MORSE RELATIONS FOR CURVATURE AND TIGHTNESS

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Abstract In chapter I we define the critical points of a continuous function $\varphi : X \to \mathbb{R}$ on a compact metric space X, and the Morse polynomial for an isolated critical point. We reproduce the proof of the Morse relations for a function with isolated critical points for this more general set-up. In chapter II we define polynomial, absolute and alternating curvature measures of certain maps or embeddings of X into E^N in terms of critical points and we obtain Morse relations for these curvature measures. In chapter III we discuss maps (and sets) in E^N of minimal total absolute curvature (tight maps) and we recall some results on tight embeddings of smooth, piecewise linear and topological manifolds.

I. Morse relations

A point $x \in X$ in a metrisable topological space X (we have in mind manifolds with or without boundary and simplicial complexes) is called non critical for a continuous function $\varphi : X \to \mathbb{R}$, if there is for some neighbourhood U of x in X a homeomorphism (call it a chart) $\pi : U \to V \times J$, J an open set in $\mathbb{R}$, and a commutative diagram with p_1 and p_2 the projections onto the factors V and J:

$$\begin{tikzcd} U \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & V \times J \arrow[r, "p_1"] \arrow[d, "p_2"] & V \\ & \mathbb{R} & \end{tikzcd} \qquad (1)$$

Otherwise the point x is called critical and $\varphi(x)$ is called a critical value. For example, the function $\varphi = z$ has on the sphere $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ and on the ball $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 1\}$ only two critical points, namely $(0, 0, 1)$

This is a considerably extended version of one of a series of four lectures given at the Liverpool Summer School on Singularities, June 1970.

and $(0, 0, -1)$ in each case.

We write

$$\varphi_s = \{x \in X : \varphi(x) \leq s\}. \tag{2}$$

We assume moreover that any pair of spaces (φ_b, φ_a) for $a < b$, has finite dimensional singular homology groups over a field $\mathbb{F}$ to be kept fixed, with the usual exact split triangle in homology:

$$\begin{tikzcd} H_*(\varphi_a) \arrow[rr, "i"] & & H_*(\varphi_b) \arrow[dl, "j"] \\ & H_*(\varphi_b, \varphi_a) \arrow[ul, "\partial"] & \end{tikzcd} \tag{3}$$

Recall that $\partial : H_k(\varphi_b, \varphi_a) \to H_{k-1}(\varphi_a)$ decreases the index. We have written $H_* = \underset{k}{\oplus} H_k$, etc.

Define the Poincare polynomials:

$$\begin{aligned} P(\varphi_a) &= \sum_k \dim H_k(\varphi_a).t^k \\ P(\varphi_b, \varphi_a) &= \sum_k \dim H_k(\varphi_b, \varphi_a).t^k \\ P(\operatorname{Im} \partial) &= \sum_k \dim (H_k(\varphi_a) \cap \operatorname{Im} \partial).t^k. \end{aligned}$$

Then (3) yields

$$P(\varphi_b, \varphi_a) - [P(\varphi_b) - P(\varphi_a)] = (1 + t)\, P(\operatorname{Im} \partial)$$

or we have the preliminary Morse formula:

$$\frac{P(\varphi_b, \varphi_a) - [P(\varphi_b) - P(\varphi_a)]}{1 + t} \succ 0. \tag{4}$$

Here $\succ$ is an order relation between polynomials such that $P_1 \succ P_2$ as well as $P_1 - P_2 \succ 0$ means that all coefficients of the polynomial $P_1 - P_2$ are non negative.

Observe that if two polynomials $\sum_{k>0} \mu_k t^k$ and $\sum_{k>0} \beta_k t^k$ obey the Morse relation

$$\frac{\Sigma\mu_k\ t^k - \Sigma\beta_k\ t^k}{1+t} \succ 0 \tag{5}$$

then we get the usual expression for the Morse inequalities:

$$t = -1 \text{ gives } \Sigma\,(-1)^k\,\mu_k = \Sigma\,(-1)^k\,\beta_k\ ; \tag{6a}$$

$$\Sigma(\mu_k - \beta_k)\,t^k \succ 0 \text{ gives } \mu_k \geq \beta_k\ ; \tag{6b}$$

and

$$\Sigma(\mu_k - \beta_k)\,t^k\,\Sigma_{i=0}^{\infty}\,(-t)^i \succ 0 \text{ gives}$$
$$\mu_j - \mu_{j-1} + \dots + (-1)^j\mu_0 \geq \beta_j - \beta_{j-1} + \dots + (-1)^j\beta_0. \tag{6c}$$

We now prove some lemmas.

__Lemma 1.__ __If__ φ_b __is compact and there is no critical value in the closed interval__ $[a, b]$, __then__ φ_a __is a deformation retract of__ φ_b.

__Proof.__ (Following M. Morse's methods) For any point $x \in \varphi_b \setminus \varphi_a$ we use a chart κ as in (1) for which $U \subset (X \setminus \varphi_a)$ and we define easily a continuous map $h_x : X \to X$ with the following properties:

$$\left.\begin{array}{ll} h_x(y) = y & \text{for } y \notin U \\ \left.\begin{array}{l} p_1\,\kappa(h_x(y)) = p_1\kappa(y) \\ p_2\,\kappa(h_x(y)) \leq p_2(\kappa y)) \\ p_2\,\kappa(h_x(x)) < p_2\kappa(x) \end{array}\right\} & \text{for } y \in U \end{array}\right\} . \tag{7}$$

A deformation $h_{x,t}$, $0 \leq t \leq 1$, leading from the identity to this map can be defined by the same equations as h_x, replacing the inequalities by

$$p_2\kappa(h_{x,t}(y)) = (1-t)\ p_2\kappa(y) + tp_2\kappa h_x(y). \tag{8}$$

For a point x with $\varphi(x) = a$ we use a chart κ as in (1) but we now define the continuous map $h_x : X \to X$ so that is has the properties:

$$\begin{array}{rcll} h_x(y) & = & y & \text{for } y \notin U \text{ and for } \varphi(y) \leq a \\ p_1\kappa h_x(y) & = & p_1\kappa(y) & \text{for } y \in U \\ a \leq p_2\kappa(h_x(y)) & \leq & p_2\kappa(y) & \text{for } y \in Y \text{ and } \varphi(y) \geq a \\ p_2\kappa h_x(y) & = & a & \text{for } y \in U \text{ and } \varphi(y) \geq a \\ & & & \text{and } y \text{ in some neighbour-} \\ & & & \text{hood of } x. \end{array} \tag{9}$$

$h_{x,t}$ is again defined as above (see (3)).

There is a finite set of continuous maps (2) and (8) such that every point y in $\varphi_b \setminus \varphi_a$ is non-invariant under at least one of them. Let $h_1, h_2, \ldots, h_L$ be such a set. Then the composition $h_L h_{L-1} \ldots h_2 h_1 = h$ is a continuous map of X onto X. By compactness there exists $\epsilon > 0$ such that $h(\varphi_{a+\epsilon}) = \varphi_a$, and there exists $\delta > 0$ such that

$$\varphi(h(y)) \leq \varphi(y) - \delta \quad \text{for } a + \epsilon \leq \varphi(y) \leq b.$$

Then it follows that

$$h^N(\varphi_b) = \varphi_a \qquad \text{for } (N-1)\delta > b - a - \epsilon,$$

and

$$h^N(y) = y \qquad \text{for } y \in \varphi_a.$$

h^N is a retraction of φ_b onto φ_a. By recalling that it is the composition of $N \times L$ elementary deformations (7) or (9) we see that h^N is a deformation retraction namely the composition (one after the other) of the corresponding deformations $h_{x,t}$.

<u>Lemma 2.</u> <u>If</u> φ_b <u>is compact and the only critical value in</u> $[c, b]$ <u>is</u> c, <u>and if all critical points with this critical value</u> c <u>are isolated, then</u> φ_c <u>is a deformation retract of</u> φ_b.

<u>Proof</u>. Let K be the (finite) set of critical points at level $= c$, and denote:

$$\overset{0}{\varphi}_c = \{x \in X : \varphi(x) < c\} \subset \varphi_c.$$

For any point x in $\varphi_b \setminus \{\overset{0}{\varphi}_c \cup K\}$ we can find a neighbourhood U not meeting K and a continuous map h_x as in either (7) or (9). Let $U_1 \supset \overline{U_2} \supset U_2 \supset \overline{U_3} \supset U_3 \ldots$ be a nested sequence of neighbourhoods of K in X with intersection $\bigcap_k U_k = K$. We may just as well assume that each U_i has one component for each point of K, namely one containing that point.

There exists a finite set of maps (7) and (9) such that every point y in the compact set $\varphi_b \setminus U_{i+1}$ is non-invariant under at least one of them. Let h'' be the composition of these maps. Of course $h''(y) = y$ for $y \in \varphi_a$. There is by the construction of h'' an $\epsilon'' > 0$ such that

$$h''(\varphi_{c+\epsilon''} \setminus U_{i+1}) \subset \varphi_a.$$

Let $\rho(\epsilon) = \sup_{y \in \varphi_{c+\epsilon}}$ distance $(h''(y), y)$ where the distance is with respect to any fixed metric on X. $\rho(\epsilon)$ is a continuous function of $\epsilon \geq 0$ and $\rho(0) = 0$. Hence we can find $0 < \epsilon' < \epsilon''$ such that

$$\rho(\epsilon') \leq \inf_{y \in U_{i+1}, z \notin U_i} \text{distance } (y, z).$$

Let h' be a deformation retraction of φ_b onto $\varphi_{c+\epsilon'}$, as constructed by lemma 1.

$$h^{(i)} = h'' \circ h'.$$

Then $h^{(i)}$ is a continuous map of φ_b into $\varphi_a \cup U_i$ (!) with

$$h^{(i)}(y) = y \quad \text{for} \quad y \in \varphi_a.$$

And $\lim_{k \to \infty} h^{(k)} h^{(k-1)} \dots h^{(2)} h^{(1)}$ is the required retraction of φ_b onto φ_a, which by its construction as a composition of deformations, and the fact that U_i for $i \to \infty$ converges to the finite point set K, is a deformation retraction as required in the lemma.

Lemma 3. Let $c \in (a, b)$ be the only critical value of the function $\varphi : X \to \mathbb{R}$ in the closed interval $[a, b]$ and let the critical points be isolated. Hence, they form a finite set K. Then we have isomorphisms for the natural maps

$$H_k(\varphi_b, \varphi_a) \xleftarrow{\cong} H_k(\varphi_c, \varphi_a) \xrightarrow{\cong} H_k(\varphi_c, \varphi_c \setminus K) \xrightarrow{\cong} \bigoplus_{x \in K} H_k(\varphi_c, \varphi_c \setminus \{x\}). \tag{10}$$

Proof. The first isomorphism is a consequence of lemma 2.
The last isomorphism follows by the excision axiom. The second homomorphism is an isomorphism, because $H_k(\varphi_c \setminus K, \varphi_a) = 0$ for all k. This is the case because every (finite) singular chain in $\varphi_c \setminus K$ is contained in some compact part of $\varphi_c \setminus K$ and the union of φ_a and that part can be homotoped inside $\varphi_c \setminus K$, keeping fixed every point of φ_a, first into $\varphi_{c-\epsilon}$ for some $\epsilon > 0$ and then onto φ_a, with a construction as before involving maps (7) and (9). Therefore the homomorphism

$$H_*(\varphi_a) \longrightarrow H_*(\varphi_c \setminus K)$$

is injective (take a singular cycle representing any element in $H_*(\varphi_a)$ and suppose it bounds a chain in $\varphi_0 \setminus K)$, as well as surjective (take a singular cycle in $\varphi_0 \setminus K)$. Hence it is an isomorphism, and so $H_*(\varphi_0 \setminus K, \varphi_a) = 0$.

If x is an isolated critical point of $\varphi : X \to \mathbb{R}$ with critical value c, then

$$M(\varphi, x) = \Sigma \mu_k(\varphi, x)\, t^k = \Sigma \dim H_k(\varphi_c, \varphi_c \setminus \{x\}) t^k \tag{11}$$

is called the <u>Morse polynomial</u> of x, and $\mu_k(\varphi, x)$ is the <u>multiplicity of index</u> k at the critical point.

$$\mu(\varphi, x) = \Sigma_k \mu_k(\varphi, x) \text{ is the (total) \underline{multiplicity} at } x. \tag{12}$$

<u>Examples</u> (The field $\mathbb{F}$ is fixed)

1) The function $\varphi = -x_1^2 - x_2^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2 : \mathbb{R}^n \to \mathbb{R}$ has a critical point 0 with

$$M(\varphi, 0) = t^k,$$

and with total multiplicity $\mu(\varphi, 0) = \mu_k(\varphi, 0) = 1$.

2) The function $\varphi = z$ on the set

$$\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \le 1\}$$

has critical points

$(0, 0, -1)$ with total multiplicity (of index 0) <u>one</u>

and

$(0, 0, 1)$ with total multiplicity <u>zero</u>.

3) The function $\varphi = x_1^3 - 3x_1x_2^2$ (= real part of $(x_1 + ix_2)^3$; monkey saddle) on $\mathbb{R}^2$ has a critical point 0 with Morse polynomial

$$M(\varphi, 0) = 2t$$

and total multiplicity = multiplicity of index $1 = \mu(\ , 0) = 2$.

4) The function $\varphi = x_1^3 - 3x_1(x_2^2 + x_3^2)$ on $\mathbb{R}^3$ has a critical point 0 with Morse polynomial

$$M(\varphi, 0) = t + t^2,$$

with $\mu_1(\varphi, 0) = \mu_2(\varphi, 0) = 1$ and with total multiplicity 2.

5) Let X be the cone with top p_0 on a space Y obtained from $Y \times I$ by identifying $Y \times 1$ to one point, and let φ be the projection into $I = [0, 1] \subset \mathbb{R}$.

Then φ has the isolated critical point p_0 with value 1 (and of course also a critical value 0).

In view of the exact sequence

$$\begin{array}{c} H_k(\text{point}) \approx H_k(\varphi_1) \to H_k(\varphi_1, \varphi_0) \to H_{k-1}(\varphi_0) \to H_{k-1}(\varphi_1) \\ \qquad\qquad\qquad\qquad\qquad\quad \| \\ \qquad\qquad\qquad\qquad\qquad\quad H_{k-1}(Y) \end{array}$$

we obtain

$$M(\varphi, p_0) = t(P(Y) - 1).$$

Y can be for example a real projective plane. We see therefore that the choice of field has an influence on the Morse polynomials. This general model occurs in some neighbourhood of an isolated critical point p_0 for any piecewise linear function φ on a simplicial complex X', hence for any real analytic function φ on an analytic manifold X'. If c is the critical value then there is a closed conical neighbourhood $X = \text{Cone}(Y)$ of p_0 in φ_c. Compare with example 4).

6) For a non critical point x the definition of the Morse polynomial applies and gives the polynomial 0.

7) If we compose a piecewise linear embedding f of a compact piecewise linear manifold in some euclidean space E, with a linear function $z : E \to \mathbb{R}$ we get a function zf with in general isolated critical points with complicated Morse polynomials.

In the smooth case we get in general only Morse polynomials of the kind t^k!

For a function $\varphi: X \to \mathbb{R}$ on a compact metrisable space X with isolated critical points, we define the Morse polynomial of the function φ:

$$M(\varphi) = \Sigma\mu_k(\varphi) \cdot t^k \overset{\text{def}}{=\!=} \sum_{x \in X} M(\varphi, x) = \sum_{x \in X} \sum_k \mu_k(\varphi, x) \cdot t^k. \tag{13}$$

Theorem (Morse relations). If $\varphi : X \to \mathbb{R}$ is a continuous function on a compact metric space X with isolated critical points, then

$$\frac{M(\varphi) - P(X)}{1 + t} \succ 0. \tag{14}$$

<u>Proof</u>. Let $c_1 < c_2 < \dots < c_L$ be the critical values of φ and let

$$-\infty = a_0 < c_1 < a_1 < c_2 < \dots < a_{L-1} < c_L < a_L = \infty.$$

By (4) we have

$$\sum_{i=0}^{L-1} \frac{P(\varphi_{a_{i+1}}, \varphi_{a_i}) - [P(\varphi_{a_{i+1}}) - P(\varphi_{a_i})]}{1 + t} \succ 0.$$

Substituting (10), (11), (13) we obtain (14).

From the interpretation (6a, b, c) of (14) we obtain in particular the Euler characteristic

$$\Sigma\ (-1)^k\ \mu_k(\varphi) = \Sigma\ (-1)^k\ \beta_k(X) = \chi(X) \tag{15}$$

and

$$\mu(\varphi) = \Sigma_k \mu_k(\varphi) \geq \beta(X) = \Sigma\beta_k(X). \tag{16}$$

If we define $\gamma(X) = \inf_\varphi \mu(\varphi)$ (17)

and

$$\bar{\gamma}(X) = \inf_\varphi \tfrac{1}{2}[\mu(\varphi) + \mu(-\ )], \tag{18}$$

with φ running over all functions with isolated critical points on X, then we have for any φ

$$\mu(\varphi) \geq \gamma(X) \geq \beta(X) \tag{19}$$

and

$$\tfrac{1}{2}[\mu(\varphi) + \mu(-\varphi)] \geq \bar{\gamma}(X) \geq \gamma(X) \geq \beta(X). \tag{20}$$

<u>Examples</u>. For a 3-manifold X which has the homology but not the homotopy type of S^3 one has

$$\gamma(X) \geq 4 > \beta(X) = 2.$$

For a space Y homeomorphic to the letter Y one has

$$\bar{\gamma}(Y) = 2 > \gamma(Y) = \beta(Y) = 1.$$

<u>Problem</u>. For a manifold X let $\gamma_{ND}(X)$ be equal to the minimal number of critical points a non degenerate function φ (that is one that gives rise to a

handle decomposition in smooth, piecewise linear or topological topology) can have. $\gamma_{ND}(X)$ is independent of the field $\mathbb{F}$. By definition:

$$\gamma_{ND}(X) \geq \gamma(X). \tag{21}$$

Is there a smooth or piecewise linear manifold X for which $\gamma_{ND}(X) > \gamma(X, \mathbb{F})$ for some or every field $\mathbb{F}$?

It is known that not every topological manifold has a handle decomposition. Does every topological manifold have a function with isolated critical points?

A space X is called a Morse - equality space in case $\bar{\gamma}(X) = \beta(X)$ for some $\mathbb{F}$.

All closed surfaces are Morse - equality spaces, as can be seen by taking $\mathbb{F} = \mathbb{Z}_2$. Hence $\bar{\gamma}(X) = \gamma(X) = \beta(X) = 4 - \chi(X)$.

II. Polynomial curvature measures

Let $f : X \to E^N$ be a continuous map of a compact metric space X into a Euclidean vector space E^N. For any linear function $z : E^N \longrightarrow \mathbb{R}$ with gradient $z^* \in S^{N-1} \subset E^N$ a unit vector, the composition zf yields a function on X and on any open set $U \subset X$. Fix again a field $\mathbb{F}$ and suppose zf has isolated critical points. Then we have the Morse polynomial

$$M(zf, U) = \sum_{x \in U} M(zf, x).$$

We now make the assumption that zf has isolated critical points for almost every unit covector z on E^N. This assumption is fulfilled for smooth immersions of smooth manifolds and for PL-immersions of PL-manifolds.

Let α be the homogeneous measure on S^{N-1} with total value $\int_{S^{N-1}} \alpha = 1$. Assume moreover that the following integral converges for every open U in X:

$$T(f, U) = \Sigma_k \tau_k(f, U)\, t^k = \mathcal{E}_{z^* \in S^{N-1}} M(zf, U) = \tag{22}$$

$$\underline{\underline{\text{def}}} \quad \int_{z^* \in S^{N-1}} M(zf, U)\,.\,\alpha.$$

Its value is a polynomial in t, called the polynomial curvature measure of f on U. In particular

$$T(f) = T(f, X) \tag{23}$$

is the <u>polynomial</u> <u>curvature</u> of f.

In particular, if $t = 1$, we get, by (22) and (20),

$$\tau(f) = \Sigma_k \tau_k(f) = \mathcal{E}_{z^* \in S^{N-1}} \mu(zf)$$

$$= \mathcal{E}_{z^* \in S^{N-1}} \tfrac{1}{2}[\mu(zf) + \mu(-zf)] \geq \bar{\gamma}(X). \tag{24}$$

We also have the Morse relations for curvature measures

$$\frac{T(f) - P(X)}{1+t} \succ 0. \tag{25}$$

By using the definition (22) and (23) of $T(f)$, and the inequality (14), we have

$$\frac{T(f) - P(X)}{1+t} = \mathcal{E}_z \left(\frac{M(zf) - P(X)}{1+t}\right) \succ 0.$$

A consequence of (25) obtained by substituting $t = -1$ in (25) is

$$\tau_{alt}(f) \overset{\text{def}}{=\!=} \Sigma_k (-1)^k \tau_k(f) = \chi(X) . \tag{26}$$

This generalises the theorem of Gauss - Bonnet.

It applies for example to a piecewise differentiable manifold with boundary, embedded in E^N.

$\tau(f)$ is called the <u>total absolute curvature</u> of f.

$\tau_{alt}(f)$ is called the <u>alternating curvature</u> of f.

The real-valued alternating curvature measure $\tau_{alt}(f, U)$ for variable U is an intrinsic metric invariant for smooth immersions f of smooth manifolds in E^N (see [5]) and also for piecewise linear embeddings or immersions of simplicial complexes (T. Banchoff [2] proved this last fact and he has interesting interpretations of τ_{alt}, which measure in this case is concentrated in the 0-skeleton!).

Some well known interpretations for curves and surfaces are as follows. For 2-dimensional smooth surfaces in E^N one has

$$\tau_{alt}(f, U) = \int_u \frac{K_s d\sigma}{2\pi} .$$

For smooth surfaces immersed in E^3 one has:

$$\tau_0(f, U) = \tau_2(f, U) = \int_{U\cap(K>0)} \frac{|Kd\sigma|}{4\pi} ;$$

$$\tau_1(f, U) = \int_{U\cap(K<0)} \frac{|Kd\sigma|}{2\pi} ;$$

$$\tau(f, U) = \int_U \frac{|Kd\sigma|}{2\pi} .$$

For curves smoothly immersed in E^N one has

$$\tau(f, U) = \int_U \frac{|\rho d\sigma|}{\pi} .$$

III. Tight maps and sets

A map $f : X \to E^N$ is called tight in case it fulfills the assumptions mentioned in II and it has minimal total absolute curvature (compare (24)):

$$\tau(f) = \vec{\gamma}(X).$$

There are several results on tight smooth embeddings of smooth manifolds without boundary into Euclidean spaces, their existence or non-existence, and their forms in case of existence. Compare [5] and [8], where other references can be found.

T. Banchoff [1] has studied tight piecewise linear embeddings of piecewise linear surfaces in Euclidean spaces. For higher dimensional PL-manifolds nothing is known. Perhaps tight PL-embeddings are very rare?

In the topological category we see easily that for any convex compact set X in E^N with interior points and boundary ∂X, X and ∂X are tight sets with total absolute curvature 1 and 2 respectively. Vice versa any compact set X in E^N with total absolute curvature 1 is convex (essentially a proof is in [4]), and any tight topological immersion of S^n in E^N is an embedding onto the boundary of a convex (n+1)-dimensional body [8].

Tightness generalises convexity therefore, and several interesting problems remain open in particular in the topological category. I mention the problem of finding all topological tight embeddings of the real projective plane substantially into Euclidean 5-space, and the problem of finding all closed topological tight embeddings (suitably defined) of the open Moebius band into Euclidean spaces.

(An example in E^4 was found by S. Carter and A. West [3].) For the Moebius band with boundary all possible tight embeddings seem to be within reach [9]. There exists no smooth tight embedding. Tight embeddings of the figure Y into Euclidean space have an image which consists of three straight segments meeting in one end point, such that that end point is not a corner point for the convex hull of the figure. The image lies therefore in a plane. The property "tightness" is of course invariant under affine transformations of the target space E^N. For Morse equality spaces the property is also invariant under projective transformations of E^N (keeping the image inside E^N of course). For a proof see [3].

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