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Research School *“Singularities : algebra, geometry and applications”*

VANISHING CYCLES OF POLYNOMIALS

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Notes for the 6 lectures

Abstract. The aim of these lectures is to review the basics on Milnor fibrations and discuss the topology of general fibres of polynomial functions. Following the new paper [ST5], we explain how to initiate a classification of polynomials $f : \mathbb{C}^n \to \mathbb{C}$ of degree d having the top Betti number of the general fibre close to the maximum, namely we find a range in which the polynomial must have isolated singularities and another range where it may have at most one line singularity of Morse transversal type. Our method uses deformations into particular pencils with non-isolated singularities.

1. Lecture 1

1.1. **Introduction.** We want to study a polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ from the local and from the global point of view. We consider its fibres $X_t = f^{-1}(t)$. These fibres do not depend on t, except for finitely many special values of t. In order to study this *failuire of local triviality*, let us recall some definitions and results.

1.2. **Local triviality.** Important ingredients are the *submersion theorem* and *Ehresmann's theorem*:

- If f is a submersion in z then f is locally trivial around z,
- If f is proper and a submersion for all $z \in f^{-1}(f(t))$ then f is a locally trivial fibration, which means that there is a neighborhood U of t, such that $f^{-1}(U)$ is diffeomorphic to the product $f^{-1}(t) \times U$.

Several examples show that there are two types of obstructions to having locally trivial fibrations:

- global aspects related to non-properness at infinity,
- local aspects related to the singular points of f.

Definition 1.1. z is a singular point of f iff $\operatorname{grad} f(z) = 0$.
Notation: $\Sigma_f = \{z \in \mathbb{C}^n \mid z \text{ is a singular point of } f\}$.
A singular point z is called isolated if it is an isolated point of Σ_f.

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1.3. **Milnor fibration.** We consider in the local case non-constant holomorphic function germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and allow arbitrary singularities (isolated or non-isolated). We recall the definition of the Milnor fibration. Let $B := B_\varepsilon \subset \mathbb{C}^{n+1}$ be an ε-ball and let $D = D_\eta \subset \mathbb{C}$ be an η-disk. For any $\varepsilon \gg \eta > 0$ small enough there exist an ε-ball $B := B_\varepsilon$ in $\mathbb{C}^{n+1}$ and an η-disc $D := D_\eta$ in $\mathbb{C}$ such that the restriction:

$$f : f^{-1}(D) \cap B \longrightarrow D$$

is a locally trivial fibre bundle over $D \setminus \{0\}$. The fibres are called *Milnor fibres*. The boundary ∂B is called the *Milnor sphere*.

Some properties of the Milnor fibration are:

1. The Milnor fibres have the homotopy type of a CW complex of real dimension $n - 1$,
2. there exist a monodromy operator which maps the Milnor fibre to itself. This operator has a representative without fixed points; its eigenvalues on the homolgy groups of the fibres are roots of unity. Some aspects of the monodromy are discussed in the series of lectures by Sabir Guzein-Zade.

The simplest example of a singularity is a Morse singularity: $f = z_1^2 + \cdots z_n^2$, also denoted by A_1 (in Arnold's notation). It is known that its Milnor fibre is diffeomorphic to the unit ball bundle of the tangentbundle to S^{n-1}. It follows that the Milnor fibre is homotopy equivalent to the sphere S^{n-1} (which may be seen as the real part of $f = z_1^2 + \cdots z_n^2 = 1$).

1.4. **Morsification of an isolated singularity.** Let z be critical point of f; consider the deformation: $f_a = f - (a_1 z_1 + \cdots + a_n z_n)$, where $a \in \mathbb{C}^n$ is a regular value of the gradient mapping $\operatorname{grad} f : (\mathbb{C}^n, z) \to (\mathbb{C}^n, z)$. This is a finite holomorhic mapping (a branched covering). Its covering degree is called the (deformation) Milnor number; notation $\mu_{def}(f)$. Let $\mathcal{O}_n$ be the ring of holomorphic functions at the origin. There is the following algebraic formula:

$$\mu_{def}(f) = \dim \frac{\mathcal{O}_n}{(\delta_1 f, \cdots, \delta_n f)}$$

where δ_i denotes partial derivitive with respect to z_i.

The critical points of f_a satisfy $\operatorname{grad} f_a = 0$ and so $\operatorname{grad} f = a$. So for generic a there are exactly $\mu_{def}(f)$ critical points. The computation of the Hessian determinant shows that all these critical points are Morse.

Example 1.2. $f = x^a + y^b + z^c$. By direct computation of the covering degree, or by the above formula, one gets: $\mu_{def}(f) = (a-1)(b-1)(c-1)$

1.5. **Bouquet theorem for isolated singularities.**

Theorem 1.3 (Milnor)**.** *Let $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ be a non-constant holomorphic germ with isolated singulariy, then its Milnor fibre is homotopy equivalent to a bouquet of $\mu(f)$ spheres of dimension $n - 1$:*

$$S^{n-1} \vee \cdots \vee S^{n-1}.$$

The proof given during the school uses the paradigma of *additivity of the vanishing homology*: Let $X_D = f_a^{-1}(D) \cap B$ and for $t \in \partial D$ denote $X_t = f_a^{-1}(t) \cap B$. The homology

groups $H_k(X_D, X_t)$ are called the *vanishing homology groups*. Milnor's theorem implies that the only non-zero vanishing homology groups are in dimension n (since X_D is a contractible space). Small deformations of f within a fixed ball B preserve the general fibres (up to diffeomorphisms). The vanishing homology is additive over the critical points. This is valid in particular for a Morsification of an isolated singulariry. Each Morse point has vanishing homology concentrated in dimension n and contributes with $\mathbb{Z}$. Summing up the $\mu_{def}(f)$ contributions, we get: $H_k(X_D, X_t) = 0$ if $k \neq n$ and $\mathbb{Z}^{\mu_{def}(f)}$ if $k = n$.
This shows Milnor's theorem on the level of homology. It also shows that $\mu(f) = \mu_{def}(f)$. An application of Whitehead's theorem (as in Milnor's book) can be used to get the conclusion of the theorem at the level of homotopy. In the lectures we also gave a direct homotopical proof which is generalized in the theorem below.

1.6. **Special Fibre Theorem.** We next consider a deformation F of f, i.e. a holomorphic map germ:

$$F : (\mathbb{C}^n \times \mathbb{C}^r, 0) \to (\mathbb{C} \times \mathbb{C}^r, 0)$$

of the form

$$F(x, a) = (f_a(x), a)$$

such that $f_0(x) = f(x)$. The map germ f_a is called a *perturbation* of f.

We require that the deformation be *topologically trivial over the Milnor sphere* ∂B. This condition implies:

- $f_a^{-1}(t)$ is (stratified) transversal to ∂B for all $|t| < \eta$ and for all $|a| < \rho$.
- $f_a^{-1}(D) \cap \partial B$ is homeomorphic to $f^{-1}(D) \cap \partial B$ and therefore contractible.
- the Milnor fibre of f and the general fibre of f_a are diffeomorphic.

Theorem 1.4 (Siersma,[Si4])**.** *Let F be a deformation of f, which is topologically trivial over the Milnor sphere. Let $a \in D_\rho$ and suppose that all fibres of f_a are smooth or have isolated singularities except for one special fibre $X_t = f_a^{-1}(t) \cap B$. Then X_t is homotopy equivalent to a wedge of spheres:*

$$X_t \overset{h}{\simeq} S^{n-1} \vee \cdots \vee S^{n-1}$$

The number of spheres is equal to the sum of the Milnor numbers in the fibres different from X_t.

Proof. In the following we use the notation:

$$g : X \to D \text{ for the perturbation } f_a : f_a^{-1}(D) \cap \partial B \to D.$$

We denote: $X_Y = g^{-1}(Y)$.

Let $x_1, \ldots, x_\sigma$ be the critical points outside X_t and $c_1, \ldots, c_\tau$ be the critical values, different from t. Take small disjoint discs $D_0, D_1, \ldots, D_\tau$ around $t, c_1, \ldots, c_\tau$ and join them with a point s on ∂D_0 with the help of a system of non-intersecting paths Γ (in the usual way, cf. Figure 1). Call the endpoints $s_1, \ldots, s_\tau$.

Figure 1. The image of the deformation

We mention the homotopy equivalence:

$$X_t \overset{h}{\simeq} X_{D_0}.$$

This equivalence is well known in the local case (i.e. in a small neighborhood of a singular point), see proposition 2.A.3.(b) of [GM]. Since our map is proper one can patch together these local equivalences to a global homotopy equivalence. One can also apply directly lemma 2.A.2. of [GM].

Next we use homotopy lifting properties and have firstly:

$$X_{D_0} \overset{h}{\simeq} X_{D_0 \cup \Gamma}$$

and secondly:

$$X_D \overset{h}{\simeq} X_{D_0 \cup \Gamma} \cup X_{D_1} \cup \cdots \cup X_{D_\tau}$$

Similar homotopy equivalences occur in [Lo] and [Si1].

All X_{c_i} contain only isolated singularities. Let μ_i be the sum of the Milnor numbers in the fibre X_{c_i}. Each X_{D_i} can be obtained (up to homotopy equivalence) from X_{s_i} by attaching μ_i cells of dimension n in order to kill the vanishing cycles.

After retraction of Γ to the point s, it follows that

$$X_D \overset{h}{\simeq} X_{D_0} \cup \cup_{i=1}^{\tau} \text{ cells of dimension } n\ .$$

Since X_D is diffeomorphic to $f^{-1}(D) \cap B$, which is contractible (as total space of the Milnor fibration), we have that:

$$\pi_k(X_t) = \pi_k(X_{D_0}) = 1, \text{ for all } k < n-1.$$

Since X_t has the homotopy type of a CW-complex of dimension $n-1$, see [GM, p.152], it follows that:

$$X_t \overset{h}{\simeq} S^{n-1} \vee \cdots \vee S^{n-1},$$

the number of spheres being equal to $\nu = \sum_i \mu_i$. □

Remark 1.5. In the case of an isolated singularity $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ the above theorem shows that for any perturbation $g : X \to D$ of f, *all fibres* X_s (including the singular fibres) have the homotopy type of a wedge of n-spheres.

Remark 1.6. If X is not smooth but if $X \setminus X_t$ is smooth, the local special fibre theorem applies since X_D is still contractible.

2. Lecture 2: Topology of the fibres of a polynomial

2.1. **Singularities at infinity.** Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree d and let $\tilde{f}(x, x_0)$ be the homogenized of f by the new variable x_0. One replaces $f : \mathbb{C}^n \to \mathbb{C}$ by a proper mapping $\tau : \mathbb{X} \to \mathbb{C}$ which depends on the chosen system of coordinates on $\mathbb{C}^n$, as follows (see [Br1]). Consider the closure in $\mathbb{P}^n \times \mathbb{C}$ of the graph of f, that is the hypersurface

$$\mathbb{X} := \{((x; x_0), t) \in \mathbb{P}^n \times \mathbb{C} \mid F := \tilde{f}(x, x_0) - t x_0^d = 0\},$$

which fits into the commuting diagram

$$\begin{array}{ccc} \mathbb{C}^n & \xrightarrow{i} & \mathbb{X} \\ {\scriptstyle f}\searrow & & \swarrow{\scriptstyle \tau} \\ & \mathbb{C} & \end{array} \quad ,$$

where i denotes the inclusion $x \mapsto (x, f(x))$ and τ is the projection on the second factor. The fibres of τ are denoted by $\overline{X_t}$.

Let H^∞ denote the hyperplane at infinity $\{x_0 = 0\} \subset \mathbb{P}^n$. The intersection $\overline{X_t} \cap H^\infty$ is independent of t.

The singularities of $\mathbb{X}$ are contained in the part "at infinity" $\mathbb{X}^\infty := \mathbb{X} \cap (H^\infty \times \mathbb{C})$, namely:

$$\mathbb{X}_{\mathrm{sing}} := S \times \mathbb{C}, \quad \text{where} \quad S := \{\frac{\partial f_d}{\partial x_1} = \cdots = \frac{\partial f_d}{\partial x_n} = 0,\ f_{d-1} = 0\} \subset H^\infty.$$

The singular set of $\mathbb{X}^\infty$ is:

$$\mathbb{X}^\infty_{\mathrm{sing}} := \Sigma^\infty_f \times \mathbb{C}, \quad \text{where} \quad \Sigma^\infty_f := \{\frac{\partial f_d}{\partial x_1} = \cdots = \frac{\partial f_d}{\partial x_n} = 0\} \subset H^\infty.$$

We have $S \subset \Sigma^\infty_f$.

The singularities of f, i.e. the affine set $\mathrm{Sing}\, f := Z(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n})$, can be identified, by the above diagram, with the singularities of τ on $\mathbb{X} \setminus \mathbb{X}^\infty$. One can prove, by an easy computation, that $\overline{\mathrm{Sing}\, f} \cap H^\infty \subset S$, where $\overline{\mathrm{Sing}\, f}$ denotes the closure of $\mathrm{Sing}\, f$ in $\mathbb{P}^n$. In particular we get $\dim \mathrm{Sing}\, f \le 1 + \dim S$.

2.2. **A global bouquet theorem.** The next result may be viewed as a global version of the local bouquet theorems of J. Milnor [Mi, Theorem 6.5] and Lê D.T. [Lê, Theorem 5.1].

Theorem 2.1. [ST1] *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial with isolated $\mathcal{W}$-singularities*[1] *at infinity. Then the general fibre of f is homotopy equivalent to a bouquet of spheres of real dimension $n - 1$.*

For the proof we refer to [ST1] and the notes from the Trieste Singularities School [ST4]. In this Kiev summer school lectures we mentioned the following sketchy arguments:

(a). Consider first the compactified situation. The function F defines a deformation $F_t(x, x_o) = F(x, x_o, t)$. Consider the polar curve of F with respect to t, $\Gamma(t, F)$, near points $p \in H^\infty$. We will discuss polar curves later in these lectures. Its intersection points $\Gamma \cap H^\infty$ are important, they are obstructions for local triviality of the family of compactified fibres. From the Special Fibre Theorem it follows that locally at (p, t_0) : $\overline{X_t} \cap B \stackrel{h}{\simeq} S^{n-1} \vee \cdots \vee S^{n-1}$, where B is a small ball of 'Milnor type' around (p, t_0), and $t \neq t_0$. In fact this is the Milnor fibre for a function germ, defined on a hypersurface, namely $\tau : \mathbb{X} \to \mathbb{C}$. Denote the number of spheres by $\lambda_{t_0}(p)$.

[1]defined in [ST1], but interpreted here in terms of the **polar curve condition**

(b). Next use the additivitity of the vanishing homology of the function $\tau : \mathbb{X} \to \mathbb{C}$ in a global way (both using the contributions of affine critical points of f with Milnor number μ_i and the λ-numbers λ_j at the points $\Gamma \cap H^\infty$).

$H_k(\mathbb{P}^n, \overline{X_t}) = \oplus \tilde{H}_{k-1}(S^{n-1} \vee \cdots \vee S^{n-1}) = \mathbb{Z}^{\mu+\lambda}$ for $k = n$ and 0 elsewhere.

Here $\mu = \sum u_i$ and $\lambda = \sum \lambda_j$. Next do homology excision (see [ST1] for details):

$H_k(\mathbb{C}^n, X_t) = \oplus \tilde{H}_{k-1}(S^{n-1} \vee \cdots \vee S^{n-1}) = \mathbb{Z}^{\mu+\lambda}$ for $k = n$ and 0 elsewhere.

(c). Do homotopy excision (see [ST1] for details) to obtain the homotopy equivalence.

(d). In case $\overline{X_t}$ has isolated singularities at $P \in \Gamma \cap H^\infty$ and no affine singularities, one can conclude, by perturbing the special fibre, that $\lambda_{t_0} = \mu(\overline{X_{t_0}}) - \mu(\overline{X_t})$.

2.3. **Lê attaching principle.** Let $f : (\mathbb{C}, 0) \to \mathbb{C}$ be a holomorphic function (with no assumptions about $\operatorname{Sing} f$). Consider a general linear function $l : \mathbb{C}^n \to \mathbb{C}$. Together they define a mapping:

$$\Phi = (l, f) : (\mathbb{C}^n, 0) \to \mathbb{C}^2$$

Let $\operatorname{Sing} \Phi$ be the set of points wher $d\Phi$ has not the maximal rank. Note that $\operatorname{Sing} f \subset \operatorname{Sing} \Phi$. The polar curve of f with respect to l is defined as the closure:

$$\Gamma_f = \overline{\operatorname{Sing}(\Phi) \setminus \operatorname{Sing} f}.$$

Its image under Φ is called the *Cerf Diagram.*

FIGURE 2. Cerf Diagram

Theorem 2.2 ([Lê]). *The difference between the Euler characteristic of the Milnor fibre X_t and its generic hyperplane section $X_t \cap \{l = 0\}$ is:*

$$\chi(X_t) - \chi(X_t \cap \{l = 0\}) = (-1)^{n-1} p_f$$

where $p_f := (\Gamma_f, \{l = 0\})$ (intersection number).
Moreover if both f and $f|\{l = 0\}$ have isolated singularities then:

$$\mu(f) + \mu(f|\{l = 0\}) = p_f$$

Proof. X_t can be obtained (up to homotopy) from $X_t \cap \{l = 0\}$ by attaching p_f cells of dimension $n - 1$. □

3. LECTURE 3: NON ISOLATED SINGULARITIES

3.1. **One dimensional singular locus.** In this section we consider singularities with a 1-dimensional critical locus (for short: 1-*isolated singularities*) and study the vanishing homology in a full neighbourhood of the origin. In this case the vanishing homology is concentrated on the 1-dimensional set $\Sigma := \operatorname{Sing} f$. As general reference we mention the summary paper [Si7]. We can write

$$\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_r$$

where each Σ_i is an irreducible curve.

At the origin we consider the Milnor fibre F of f and on each $\Sigma_i - \{0\}$ a local system of transversal singularities, as follows: take at any $x \in \Sigma_i - \{0\}$ the germ of a generic transversal section. This gives an isolated singularity whose μ-class is well-defined. We denote a typical Milnor fibre of this transversal singularity by F'_i.

More precisely we consider in the 1-isolated case the following data:

The **Milnor fibre** F. The homology is concentrated in dimensions $n-1$ and $n-2$ (see the Proposition below):

$$\begin{cases} H_{n-1}(F) = \mathbb{Z}^{\mu_{n-1}}, & \text{which is free.} \\ H_{n-2}(F), & \text{which can have torsion.} \end{cases}$$

The **transversal Milnor fibres** F'_i. The homology is concentrated in dimension $n-2$:

$$\tilde{H}_{n-2}(F'_i) = \mathbb{Z}^{\mu'_i} \text{ ,which is free.}$$

We don't discuss here the monodromies, which act on these space, we refer to [Si7].

Example 3.1. D_∞-singularity: $f = xy^2 + z^2$.
Σ is given by $y = z = 0$ and is a smooth line. The transversal type is A_1.
It is known that F is homotopy equivalent to S^2 (cf. [Si1]).

Example 3.2. $T_{\infty,\infty,\infty}$-singularity: $f = xyz$
$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and consists of the three coordinate axes in $\mathbb{C}^3$.
The transversal type is again A_1.
It is known that F is homotopy equivalent to the 2-torus $S^1 \times S^1$ (cf. [Si3])

Proposition 3.3. *If f has a 1-dimensional singular set then its Milnor fibre is $n-3$ connected*[2] *and therefore $\tilde{H}_k(F)$ has only non-zero contributions for $k = n-1$ and $n-2$.*

Proof. The Milnor fibre X_t is constructed (up to homotopy) by adding cells of dimension $n-1$ (Lê attaching) to the hyperplane section $X_t \cap \{l = 0\}$. Because $f|\{l = 0\}$ has an isolated singularity we have $X_t \cap \{l = 0\} \overset{h}{\simeq} S^{n-2} \vee \cdots \vee S^{n-2}$. Attaching cells of dimension $n-1$ does not influences homotopy and homology groups of dimension $\leq n-3$. □

Using such a proof by induction, one finds the known result:

Proposition 3.4. (Kato-Matsumoto)
If $\dim \Sigma_f = s$ then its Milnor fibre is $n-s-2$ connected.

3.2. **Series of singularities.** Let again $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function. Let f have a 1-dimensional critical locus $\Sigma = \operatorname{Sing} f$. One considers for each $N \in \mathbb{N}$ the series of functions:

$$f_N = f + \epsilon l^N$$

where l is an admissible linear form, which means that the hyperplane section $f^{-1}(0) \cap \{l = 0\}$ has an isolated singularity. One calls this series of function germs a *Yomdin series* of the

[2] this means all homotopy groups up to $n-3$ are trivial

hypersurface singularity f. Under the above condition all members of the Yomdin series have isolated singularities. Moreover their Milnor numbers can be computed using the so-called Lê-Yomdin formula:

Theorem 3.5. *Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ have 1-dimensional critical locus $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$ (irreducible components). Let l be an admissible linear form.*

$$\mu(f + \epsilon l^N) = b_{n-1}(f) - b_{n-2}(f) + N e_0(\Sigma).$$

Here b_{n-1}, resp b_{n-2} are the corresponding Betti-numbers of the Milnor fibre F of the non-isolated singularity f and $e_0(\Sigma)$ is the intersection multiplicity of Σ and $l = 0$. The formula holds for all N sufficiently large. Moreover $e_0(\Sigma) = \sum d_i \mu_i'$, where d_i is the intersection multiplicity of Σ_i (with reduced structure) and l.

Proof. The idea behind the proof is to use polar methods and to consider the map germ

$$\Phi = (f, l) \ : \mathbb{C}^{n+1} \to \mathbb{C} \times \mathbb{C}.$$

The Milnor fibres F of f, resp F^N of f_N occur as inverse images under Φ of the sets $\{f = t\}$, resp $\{f + \epsilon l^N = t\}$. Next one constructs via a (stratified) isotopy an embedding $F \subset F^N$.

FIGURE 3. Both Cerf diagrams in one picture

From the corresponding homology sequence one gets the following 4-term exact sequence

$$0 \to H_n(F) \to H_n(F^N) \to H_n(F^N, F) \to H_{n-1}(F) \to 0.$$

The difference $F^N \setminus F$ is (by excision and homotopy equivalence) related to the part of F^N located near the d_i intersection points of Σ_i and F^N. One obtains:

$$H_q(F^N, F) = \oplus_{i=1}^{r} \oplus_{k=1}^{N d_i} \tilde{H}_{n-1}(F_{i,k}),$$

FIGURE 4. Situation in the l-plane

where each $F_{i,k}$ is a copy of the Milnor fibre of the transversal singularity F_i'. From this one gets

$$b_n(F) - b_{n-1}(F) = b_n(F^N) - N \sum d_i \mu_i'.$$

□

REMARK 3.6. In [Si5] there is a formula that relates the characteristic polynomials of the monodromies of f and f_N. Other ingredients are the horizontal and vertical monodromies. The eigenvalues of the monodromy satisfy Steenbrink's spectrum conjecture, cf [Stb]. This conjecture was later proved by M. Saito [Sa], using his theory of Mixed Hodge Modules.

3.3. **Euler characteristic of a projective hypersurface with isolated singularities.** We consider first smooth hypersurfaces V of degree d in the projective space $\mathbb{P}^n$. It is well-known, that they have all the same topological type, in particular the same Euler characteristic $\chi^{n,d}$. A typical example is given by $x_0^d + \cdots + x_n^d = 0$.

Proposition 3.7. *Let V be a smooth hypersurface of degree d in $\mathbb{P}^n$. Then*

$$\chi(V) = \chi^{n,d} = (n+1) - \frac{1 + (-1)^n (d-1)^{n+1}}{d}.$$

Proof. Consider the projection map $\pi : \mathbb{C}^{n+1} \to \mathbb{P}^n$ and restrict π to $\{x_0^d + \cdots + x_n^d = 1\}$. This restriction defines a d-fold cover onto $\mathbb{P}^n \setminus V$. The source space can be identified with the Milnor fibre of $x_0^d + \cdots + x_n^d$, which has Milnor number $(d-1)^{n+1}$. It follows:

$$d\chi(\mathbb{P}^n \setminus V) = 1 + (-1)^n (d-1)^{n+1}.$$

□

Proposition 3.8. *Let V be a smooth hypersurface of degree d in $\mathbb{P}^n$, which has only isolated singularities. Then*

$$\chi(V) = \chi^{n,d} + (-1)^n \sum_p \mu(V,p).$$

where $\mu(V,p)$ is the Milnor number of the isolated singularity at p, and p runs over all singular points.

Proof. Consider the family of projective hypersurfaces:

$$V_s = \{f_d + s h_d = 0\},$$

where $f_d = 0$ defines V and h_d defines a hypersurface of generic type.
Note that V_s is a smoothing of $V = V_0$. Outside neigborhoods of the singular points we have local diffeomorphisms of pieces of V and V_s.

Figure 5. Deformation to generic hypersurface

In neigbourhouds B of the singular points p we have that $V_s \cap B$ is equal to the Milnor fibre of the singularity (V,p). The difference in Euler characteristic between $V_s \cap B$ and $V \cap B$ is just (up to sign) its Milnor number. □

We will treat in Lecture 4 the more complicated situation of hypersurfaces with a 1-dimensional singular set.

3.4. **Top Betti Defect.** Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial function of degree $d \geq 2$, where $n \geq 2$. It is well-known that f is a locally trivial fibration over $\mathbb{C}$ outside a finite number of *atypical values*, [Th, Br1]. It's *general fibre* G is a Stein manifold and therefore homotopy equivalent to a CW-complex of dimension $n-1$, by [Ka, Ha]. We shall call *top Betti number of* f and denote $b_{n-1}(f) := b_{n-1}(G)$ the $(n-1)$th Betti number of the general fibre. While this number is clearly bounded in terms of n and d, our aim is to find what are the special properties of f which make $b_{n-1}(f)$ approach the maximum $(d-1)^n$. Let us call *top Betti defect of* f the difference $\Delta_{n-1}(f) := (d-1)^n - b_{n-1}(f)$.

We shall show in the next lectures that if $\Delta_{n-1}(f)$ is small enough, then the polynomial has special types of singularities.

While considering the Betti numbers of a fibre X_t of a polynomial mapping f, it is usefull to compare $\overline{X_t}$ and $\overline{X_t} \cap H^\infty$. A natural concept is the following:

Definition 3.9. Let (Y,p) be a germ of hypersurface in $\mathbb{C}^n$ and H a hyperplane. The pair $(Y, Y \cap H)$ has an isolated *boundary singularity* at p if both (Y,p) and $(Y \cap H, p)$ have isolated singularity. Its *boundary Milnor number* is defined by:

$$\mu_p(Y) + \mu_p(Y \cap H)$$

The local theory of boundary singularities has been studie by V. Arnol'd [Ar1] and his school.

Returning to our global situation we have the equivalence: the boundary pair $(\overline{X_t}, \overline{X_t} \cap H^\infty)$ has isolated singularities if and only if X_t has isolated singularities and $\dim \Sigma_f^\infty \leq 0$. Then $S = \Sigma_f^\infty \cap \{f_{d-1} = 0\}$ is the subset of points of H^∞ where $\overline{X_t}$ is singular, and this does not depend on the value $t \in \mathbb{C}$.

Proposition 3.10 ([ST5]). *Let f be a polynomial of degree d with isolated singularities, having general fibre X_0 and satisfying* $\dim \Sigma_f^\infty \cap \{f_{d-1} = 0\} \leq 0$. *Then:*

$$\Delta_{n-1}(f) = \sum_{p \in \Sigma_f^\infty \cap \{f_{d-1}=0\}} \mu_p(\overline{X_0}) + (-1)^n \Delta\chi^\infty \tag{1}$$

where $\Delta\chi^\infty := \chi^{n-1,d} - \chi(\{f_d = 0\})$ *and* $\chi^{n-1,d} = n - \frac{1}{d}\{1 + (-1)^{n-1}(d-1)^n\}$ *denotes the Euler characteristic of the smooth hypersurface $V_{gen}^{n-1,d}$ of degree d in $\mathbb{P}^{n-1}$.*

In particular, if $\dim \Sigma_f^\infty \leq 0$ *then:*

$$\Delta_{n-1}(f) = \sum_{p \in \Sigma_f^\infty} [\mu_p(\overline{X_0}) + \mu_p(\overline{X_0} \cap H^\infty)]. \tag{2}$$

Proof. One uses the fact that X_0 is the difference of two projective varieties. Consequently $\chi(X_0) = \chi(\overline{X_0}) - \chi(\overline{X_0} \cap H^\infty)$. We treat these terms separately and compare them to the corresponding terms for a general hypersurface. Then applying Proposition 3.8 in case of isolated singularities yields (1) and (2). Note that in (2) the contributions are exactly the boundary Milnor numbers defined above. □

3.5. **Isolated Line Singularities: local case.** The fundamental paper on this is [Si1]. We consider the germ $f : (\mathbb{C}^n, 0) \to \mathbb{C}$ where the singular set is a straight line L. Choose coordinates $(x, y_1, \cdots, y_{n-1})$; for short (x,y). The line L is given by $y_1 = \cdots = y_{n-1} = 0$.

Lemma 3.11. *f is singular on L iff there exist $h_{ij}(x,y)$ such that $f = \sum h_{ij} y_i y_j$.* □

An important ingredient is the transversal Hessian: $h_f(x) = \det(h_{ij}(x,0)$. Note that $h_f(x) \neq 0$ in x implies that the transversal singularity (given by $f(x,y)$ as function of y) is of type A_1.

Definition 3.12. Let $\operatorname{Sing} f = L$ and $h_f(x) \neq 0$ whenever $x \neq 0$. We call f an *isolated line singularity.*

Isolated line singularities are classified in low codimension (including all simple line singularities). The beginning of the list is:

codim	type	normal form	hessian
0	A_∞	$y_1^2 + \cdots y_{n-1}^2$	$h_f(x) = 1$
1	D_∞	$xy_1^2 + y_2^2 + \cdots y_{n-1}^2$	$h_f(x) = x$
2	J_∞	$x^2y_1^2 + y_2^2 + \cdots y_{n-1}^2$	$h_f(x) = x^2$

For isolated line singularities one can perform a generic deformation by perturbing the h_{ij} in a generic way. The resulting deformed function has generically only A_∞-,D_∞- and A_1-singularities.

Theorem 3.13. *If f has an isolated line singularity then its Milnor fibre is a bouquet of spheres:*

1. *S^{n-2} if f is of type A_∞,*
2. *$S^{n-1} \vee \cdots \vee S^{n-1}$ else.*

In the second case the number of spheres is equal to $A_1 + 2D_\infty - 1$, where A_1 and D_∞ denote the number of A_1-points and D_∞- points in the generic deformation.

Proof. The main idea is to use the generic deformation mentioned above. Start with a transversal Milnor fibre S^{n-2}, each D_∞-point kills this sphere with 2 balls of dimension $n-1$. Next attach $(n-1)$-spheres for each A_1 singularity. For details cf [Si1]. □

Figure 6. Generic deformation of an isolated line singularity

3.6. **Isolated Line Singularities: projective case.** We intend to use the theory of isolated line singularities in the projective space $\mathbb{P}^{n-1}$. Assume now we have a hypersurface $V = \{f = 0\}$, where f is homogeneous of degree d. Let its singular set be a projective line L. One may assume:

$$f = \sum h_{ij}(x, y) y_i y_j$$

where we use homogeneous coordinates $(x_0, x_1, y_2, \cdots, y_{n-1})$, for short (x, y). Assume now that the transversal type at general points of L is A_1. Note that the Hessian determinant $h_f(x_0, x_1)$ has homogeneous degree $(d-2)(n-2)$. A generic deformation will produce a hypersurface with singular set a projective line and $(d-2)(n-2)$ D_∞-points and of type A_∞ elsewhere along the line. We will use this in Lecture 5.

4. Lecture 4

The references for the statements in this lecture are [ST2] and [ST5], unless otherwise stated.

By *one-parameter deformation of f* we mean a holomorphic map $P : \mathbb{C}^n \times \mathbb{C} \to \mathbb{C}$, where $P_s := P(\cdot, s)$ is a polynomial of degree d for any $s \in \mathbb{C}$, and such that $P_0 = f$. We shall work with germs at $s = 0$ of such families of polynomials. Let G_s denote the general fibre of P_s. We start from the following key result.

Proposition 4.1. [ST3] *For $s \neq 0$ close enough to 0, the general fibre G_0 of P_0 can be naturally embedded in the general fibre G_s of P_s such that the embedding $G_0 \subset G_s$ induces an injective map $H_{n-1}(G_0) \hookrightarrow H_{n-1}(G_s)$.*

In particular, we have the following semi-continuity principle for the top Betti number:

$$\Delta_{n-1}(P_s) \leq \Delta_{n-1}(P_0), \text{ for } s \neq 0 \text{ close enough to } 0. \tag{3}$$

□

An affine hypersurface in $\mathbb{C}^n$ will be called *general-at-infinity* if its projective closure is non-singular in the neighbourhood of the hyperplane at infinity H^∞ and intersects it transversely. The polynomial f will be called *general-at-infinity* (or *of $\mathcal{G}$-type*) iff all its fibres are general-at-infinity. Since this is equivalent to $\Sigma_f^\infty = \emptyset$, it means that f is general-at-infinity iff some fibre of f is so.
Another key fact is that any polynomial is deformable into a general-at-infinity polynomial.

Proposition 4.2. *Any polynomial can be deformed into a general-at-infinity polynomial of the same degree. More precisely, let h_d be some general-at-infinity polynomial of degree d. Then the deformation $f_\varepsilon := f + \varepsilon h_d$ transforms any given polynomial f of degree d into a general-at-infinity polynomial f_ε, for any $\varepsilon \neq 0$ close enough to 0.* □

Using such a deformation and the additivity of the Euler characteristic, we may prove Proposition 3.10, cf [ST5].

4.1. **Euler characteristic of projective hypersurfaces with one-dimensional singularities.** Let $V := \{f_d = 0\}$ denote a hypersurface in $\mathbb{P}^{n-1} = H^\infty$ of degree d with singular locus $\hat{\Sigma}$ of dimension one, more precisely $\hat{\Sigma}$ consists of a union Σ of irreducible curves and eventually a finite number of points $\{R_1, \ldots, R_\delta\}$. Let h_d be a general-at-infinity homogeneous polynomial of degree d and consider the deformation $f_\varepsilon = f + \varepsilon h_d$. This is general-at-infinity for $\varepsilon \neq 0$ in some small enough disk centered at 0, by Proposition 4.2. For any $\varepsilon \in \mathbb{C}$, let $V_\varepsilon := \{f_{\varepsilon,d} := f_d + \varepsilon h_d = 0\}$ be a pencil of projective hypersurfaces.

The genericity of h_d ensures that V_ε is nonsingular for all $\varepsilon \neq 0$ in a small enough disk $\Delta \subset \mathbb{C}$ centered at the origin. Let us consider the total space of the pencil:

$$\mathbb{V}_\Delta := \{f_d + \varepsilon h_d = 0\} \subset \mathbb{P}^{n-1} \times \Delta$$

as germ at V_0 and the projection $\pi : \mathbb{V}_\Delta \to \Delta$. We denote by $A = \{f_d = h_d = 0\}$ the axis of the pencil. One considers the polar locus of the map $(h_d, f_d) : \mathbb{C}^n \to \mathbb{C}^2$ and since this is a homogeneous set one takes its image in $\mathbb{P}^{n-1}$ which will be denoted by $\Gamma(h_d, f_d)$.

We have:

Lemma 4.3. *The space $\mathbb{V}_\Delta$ has isolated singularities:* $\operatorname{Sing} \mathbb{V}_\Delta = (A \cap \Sigma) \times \{0\}$, *and $\pi : \mathbb{V}_\Delta \to \Delta$ is a map with 1-dimensional singular locus:* $\operatorname{Sing}(\pi) = \hat{\Sigma} \times \{0\}$. □

We shall use the following notations: $A \cap \Sigma = \{P_1, \ldots, P_\nu\}$, $\Sigma^* := \Sigma \setminus (\{P_i\}_{i=1}^\nu \cup (\{Q_j\}_{j=1}^\gamma)$, $\mathcal{N} :=$ small enough tubular neighbourhood of Σ^*, and B_i, B_j, B_k are small enough Milnor balls within $\mathbb{V}_\Delta \subset \mathbb{P}^{n-1} \times \Delta$ at the points P_i, Q_j, R_k, respectively.

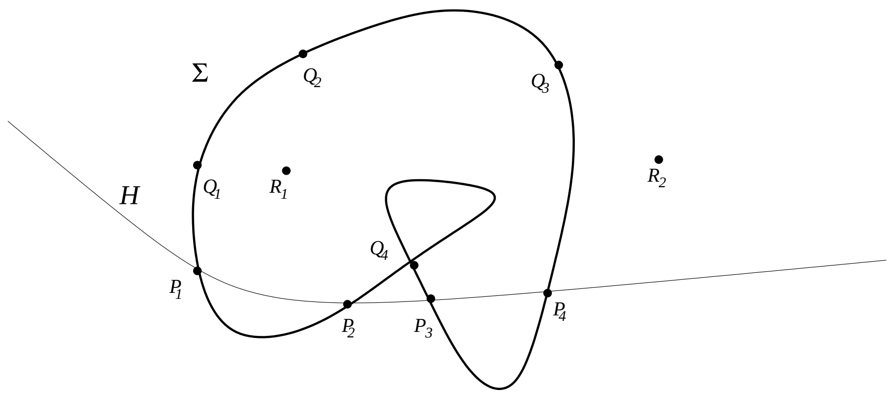

FIGURE 7. *Point-strata of V and the intersection $H \cap \Sigma$*

Since the Euler characteristic χ is a constructible functor, we have the following decomposition into a sum:

$$\chi(\mathbb{V}_\Delta, V_\varepsilon) = \chi(\mathcal{N}, \mathcal{N}\cap V_\varepsilon) + \sum_{i=1}^{\nu} \chi(B_i, B_i\cap V_\varepsilon) + \sum_{j=1}^{\gamma} \chi(B_j, B_j\cap V_\varepsilon) + \sum_{k=1}^{\delta} \chi(B_k, B_k\cap V_\varepsilon) \tag{4}$$

The pair $(B, B \cap V_\varepsilon)$ in all of the above three sums represents the local Milnor data of a hypersurface germ of dimension $n-2$ in a space of dimension $n-1$. The last one $(B_k, B_k \cap V_\varepsilon)$ corresponds to the isolated hypersurface singularity of V at R_k with Milnor number $\mu_k \geq 1$, of which π is a smoothing, and therefore we have:

$$\chi(B_k, B_k \cap V_\varepsilon) = (-1)^{n-1}\mu_k$$

For the first term, since the map $\pi : \mathbb{V}_\Delta \to \Delta$ has a trivial transversal structure along Σ_r^*, where Σ_r is some irreducible component in the decomposition of Σ, we have the equality:

$$\chi(\mathcal{N}, \mathcal{N} \cap V_\varepsilon) = \sum_r \chi(\Sigma_r^*)\chi(B_r, F_r^\pitchfork).$$

where $(B_r, F_r^\pitchfork)$ is the transversal Milnor data at some point of Σ_r^*, namely B_r is a Milnor ball of the transversal singularity and $F_r^\pitchfork$ is the corresponding transversal Milnor fibre. Note that this is the Milnor data of an isolated hypersurface singularity of dimension $n-3$; its Milnor number will be denoted by $\mu_r^\pitchfork$ and this does not depend on the choice of the point on Σ_r^*. We therefore have: $\chi(B_r, F_r^\pitchfork) = (-1)^{n-2}\mu_r^\pitchfork$.

We also have $\chi(\Sigma_r^*) = 2 - 2g_r - \nu_r - \gamma_r$ where g_r is the genus of Σ_r, and where ν_r and γ_r are the numbers of points P_i and Q_j on Σ_r, respectively. Then:

$$\chi(\mathcal{N}, \mathcal{N} \cap V_\varepsilon) = (-1)^{n-1} \sum_r (\nu_r + \gamma_r + 2g_r - 2)\mu_r^\pitchfork. \tag{5}$$

The contribution of the axis in the formula (4) is null, since one shows:

Lemma 4.4. $\chi(B_i, B_i \cap V_\varepsilon) = 0$. □

This follows since the germ of the polar locus $\Gamma_p(h_d, f_d) \subset \mathbb{P}^{n-1}$ is empty at any point $p \in A \times \{0\}$.

Let us finally remark that $\chi(\mathbb{V}_\Delta, V_\varepsilon) = \chi(V) - \chi^{n-1,d}$ since V_ε is a general hypersurface of degree d in $\mathbb{P}^{n-1}$ and since $\mathbb{V}_\Delta$ retracts to its central fibre $V = \{f_d = 0\}$. Then the preceding considerations prove the following:

Theorem 4.5. *Let $V := \{f_d = 0\} \subset \mathbb{P}^{n-1}$ be a hypersurface of degree d where $\operatorname{Sing} V$ is a union of curves and isolated points. Then, in the above notations:*

$$\chi(V) = \chi^{n-1,d} + (-1)^{n-1} \sum_r (\nu_r + \gamma_r + 2g_r - 2)\mu_r^{\pitchfork} + \sum_{j=1}^{\gamma} \chi(B_j, B_j \cap V_\varepsilon) + (-1)^{n-1} \sum_{k=1}^{\delta} \mu_k. \tag{6}$$

□

5. Lecture 5. Polynomials with line singularities at infinity

Our aim is to compute, via formula (1), the top Betti defect of polynomials f with $\dim \operatorname{Sing} f \le 0$ and Σ_f^∞ is a union of curves and isolated points, such that $\dim \Sigma_f^\infty \cap \{f_{d-1} = 0\} \le 0$. We are considering the deformation $f_\varepsilon = f + \varepsilon h_d$, where h_d is a general-at-infinity homogeneous polynomial of degree d. By Proposition 4.2, f_ε is a general-at-infinity polynomial for $\varepsilon \neq 0$. In the notations of Proposition 3.10, $\Delta\chi^\infty = \chi(\mathbb{V}_\Delta, V_\varepsilon)$. Then Theorem 4.5 reads:

Corollary 5.1. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree d with $\dim \operatorname{Sing} f \le 0$ and Σ_f^∞ is a union of curves and isolated points. Then:*

$$\Delta\chi^\infty = (-1)^n \sum_r (\nu_r + \gamma_r + 2g_r - 2)\mu_r^{\pitchfork} - \sum_{j=1}^{\gamma} \chi(B_j, B_j \cap V_\varepsilon) + (-1)^n \sum_{k=1}^{\delta} \mu_k \tag{7}$$

□

The only part of the formula (7) which is not explicitly computed is the sum of $\chi(B_j, B_j \cap V_\varepsilon)$ which runs over the Whitney point-strata Q_j of the hypersurface V. One may compute it in particular cases, like in the natural class of polynomials where Σ_f^∞ is a reduced line with Morse generic transversal type.[3] We then prove:

Theorem 5.2. *If f has at most isolated affine singularities and Σ_f^∞ is a reduced projective line with generic Morse transversal type, then*

$$\Delta_{n-1}(f) \ge 2(n-1)(d-2) + 1.$$

The existence of such singularities implies $n \ge 3$. For $n = 3$ our formula reads: $\Delta_{n-1}(f) \ge 4d - 7$ which shows that our result specialises to the estimation proved in [ALM] for a particular class of polynomials in 3 variables, with $\dim \operatorname{Sing} f \le 0$ and $\dim \Sigma_f^\infty = 1$, and with no singularities at infinity in the sense of [ST1].

[3]We shall see in the proof of Proposition 6.4 that, if the top Betti defect of f is between d and $2d-3$, then f might have such type of singularities.

Proof of Theorem 5.2. By eventually deforming the $d-1$ homogeneous part of f we get that the intersection $\Sigma \cap \{f_{d-1}=0\}$ is of dimension ≤ 0. Using the definitions in §4.1 in our particular setting, we have by assumption $\delta=0$, $r=1$, $g_1=0$, $\mu_1^{\pitchfork}=1$, and $\nu_1=\nu=\mathrm{mult}(\Sigma,\{h_d=0\})=\deg h_d=d$. We apply formula (1) and Corollary 5.1 and we get:

$$(8)\qquad \Delta_{n-1}(f)=\sum_{p\in\Sigma_f^{\infty}\cap\{f_{d-1}=0\}}\mu_p(\overline{X_0})+(d+\gamma-2)+(-1)^{n-1}\sum_{j=1}^{\gamma}\chi(B_j,B_j\cap V_\varepsilon).$$

Let us first evaluate the sum of Milnor numbers of the general fibre X_0 of f. For a general $d-1$ homogeneous part of f we get that the intersection $\Sigma\cap\{f_{d-1}=0\}$ consists of $d-1$ simple points, each of which being an A_1 singularity of X_0. This implies that the above first sum is bounded from below by $d-1$.

The number γ counts the special points Q_j on the singular line $\Sigma := \Sigma_f^{\infty}$. Then the last sum has γ terms and we need to determine for each of them the contribution $\chi(B_j,B_j\cap V_\varepsilon)$. For this we need the deformation theory of line singularities, founded by Siersma [Si1] and subsequently developed by several authors. Let us assume without loss of generality that the line $\Sigma\subset\mathbb{P}^{n-1}$ is the zero locus of the ideal $I=(x_1,\ldots,x_{n-2})$. We remark first that the ideal of homogeneous polynomials $g:(\mathbb{C}^n,0)\to(\mathbb{C},0)$ such that $\mathrm{Sing}\, g\supset\Sigma$ is spanned by the polynomials of the form $g(x)=\sum_{i,j=1}^{n-2}h_{ij}(x)x_ix_j$ where $h_{ij}(x)$ are polynomials depending on all variables $x_1,\ldots,x_n$. This was established in [Si1] and [Pe1] in the germ case; our situation is slightly different but the same proof applies.

In our setting the functions $h_{ij}(x)$ are of degree $\leq d-2$. Following the deformation theory in [Si1] (see also Lecture 3), by deforming $h_{ij}(x)$ we get a generic *transversal Hessian* $\mathcal{H}(x):=det(h_{ij})_{i,j}(x)$ and this implies that g has generic singularity type A_∞ along Σ. Then the point-strata Q_j are precisely the type D_∞ singularities. Following Siersma's theory [Si1], the number of D_∞ points is equal to the degree of the Hessian $\mathcal{H}(0,\ldots,0,x_{n-1},x_n)$. In the generic case this degree turns out to be equal to $(d-2)(n-2)$. We may then take this value of γ in the formula (8) as a minimum. We also get from [Si1] that the Milnor fibre of a D_∞-singularity is homotopy equivalent to a $(n-2)$-sphere, therefore $\chi(B_j,B_j\cap V_\varepsilon)=(-1)^{n-1}$.

Finally, putting together the lower bounds we get:

$$\Delta_{n-1}(f)\geq d-1+d+2(d-2)(n-2)-2=2(n-1)(d-2)+1.$$

□

Example 5.3. The proof actually shows that whenever h_{ij} and f_{d-1} are generic we have the equality $\Delta_{n-1}(f)=2(n-1)(d-2)+1$, which means that the bound of Theorem 5.2 is sharp for any $n\geq 3$. An explicit example for $n=3$ is the following.

Let $f=z^d+z^2x^{d-2}+z^2y^{d-2}+xy(x^{d-3}-y^{d-3})$, where $n=3$ and $d\geq 3$. Then f has isolated affine singularities and Σ_f^{∞} is a reduced projective line with generic transversal type A_1. Let us compute $\Delta_2(f)$ using Proposition 3.10 and its notations.
Let X_t denote a general fibre. Then $H^\infty\cap\mathrm{Sing}\,\overline{X_t}=\{z=0,xy(x^{d-3}-y^{d-3})=0\}$. By computing in coordinate charts it follows that the set $\mathrm{Sing}\,\overline{X_t}$ consists of $d-1$ Morse singularities. According to formula (1), this contributes with $d-1$ to the top Betti defect.

Next we compute $\Delta\chi^\infty$ from the same formula. We have $f_d = z^2(z^{d-2} + x^{d-2} + y^{d-2})$ and the reduced hypersurface $\{f_d = 0\} \subset \mathbb{P}^2$ has degree $d-1$ and precisely $d-2$ singular points which are Morse. This implies the equality $\chi(\{f_d = 0\}) = \chi^{2,d-1} + d - 2$. Since $\chi^{2,d-1} = -(d-1)^2 + 3(d-1)$, we get $\chi(\{f_d = 0\}) = -d^2 + 6d - 6$, which yields $\Delta\chi^\infty = \chi^{2,d} - \chi(\{f_d = 0\}) = -3d+6$. By formula (1) we then get $\Delta_2(f) = d-1+3d-6 = 4d-7$. This corresponds to the equality in Theorem 5.2, showing that the bound is sharp for $n = 3$.

6. Lecture 6. Polynomials with non-isolated singularities

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a polynomial of degree $d \geq 2$ with singular loci of dimension ≥ 2, more precisely $\dim \mathrm{Sing}\, f \geq 2$ or $\dim \Sigma_f^\infty \geq 2$. If one deforms f directly to general-at-infinity polynomials, then it appears that comparing the general fibres becomes a difficult task. A better strategy would be to deform f in two steps and use the semi-continuity principle (3) according to the following program: (a). deform such that the dimension of the singularity locus decreases to one, and then (b). compare the new polynomial to another deformation of it into a polynomial satisfying the hypothesis of Proposition 3.10 or directly use results from the theory of one-dimensional singularities. The reason is that one-dimensional singularities and their deformations are quite well understood, due to the work of Lê [Lê], Yomdin [Yo] and the detailed study by Siersma [Si1] and his school, see e.g. the survey [Si7].

Let $l : \mathbb{C}^n \to \mathbb{C}$ be a linear function. We denote by:

$$\Gamma(l, f) := \mathrm{closure}[\mathrm{Sing}\,(l, f) \setminus \mathrm{Sing}\, f] \subset \mathbb{C}^n$$

the *polar locus of f with respect to l*. One has the following Bertini type result, proved in [Ti1, Ti2], [Ti3, Thm. 7.1.2]:

Lemma 6.1. *There is a Zariski-open subset Ω_f of the dual projective space $\check{\mathbb{P}}^{n-1}$ such that, for any $l \in \Omega_f$, the polar locus $\Gamma(l, f)$ is a reduced curve or it is empty.* □

We may and shall also assume (by eventually restricting Ω_f to some open Zariski subset of it) that if $\dim \mathrm{Sing}\, f \geq 1$ then $\dim \mathrm{Sing}\, f \cap \{l = 0\} = \dim \mathrm{Sing}\, f - 1$ for any $l \in \Omega_f$. We then say that l is *general with respect to f* whenever $l \in \Omega_f$. With these settings we may start our program.

Lemma 6.2. *Let l be general with respect to f and to f_d. If $\dim \mathrm{Sing}\, f \geq 1$, or if $\dim \Sigma_f^\infty \geq 1$, then the deformation $f_\varepsilon = f + \varepsilon l^d$ reduces by one the dimension of the respective singular locus. If f has the property that $\dim \mathrm{Sing}\, f \leq 0$ or $\dim \Sigma^\infty \leq 0$ then f_ε preserves this property.* □

Lemma 6.3. *Let l be general with respect to f and consider the deformation $f_\varepsilon = f + \varepsilon l$. If $\dim \mathrm{Sing}\, f \geq 1$ then there exists a small disk centered at the origin $D \subset \mathbb{C}$ such that $\dim \mathrm{Sing}\, f_\varepsilon \leq 0$ and $\Sigma_{f_\varepsilon}^\infty = \Sigma_f^\infty$, for any $\varepsilon \in D^*$.* □

With these preparations we may consider in the next statements the two cases of dimension one singular locus.

Proposition 6.4. *If f is a polynomial of degree d with* $\dim \Sigma_f^\infty = 1$ *and* $\dim \operatorname{Sing} f \leq 0$ *then*

$$\Delta_{n-1}(f) \geq d-1.$$

Proof. We consider the deformation $f_\varepsilon = f + \varepsilon l^d$ for general l as in Lemma 6.2. It then follows that $\dim \operatorname{Sing} f_\varepsilon \leq 0$ and that $\operatorname{Sing} f_{\varepsilon,d} \subset \mathbb{P}^{n-1}$ is the union of $\Sigma_f^\infty \cap \{l=0\}$ and eventually some finite set of points. We may assume that the hyperplane $\{l=0\}$ slices Σ_f^∞ at regular points only; this property is generic too. Let then $\Sigma_f^\infty = \cup_r \Sigma_r$ be the decomposition into irreducible components and let $p \in \Sigma_r \cap \{l=0\}$ for some r. In order to compute the top Betti defect $\Delta_{n-1}(f_\varepsilon) = (d-1)^n - b_{n-1}(X_\varepsilon)$ for some general fibre X_ε of f_ε, we may use formula (2) in which one of the ingredients is $\mu_p(\overline{X_\varepsilon} \cap H^\infty)$ and observe that this is equal to $\mu_p(f_{\varepsilon,d})$.

We denote by $\mu_r^\pitchfork$ the Milnor number of the transverse singularity of Σ_r. By the local Lê *attaching formula*, see [Lê], [Yo] and [Si5], we have:

$$\mu_p(f_{\varepsilon,d}) + \mu_p((f_{\varepsilon,d})_{|l=0}) = \operatorname{mult}_p(\hat{\Gamma}_p(l, f_{\varepsilon,d}), \{f_{\varepsilon,d}=0\}) + \operatorname{mult}_p(\Sigma_r, \{f_{\varepsilon,d}=0\})\mu_r^\pitchfork \tag{9}$$

where $\hat{\Gamma}_p(l, f_{\varepsilon,d})$ denotes the union of the components of the germ at p of the polar curve of the map $\psi := (l, f_{\varepsilon,d}) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ other than the singular locus Σ_r.

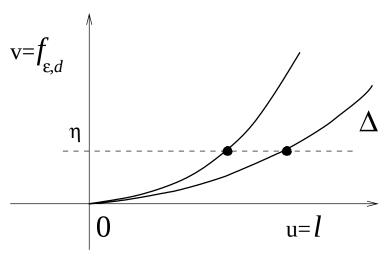

FIGURE 8. *Polar multiplicities*

By the regularity of p, it follows that $\mu_p((f_{\varepsilon,d})_{|l=0}) = \mu_r^\pitchfork$. In local coordinates at the regular point p the germ of the singular locus Σ_r is a line and the restriction to Σ_r of the map ψ is one-to-one. The germ at $\psi(p)$ of the image $\Delta := \psi(\Sigma_r)$ is parametrised by $(l, \varepsilon l^d)$ since $f_{\varepsilon,d} = f_d + \varepsilon l^d$ and $\Sigma_r \subset \{f_d = 0\}$. This multiplicity is represented in Figure 8 by the number of intersection points of $\{f_{\varepsilon,d} = \eta\}$ with the curve Δ. Therefore $\operatorname{mult}_p(\Sigma_r, \{f_{\varepsilon,d}=0\}) = \operatorname{mult}_{\psi(p)}(\Delta, \{v=0\}) = d$, where (u,v) are the coordinates of the target $(\mathbb{C}^2, 0)$. Then formula (9) becomes:

$$\mu_p(f_{\varepsilon,d}) = (d-1)\mu_r^\pitchfork + \operatorname{mult}_p(\hat{\Gamma}_p(l, f_{\varepsilon,d}), \{f_{\varepsilon,d}=0\}). \tag{10}$$

We next need to sum up over all the points $p \in \Sigma_f^\infty \cap \{l=0\}$. The number of points of $\Sigma_r \cap \{l=0\}$ is equal to the degree $d_r := \deg \Sigma_r$ and we get:

$$\sum_{p \in \Sigma_f^\infty \cap \{l=0\}} \mu_p(f_{\varepsilon,d}) = (d-1)\sum_r d_r \mu_r^\pitchfork + \sum_r \sum_{p \in \Sigma_r \cap \{l=0\}} \operatorname{mult}_p(\hat{\Gamma}_p(l, f_{\varepsilon,d}), \{f_{\varepsilon,d}=0\}), \tag{11}$$

hence

$$\sum_{p \in \Sigma_f^\infty \cap \{l=0\}} \mu_p(f_{\varepsilon,d}) \geq (d-1) \sum_r d_r \mu_r^\pitchfork \tag{12}$$

with equality if and only if $\hat{\Gamma}_p(l, f_{\varepsilon,d}) = \emptyset$ for all $p \in \Sigma_f^\infty \cap \{l = 0\}$. We finally get from formulas (2) and (12):

$$\Delta_{n-1}(f_\varepsilon) = (d-1)^n - b_{n-1}(f_{\varepsilon,d}) \geq \sum_{p \in \Sigma_f^\infty \cap \{l=0\}} \mu_p(\overline{X_\varepsilon} \cap H^\infty) \geq (d-1) \sum_r d_r \mu_r^\pitchfork \geq d-1, \tag{13}$$

where X_ε denotes the general fibre of f_ε. The first inequality becomes an equality if and only if $\overline{X_\varepsilon}$ has no singularities in the neighbourhood of H^∞. The last one becomes an equality if and only if $r = 1$ and $d_1 = 1$.

Our claim follows since we have $\Delta_{n-1}(f) \geq \Delta_{n-1}(f_\varepsilon)$ by the semi-continuity principle (3). □

Proposition 6.5. *If f is a polynomial of degree d with* $\dim \operatorname{Sing} f = 1$ *and* $\dim \Sigma_f^\infty = 0$ *then*

$$\Delta_{n-1}(f) \geq d - 1.$$

□

These preparations lead to the following principal result:

Theorem 6.6. *Let $f : \mathbb{C}^n \to \mathbb{C}$ be some polynomial of degree $d \geq 2$. Then:*

(a) $\Delta_{n-1}(f) \geq 0$ *and the equality holds if and only if f is general-at-infinity.*
(b) *If* $0 < \Delta_{n-1}(f) \leq d - 1$ *then* $\dim \operatorname{Sing} f \leq 0$ *and* $\dim \Sigma_f^\infty \leq 0$.
(c) *If* $d \leq \Delta_{n-1}(f) < 2d - 2$ *for* $d \geq 3$, *then* $\dim \Sigma_f^\infty \leq 0$ *and either* $\dim \operatorname{Sing} f \leq 0$ *or* $\operatorname{Sing} f$ *is one line with generic Morse transversal type and transverse to the hyperplane at infinity.*

□

Example 6.7. $f = x + x^2 y$. We have $n = 2$, $d = 3$, $\operatorname{Sing} f = \emptyset$ and $\Sigma_f^\infty = [0; 1] \in \mathbb{P}^1$. The computation yields $b_1(f) = 1$ and therefore $\Delta_1(f) = 4 - 1 = 3$ which corresponds to the situation in Theorem 6.6(c).

Example 6.8. $f = x^2 y$. Here $n = 2$, $d = 3$, $\operatorname{Sing} f = \{x = 0\}$ is a line with transversal type A_1 and $\Sigma_f^\infty = [0; 1] \in \mathbb{P}^1$. By computation we have $b_1(f) = 1$ and so $\Delta_1(f) = 4 - 1 = 3$. This corresponds to the situation in Theorem 6.6(c) and also shows that the estimation is sharp. In full generality, for any $n \geq 2$ and $d > 2$, let $f = (a_1 z_1^2 + \cdots + a_{n-1} z_{n-1}^2) x^{d-2} + c_1 z_1^d + \cdots + c_{n-1} z_{n-1}^d$. This is a homogeneous polynomial and has a line singularity $L = \{z_1 = \ldots = z_{n-1} = 0\}$. For general coefficients a_i, c_i, this polynomial has no other singular point and the line L is transversal to H^∞ and has Morse generic transversal singularity type. The projective hypersurface $\{f_d = 0\}$ has a single singular point at $p := [1; 0; \cdots ; 0]$. By a local computation, the singularity type of $\overline{X_t}$ at p is A_{d-1} for $t \neq 0$. Then we may apply formula (2) of Proposition 3.10 since this works in our situation too. Indeed, the fibre X_t of f has reduced homology concentrated in dimension $n-1$ since it is diffeomorphic to the Milnor fibre of f at the origin (by the homogeneity) and since by [Si1] this line singularity Milnor fibre is homotopy equivalent to a bouquet of spheres of dimension $n-1$. So by (2) we get $\Delta_{n-1}(f) = d - 1 + 1 = d$, which also shows that the lower bound in Theorem 6.6(c) is sharp.

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