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CYCLIC CONFIGURATIONS OF PLANAR MULTIPLE PENDULI

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Abstract

We consider the signed area function A on the moduli space $M(P)$ of mechanical linkage P representing a planar multiple pendulum. For generic lengths of the sides of P, it is proved that A is a Morse function on $M(P)$ and its critical points are given by the cyclic configurations of P satisfying an additional geometric condition. For triple penduli, the main result is complemented by a rather comprehensive analysis of the structure of cyclic configurations. A number of related results and open problems are also presented.

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Introduction

We deal with certain *planar mechanical linkages* [3] which we call *planar multiple penduli.* Recall that a planar mechanical linkage is defined as a mechanism built up from rigid bars (sides) consecutively joined at flexible links (vertices) in such a way that each bar may freely rotate around any of its endpoints (pin-joints) and the whole system is confined to stay in a certain plane (reference plane) [3]. A linkage can move and make various configurations so that lengths of segments representing bars should remain unchanged but they may intersect and vertices may coincide. If it is required that the last vertex coincides with the first one then we speak of a *polygonal linkage* [3]. If there is no such requirement, in order to distinguish from the preceding case, one usually speaks of a *planar kinematic chain* [3] but we prefer to think of it as a *planar multiple pendulum* which seems more intuitive and adequate to the setting accepted in this paper.

Such linkages are also considered as useful models for a robot arm or mechanical manipulator and therefore often called *mechanical n-arms* [4]. However, we will exclusively use the term *planar multiple pendulum* (PMP) to avoid possible misleading interpretations and associations. Actually, we will basically work with the moduli space of a PMP which appears as a particular case of the general concept of moduli space of mechanical linkage [3].

Moduli spaces of mechanical linkages of various types were actively studied in the last few decades (see, e.g., [8], [18], [11]). In particular, critical points of various functions on moduli spaces have been discussed in [8], [11], [16]. Along these lines, we consider the signed (oriented) area as a function A on moduli space of a PMP and show that its critical points are given by the cyclic configurations satisfying some additional conditions of orthogonality and that, for $n = 3$, A is a Morse function on a generic moduli space. As usual under a cyclic configuration we understand a configuration of linkage such that all the vertices lie on the same circle. The study of cyclic polygons has a long history starting with elementary classical results such as the Ptolemy theorem and Brahmagupta formula (see, e.g., [5]). This topic continues to attract considerable interest (see, e.g., [7], [19]), in particular, due to the results and conjectures of D. Robbins concerned with computing the area of cyclic polygon [17]. The aim of this paper is to show that cyclic configurations also arise in the study of planar multiple penduli in a quite natural way.

We tried to make the exposition (reasonably) self-contained. To this end, in the first section we present various auxiliary results and comments about the cyclic configurations of multiple penduli as well as our main result (Theorem 1) which shows that cyclic configurations are the critical points of the signed area function. This result suggests a number of natural questions some of them are addressed in the sequel. Detailed results are only presented for *triple* penduli despite most of the statements and arguments admit quite straightforward generalizations for generic PnP's with arbitrary n. However, proving all results in full generality would require much more space and time. So only the key result (Theorem 1) is formulated and proved for multiple n-penduli with arbitrary n, while all other considerations are performed for *triple* penduli. The results for triple penduli seem to be of interest by themselves and may also serve as a paradigm for future research in this direction. With this in mind, in the last section we briefly discuss open problems and arising research perspectives.

1. Cyclic configurations of planar penduli

We begin with giving a rigorous definition of moduli space of a *planar multiple pendulum* (PMP). Let $l = (l_1, \ldots, l_n) \in \mathbb{R}^n$ be a collection of positive real numbers. A planar n-pendulum (PnP) $P_n(l)$ is defined as a linkage consisting of n-sides and $n-1$ pin-joints. For $i = 1, \ldots, n$, the i-th side $v_{i-1}v_i$ has length l_i and is pin-joint with the $(i+1)$-th side at their common endpoint v_i which is the $(i+1)$-th vertex v_i of the linkage $P_n(l)$. The last side has length l_n and can freely rotate about the vertex v_{n-1}. Notice that the first vertex is denoted by v_0. l is called the *sidelength vector* of P.

The definition of the moduli space of a planar multiple pendulum $P = P(l)$ runs as follows. One first introduces the set $C_2(P)$ defined as the collection of all n-tuples of points v_i in the Euclidean plane $\mathbb{R}^2$ such that the distance between v_{i-1} and v_i is equal to l_i, where $i = 1, \ldots, n$. Each such collection V of points, as well as the chain of line segments joining the consecutive vertices, is called a configuration of P. We assume that the corresponding (piecewise linear) curve is oriented by the given ordering of vertices. A configuration is called *cyclic* if all vertices lie on a certain circle and *aligned* if all vertices lie on the same straight line. Obviously, the latter type of configuration is a sort of limiting case of the former.

Factoring the configuration space $C_2(P)$ by the natural diagonal action of the group of orientation preserving isometries $Iso_+(2)$ of the plane $\mathbb{R}^2$ one obtains the *moduli space* $M(P)$ of a given PMP [11]. Moduli spaces are endowed with the natural topologies induced by Euclidean metric. It is easy to see that the moduli space $M(P)$ can be naturally identified with the subset of configurations such that $v_0 = (0,0), v_1 = (l_1, 0)$. Thus $M(P)$ can be considered as embedded in $\mathbb{R}^{2n-2}$. This embedding endows it with a differentiable space structure so that one can speak of smooth mappings and diffeomorphisms in this context [2].

After these preparations it is obvious that the moduli space of any planar n-pendulum is diffeomorphic to $(n-1)$-torus T^{n-1}. In the sequel we will encounter certain subsets of the moduli space. In order to describe these subsets we first make some comments about the cyclic configurations of points in the Euclidean plane.

Lemma 1. *If four points $v_i = (x_i, y_i)$ lie on the same circle then the following determinant vanishes*

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 & x_4^2 + y_4^2 \end{vmatrix} = 0. \tag{1}$$

Notice that condition (1) is necessary but not sufficient for four points to form a cyclic configuration because this determinant also vanishes in the case when they lie on a straight line.

For this reason we will say that a configuration of four points $v_i \in \mathbb{R}^2$ is *quasicyclic* if the above determinant vanishes. Such configurations are relevant for our considerations and so it seems natural to consider the set of all quasicyclic configurations $QC(P)$ of a given PMP P. In the sequel we'll describe its geometric structure for planar triple penduli. Let us add that many of the results in the sequel are formulated for *generic* PMP or PMP having *generic sidelength vector* l. As usual this means that the corresponding statements are true for all vectors l in a certain (non-specified) open dense subset of the parameter space $\mathbb{R}^n_+$ (cf. [9], [10]).

Since the moduli space of a PMP is a smooth manifold (the $(n-1)$-dimensional torus) it is natural to consider various geometrically meaningful functions on the moduli space and study critical points of those functions. In particular, taking into account the aforementioned embedding of $M(P)$ into $\mathbb{R}^{2n-2}$ we can consider restrictions to $M(P)$ of polynomial functions on $\mathbb{R}^{2n-2}$. If a function $f : M(P) \to \mathbb{R}$ arises as a restriction to $M(P)$ of a certain smooth function F on $\mathbb{R}^{2n-4}$ then the critical points of f can be found by the Lagrange method as the points $V \in M(P)$ such that $grad\, F$ is orthogonal to the tangent space $T_V(M(P))$ [1]. The main aim of this paper is to develop critical point theory for *the signed (oriented) area* ([5]) considered as a function on $M(P)$. Alternatively, since the torus is smooth, we can use a parametrization by angle-coordinates.

To this end recall that, for any configuration V of P with vertices $v_i = (x_i, y_i), i = 0, \ldots, n$, its (doubled) signed area $A(V)$ is defined by

$$A(V) = (x_0y_1 - x_1y_0) + \ldots + (x_ny_0 - x_0y_n). \tag{2}$$

In other words, we add the "connecting" side v_nv_0 turning a given configuration V in a $(n+1)$-gon $\overline{V}$ and compute the oriented area of the latter. Obviously, formula (2) defines a smooth function on $\mathbb{R}^{2n-2}$ and also on the moduli space $M(P)$ of any PMP P. Thus we can consider its critical points. Since the Lusternik-Schnirelmann category of an n-torus is equal to $n+1$ [10], from the Lusternik-Schnirelmann theory it follows that A certainly has critical points different from maxima and minima. Thus one may wish to find their amount and describe the behaviour of A near its critical points using standard paradigms of singularity theory [1].

Our main result (Theorem 1) states that critical points of A in $M(P)$ are given by certain cyclic configurations of P. For triple penduli, we will also show that, in fact, A is a Morse function on generic moduli space and so one may deal with it in the framework of Morse theory [10]. This suggests a number of natural problems for PnPs with arbitrary n, some of which are investigated in Section 3 for triple penduli. Analogous settings and results have earlier been discussed for polygonal linkages [15], [16].

Before presenting the main result we add a few remarks about the signed area which will appear useful in the sequel. With a given n-pendulum P we associate a $2n$-gon linkage Φ with the sidelength vector $(l_1, \cdots, l_n, l_1, \cdots, l_n)$. Then each configuration V of P defines a "doubled" configuration W of Φ by adding the result of reflection of V in the midpoint of the connecting side v_0v_n. By additivity of the area function we get $A(W) = 2A(V)$. By cutting W along other diagonals, connecting opposite points, we get several PMP's with the same area as V. The components of their sidelength vectors are any cyclic permutation of $(l_1, \cdots, l_n)$. The moduli space is again the $(n-1)$-torus and the area function is the same. So it makes no difference which of those PMP's we study, e.g. we can (if we want) suppose that the first arm has the biggest length, etc. Moreover, the "opposite" pendulum with sidelength $(l_n, \cdots, l_1)$ from v_n to v_0 with vertices in the opposite order has area function $-A(V)$.

To formulate the main result we need one more definition. Let us say that a configuration V of a planar n-pendulum P is *diacyclic* if it is cyclic and the "connecting side" $v_n v_0$ is a diameter of the circumscribed circle ("diacyclic" is a sort of shorthand for "diametrally cyclic"). In other words, the "connecting" side $v_n v_0$ passes through the center of the circumscribed circle or, equivalently, each interval $v_0 v_k$ is orthogonal to the interval $v_k v_n$ (whenever the last phrase is meaningful).

Theorem 1. *For any sidelength vector $l \in \mathbb{R}^3_+$, critical points of A on $M_2(P(l))$ are given by the diacyclic configurations of $P(l)$.*

Proof. As above, we assume that $v_0 = (0,0), v_1 = (l_1, 0)$. For a configuration $V = (v_0, \ldots, v_n)$ we put $e_i = v_i - v_{i-1}, i = 1, \ldots, n$. Obviously, $v_i = e_1 + \ldots + e_i$ and $e_i = l_i(\cos\beta_i, \sin\beta_i)$. Denote by $a \times b$ the signed area of the parallelogram spanned by vectors a and b (i.e., we take the third coordinate of their vector product). The differentiation of vectors e_i with respect to angular coordinates β_j will be denoted by upper dots (i.e. there will appear terms of the form $\dot{e_i}$).

With these assumptions and notations we can write

$$A = \sum_{j=1}^{n} v_{j-1} \times v_j = \sum_{j=2}^{n} (e_1 + \cdots + e_{j-1}) \times e_j = \sum_{1 \le i < j \le n} e_i \times e_j.$$

Taking partial derivatives with respect to β_k, $k = 2, \ldots, n$ we get

$$\partial A / \partial \beta_k = \sum_{i=1}^{k-1} e_i \times \dot{e_k} + \sum_{i=k+1}^{n} e_k \times \dot{e_i}.$$

Notice now the identities:

$$\dot{e_i} \times e_j = e_i \cdot e_j = -e_i \times \dot{e_j}.$$

Eventually we get:

$$\partial A / \partial \beta_k = -\sum_{i=1}^{k-1} e_k \cdot e_i + \sum_{i=k+1}^{n} e_k \cdot e_i = (-\sum_{i=1}^{k-1} e_i + \sum_{i=k+1}^{n} e_i) \cdot e_k.$$

Consider now the equations $\partial A / \partial \beta_k = 0$, $k = 2, \ldots, n$ defining the critical set of A. By taking appropriate linear combinations of equations, this system of $n-1$ equations is easily seen to be equivalent to the system of equations:

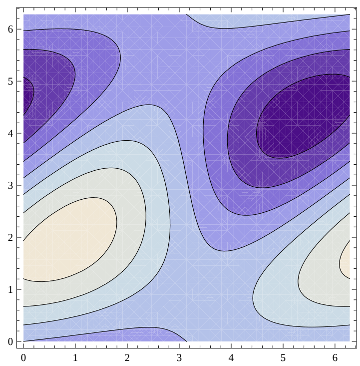
$$(\sum_{i=1}^{k-1} e_i) \cdot (\sum_{i=k}^{n} e_j) = 0, \; k = 2, \ldots, n.$$

In geometric terms this means that the intervals v_0v_{k-1} and $v_{k-1}v_n$ are orthogonal for $(k = 2, \ldots, n)$. It remains to refer to Thales theorem ([5]) to conclude that the points $v_0, \ldots, v_n$ lie on a circle with diameter v_0v_n. The proof is complete.

This theorem reveals the role of (dia)cyclic configurations and serves as a starting point for our further considerations. Notice an analogy with the case of polygonal linkages considered in [15], [6], [16].

2. Cyclic configurations of triple penduli

From now on we put $n = 3$ and deal exclusively with planar 3-penduli (P3P). We'll complement Theorem 1 by obtaining a rather complete description of critical points and critical values of the signed area function A on the moduli space of a P3P. It is again convenient to use the term "generic sidelength vector" in the aforementioned sense. Namely, we say that a statement holds for generic sidelength vector l if it holds for each collection l from an open dense subset of the parameter space $\mathbb{R}^3_+$. In the sequel we freely use a number of standard concepts of Morse theory and singularity theory which can be found in [1] and [10].



Level curves of A for generic sidelengths

For convenience, in the P3P case we modify the notation a bit. Vertices v_0, v_1, v_2, v_3 will be denoted by O, A, B, C, respectively. The sidelength vector is written as $(l_1, l_2, l_3) = (a, b, c)$, while side vectors are denoted by $\vec{e_1} = \vec{a}, \vec{e_2} = \vec{b}, \vec{e_3} = \vec{c}$. Thus we have $\vec{a} = (a, 0)$, $\vec{b} = b(\cos\beta, \sin\beta)$, $\vec{c} = c(\cos\gamma, \sin\gamma)$. A certain number of our computations were done (and can be verified) using the Mathematica package and for this reason in a few places below we (re)denote $\beta = x, \gamma = y$.

Now the signed area function (modulo a constant factor) can be written in the form

$$A = \vec{a} \times (\vec{a} + \vec{b}) + (\vec{a} + \vec{b}) \times (\vec{a} + \vec{b} + \vec{c}) =$$

$$= \vec{a} \times \vec{b} + (\vec{a} + \vec{b}) \times \vec{c} = ab\sin[x] + ac\sin[y] + bc\sin[y - x].$$

Below we make an essential use of a well-known paradigm of singularity theory which is referred to as the "parametric transversality paradigm" (PTP). There are several different formulations of parametric transversality paradigm (see, e.g., [9]). We cannot dwell upon the PTP here and just mention that for our purposes it is sufficient to use a rather elementary version of PTP described in [1]. Now we are ready to formulate and prove the three results we aimed at.

Theorem 2. *For a generic sidelength vector $l \in \mathbb{R}^3_+$, the function A has only non-degenerate critical points on $M_2(P(l))$.*

Proof of Theorem 2. The proof is achieved by merely applying the parametric transversality paradigm to the gradient mapping ∇A of function A. In our notation ∇A depends on parameters b and c and we want to show that for almost all (b, c) the mapping $\nabla A_{(b,c)}$ is a submersion over the origin. To this end we consider the Jacobi matrix of ∇A with respect to all of its variables (x, y, b, c) which obviously has the form

$$\begin{pmatrix} A_{xx} & A_{xy} & A_{xb} & A_{xc} \\ A_{yx} & A_{yy} & A_{yb} & A_{yc} \end{pmatrix}. \tag{3}$$

We call it the *extended hessian matrix* (EHM) of A. In order to apply the parametric transversality paradigm to the gradient map ∇A we need to find out at which points (b, c) the rank of the extended hessian matrix is equal to 2. Consider first the minor obtained by deleting the first two columns:

$$\begin{pmatrix} A_{xx} & A_{xy} & cCos[y-x] - aCos[x] & -cCos[y-x] \\ A_{yx} & A_{yy} & bCos[y-x] & aCos[y] - bCos[y-x] \end{pmatrix}. \tag{4}$$

It is easily seen that at the critical point the EHM takes the form

$$\begin{pmatrix} A_{xx} & A_{xy} & 0 & -cCos[y-x] \\ A_{yx} & A_{yy} & bCos[y-x] & 0 \end{pmatrix}. \tag{5}$$

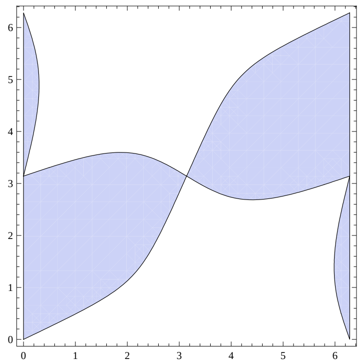
So the rank is maximal if $Cos[y - x] \neq 0$. In case of equality it follows that

$$Cos[y - x] = Cos[x] = Cos[y] = 0,$$

which gives a contradiction. Thus we can apply PTP to conclude that for generic (a, b, c) the area function is Morse. QED

Remark. For further use we notice that in our notation the Hessian (determinant of the hessian matrix) of A has the form:

$$H(x, y) = a\sin[x]\sin[y] + b\sin[x]\sin[y - x] + c\sin[y - x]\sin[y].$$



Zero locus of the signed area Hessian

Theorem 3. *For a generic sidelengths vector* $l \in \mathbb{R}^3_+$*, the set of quasicyclic configurations* $QC(P(l))$ *of planar triple pendulum* $P(l)$ *is a closed smooth one-dimensional submanifold of* $M_2(P(l))$*.*

Proof of Theorem 3. Let us denote by g the determinant (1) written in the natural angular coordinates on $M_2(P) \cong T^2$. In our notation one has

$$g[x, y] = c \sin[x] - b \sin[y] - a \sin[y - x] + 2b \sin[x] cos[y - x].$$

So we need to analyze the equation

$$c \sin[x] - b \sin[y] - a \sin[y - x] + 2b \sin[x] cos[y - x] = 0.$$

We can again fix the length a of the first side and consider g as depending on parameters b and c. To show that this g is submersion over the origin for almost all (b, c), we have to analyze the system:

$$\partial g/\partial b = - \sin[y] + 2 \sin[x] cos[y - x] = 0,$$

$$\partial g/\partial c = c \sin[x] = 0.$$

These equations are equivalent to $\sin[x] = \sin[y] = 0$. Substituting the solutions in the first two componets of ∇g we get a system

$$\partial g/\partial x = c cos[x] + a cos[y - x] + 2b cos[x] cos[y - x] 2b \sin[x] \sin[y - x] = 0,$$

$$\partial g/\partial y = -b cos[y] - a cos[y - x] + 2b \sin[x] \sin[y - x] = 0,$$

which can only happen if

$$\pm c \pm b \pm 2b = 0,$$

$$\pm b \pm a = 0.$$

Hence the conditions of the parametric transversality theorem [1] are fulfilled for all (b, c) which do not satisfy the above linear relations. Thus we can again apply PTP to finish the proof.

Theorem 4. *For a generic sidelength vector* $l \in \mathbb{R}^3_+$*, there exists a polynomial* P_l *with real coefficients such that the critical values of* A *on* $M_2(P(l))$ *are the roots of* P_l*. The coefficients of* P_l *can be polynomially expressed via the lengths of the sides of* $P(l)$*.*

Proof of Theorem 4. This follows from the results of [17] and [7]. To show this we introduce a "double" of P defined as the hexagon linkage Φ with the sidelength vector $(l_1, l_2, l_3, l_1, l_2, l_3)$. Then each diacyclic configuration V of P gives a cyclic configuration W of Φ by adding the result of reflection of V in midpoint of the connecting side (which coincides with the diameter of circumscribed circle). By additivity of the signed area function we get $A(W) = 2A(V)$. On the other hand, from the results of [17] it follows that the signed area of each such W coincides with a certain root of the real polynomial introduced in [17]. Thus the first statement follows from an analogous result for inscribed hexagons established in [17]. The proof in [17] did not give an effective way of constructing P_l but explicit algebraic formulae for the coefficients of P_l were given in [7]. These observations complete the proof.

Our results give a sufficiently visual picture of the critical points of A for a triple pendulum. In the next section we give a few comments and outline some possible generalizations for PnPs with arbitrary n.

3. Concluding remarks

We wish to add several comments on how one could complement and extend the above results about the critical configurations of a triple pendulum. One obvious perspective of further research is to verify if straightforward analogues of our three theorems for PsPs hold for generic PnPs with arbitrary $n \geq 4$. This leads to a number of natural and pleasantly looking problems. In particular, one could wish to find the maximal number of diacyclic configurations of a planar PnP. We point out that certain estimates can be derived from the results of [17] and [7] but we could not prove that they are exact for arbitrary n and so the problem remains largely open.

For completeness let us also say a few words about the case of double pendulum. It is obvious that the moduli space of a P2P is homeomorphic to the circle and all configurations are quasicyclic. Moreover, we can use all the above formula for hessian of A with $c = 0$. This immediately gives that A has one maximum ab (for $\beta = \pi/2$) and one minimum $-ab$ (for $\beta = -\pi/2$). Both these critical points are Morse and there are no other critical points.

For a generic planar triple pendulum we can explicate the results of the previous section as follows.

Proposition 1. *Generically, A has exactly 4 critical points on $M(P)$: two extrema and two saddles. Extrema are given by the convex diacyclic configurations. Each of these points is non-degenerate.*

Proof. Partial derivatives give the conditions for critical point:

$$\partial A/\partial x = abcos[x] - bccos[y-x] = 0,\ \partial A/\partial y = accos[y] + bccos[y-x] = 0.$$

The orthogonality conditions are simply $\vec{a} \perp (\vec{b} + \vec{c})$, $(\vec{a} + \vec{b}) \perp \vec{c}$.

The next step is to show that there are exactly 4 critical points. This can be done as follows. One uses elementary geometry to obtain a cubic equation for the length d of the connecting edge OC:

$$d^3 = (a^2 + b^2 + c^2)d \pm 2abc.$$

One has to solve these equations in d, taking into account $d \geq a$ and $d \geq c$. Elementary calculations show that both the + equation and the - equation have one solution satisfying these conditions. From $\cos\beta = \pm c/d$, $\cos(\gamma - \beta) = \pm a/d$ it follows that there are exactly two solutions in each case. They occur in pairs (β, γ), $(-\beta, -\gamma)$, which gives the result.

Notice that this reasoning applies to all cases, except when $a = b = c$. Then there are 3 critical points (one of which is a "monkey saddle" [1]). The proof of the next result is quite elementary and therefore we omit it.

Proposition 2. *Generically, the set of quasicyclic configurations $QC(P(l))$ has 2 connected components. The extrema always belong to the same component and the two saddles belong to another one.*

We add a few words about the "double" of a PnP P as defined at the end of section 1. This is a $2n$-gon linkage Φ with the sidelength vector $(l_1, \cdots, l_n, l_1, \cdots, l_n)$ and its moduli space has dimension $2n - 3$. It follows from [12] that this moduli space is smooth except at the aligned configurations; and its singularities are isolated and all of Morse type.

The proof of theorem 4 tells us: each diacyclic configuration V of P gives a cyclic configuration W of Φ. This links the critical points of the signed area function for a pendulum to critical points of the signed area function for the corresponding "double" $2n$-gon, since the cyclic configurations of the latter are just critical points of A. Note that all these points belong to the smooth part of the moduli space of Φ.

There are more critical points of A on this moduli space; there are even nonisolated singularities. They occur as follows: start with a planar pendulum in cyclic configuration (but not necessarily diacyclic). Consider its "pseudo-double" by reflecting the pendulum into the perpendicular bisector of v_0v_n. We get a configuration of $2n$-gon that starts with our pendulum and returns with the same pendulum in opposite order. All vertices are still on the same circle. This corresponds to a critical point of the signed area function with value 0 in the smooth part of the moduli space of Φ. Since the radius of the cylic configuration is a free parameter in the construction, we get a 1-parameter family of critical points. Thus A has nonisolated critical points on the moduli space of Φ.

Another wide perspective for further research is related to the calculation of the Morse indices of diacyclic configurations. This topic is poorly understood also for polygon linkages (cf. [15]). Given formulae for Morse indices considerable information about the topology of polygon spaces can be derived from a variety of results on the geometry of cyclic configurations obtained in [17], [19], [7].

It would also be interesting to investigate what happens for an arbitrary (not necessarily generic) sidelength vector. All this shows that the relation between cyclic and critical configurations discussed in this paper has a number of interesting aspects and we intend to address them in a forthcoming publication.

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