer experiments suggest that there are no more critical points.

If one of the aligned critical points is a minimum, then two saddles and one maximum could suffice. This is supported by computer tests. Direct computations show that no more aligned minima appear. The existence of the pitchfork bifurcation also implies the existence of (at least) 2 minima for certain values of the  $q_i$ 's.

Observe that the Hessian also depends on  $r_1, r_2, r_3$ . Changing these values will also influence the position of critical configurations, including the effect of pitchfork bifurcation. An interesting point is also to take the limit to the equal radius case. In this case we have the following proposition.

**Proposition 3.** *For three positive equal radii and charges, the Coulomb energy has a unique minimal value, which is achieved at a equilateral triangle. This corresponds to two symmetric non-degenerate critical points in the state space.*  $\square$

*Proof.* We use the equalities from Proposition 2 and reduce them by direct computation to  $\cos \alpha_1 = \cos \alpha_2 = \cos \alpha_3$ . We find as only non-aligned solution the equilateral triangle. The Hessian at the critical points takes value  $\frac{25}{144}$ , with non-degeneracy a consequence. In the aligned cases at least two points coincide and the energy has poles with value infinity.  $\square$

As a consequence we have that, for almost equal radii and almost equal charges, the global minimum is achieved at two mutually symmetric triangles that are close to the equilateral one.

## 6. CONCLUDING REMARKS

The considerations and results of the present paper suggest a few immediate remarks.

(1) The same qualitative behaviour holds for several other central forces, that is, e.g. for the potentials

$$E = \sum_{i < j} \frac{q_i q_j}{d_{ij}^k} \quad (k > 1), \quad \text{or}$$

$$E = \sum_{i < j} q_i q_j \log d_{ij} \quad (\text{logarithmic force}).$$

(2) A natural generalization of the setting suggested in this paper arises if one considers equilibria of point charges lying on several different contours in the plane. In this case Coulomb energy defines a differentiable function on a torus so there are topological restrictions

on the number of critical points. It would be interesting to find examples of charges for which the number of equilibria is minimal or maximal.

(3) There apparently exist many other developments for which the results of this paper may serve as paradigms.

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