On gradient curves of an analytic function near a critical point.

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Summary. René Thom [T] conjectured that a gradient curve $x(t)$ of an analytic function on $\mathbb{R}^n$, which descends to the critical point $x(\infty) = 0 \in \mathbb{R}^n$, called a *path*, has there a tangent. We prove this in case $n = 3$ for *"standard" paths* and *"standard" functions*.

For the remaining *rare paths* and *rare functions* we reduce the conjecture for irreducible Σ to an "evident" conjecture and give some arguments in favour of CRT for reducible Σ. We did not succeed in elaborating these arguments to a complete proof that covers all cases.

[1] Notes for four lectures 1990/1991 at the Ecole Normale Supérieure, Paris, with a survey as guide.

Survey and guide

Let $f(x) = \Sigma_{\ell \geq k} \bar{H}_\ell(x)$, $x \in \mathbb{R}^n$, $k \geq 2$, $\bar{H}_\ell$ homogeneous of degree ℓ in $x = (x_1,\ldots,x_n)$, be an analytic function near the critical point $0 \in \mathbb{R}^n$. A descending gradient curve $x(t)$ with limit $x(\infty) = 0$, called a *path*, is a solution for $o \leq t \leq \infty$, of the equation $dx/dt = \dot{x} = -\operatorname{grad} f$ with $x(\infty) = 0$. *R. Thom* proposed the *conjecture* (CRT) that every path has a tangent at the limitpoint $x(\infty) = 0$. For two "polar coordinates", $r = |x| \geq 0$ a real number, *and* $\omega = x/|x|$ a point in the unit sphere S^{n-1} , we call the curve $\omega(t) = \omega(x(t))$ the *celestial trace* in S^{n-1} of the path $x(t)$. Clearly the tangent at $x(\infty) = 0$ exists if and only if *the limitset* $\Omega = \lim_{t=\infty} \omega(t)$ of the celestial trace *is one point.*

In order to prepare for Thom's conjecture we study paths and celestial traces near the critical point, and in particular for $n \leq 3$. The local theory of traces has interest in its own right.

From the *fundamental differential equations* in polar coordinates in section 1 we conclude that the limitset Ω is a compact connected set in a topological component Σ of the algebraic set of critical points, crit H_k , in S^{n-1} . The function H_k on S^{n-1} is by definition the restriction $\bar{H}_k \backslash S^{n-1}$. Thom's conjecture is then proved (in section 2) in the known cases, namely if $n = 2$, and for any n if H_k is constant and if $f \equiv \bar{H}_k$ is homogeneous. This is **Theorem 1.**

In section 3 we consider the case $n = 3$ and paths with $H_k(\Omega) = H_k(\Sigma)$ (= constant) > 0 . Let the algebraic set $\Sigma \subset S^2$ have dimension one. It can be suitably stratified as a graph with edges and vertices. Then it is shown, for large t_0 and $\omega(t_0)$ near to a one-stratum of Σ , that $\omega(t)$ hardly moves for $t > t_0$ in the direction parallel to the stratum, and more precisely that $\omega(t)$ converges to one point $\omega(\infty)$ on Σ , near to $\omega(t)$. We conclude to

Theorem 2. CRT holds for paths for which $H_k(\Omega) > 0$. There remains for $n = 3$ the case, which we will *assume* from now on,

$$H_k(\Omega) = H_k(\Sigma) = 0 .$$

Any one-stratum of Σ , lies on an irreducible algebraic curve $R_u = 0$ on S^2 . We introduce near a one-stratum on S^2 one real local coordinate σ (for example equal to R_u) and an *orthogonal* coordinate λ , which parametrizes on the one-stratum the curve $R_u = 0$, and could be its arc-length. Useful tools defined in section 5.1 are the *Newton polygon* P(f) of f and the Newton polygons of $\dot{r}$, $\dot{\sigma}$ and $\dot{\lambda}$. They are defined *with respect to* r *and* σ and concern the powerseries in r and σ , with analytic functions of λ as coefficients. For example $f = \Sigma_{i \geq k} r^i H_i = \Sigma A_{ij}(\lambda) r^i \sigma^j$. See (5.4) in section 5 and examples in figures 6, 7, 8 .

The Newton polygon P(f) for f and R_u can have *two kinds of declining sides.* A side is called a *special side* if it ends on the horizontal axis of the Newton i-j-plane. Other declining sides are called *general sides* of P(f) .

<u>Special side ETS (standard paths, standard functions).</u>

The example in section 4 illustrates the common situation for a special N(f) polygon side. N(f) is here very simple and has only one declining side. What we find is that *for arbitrarily small* $r > 0$ *and* $\sigma > 0$, but as a pair , (r,σ) , restricted to be in a wedge bundle, the λ-coordinate of the path can *even so essentially* change. The trace $\omega(t)$ follows then for large t the decrease of some potential function on S^2 restricted to the stratum, and ends eventually in the example at the minimum of that potential in a point $\Omega = \omega(\infty) \in \Sigma$. In general, for a special polygon side, the potential is H_y for some $y > k$, and $y = y(u)$ depends on the algebraic curve R_u . There may be more than one wedge for a given one-stratum and given R_u . If so then we use the expression "wedges-bundle". But in the case of a special Newton polygon side there is one common potential H_y to guide. Given the essential change in $\lambda(t)$ we say that the stratum carries *Essential Trace Speed* (ETS) (For the definition of ETS see lemma 8 in section 5). This ETS is in our example guided by one potential. Essential trace speed is clearly necessary if Ω is not a point. Hence ETS is crucial for Thom's conjecture.

Also the next example (I in section 5.2) has only one declining polygon side and it is special. The example is exceptional because the Newton polygon of $\dot{y}$ has one point which lies outside (under) the Newton polygon of $\dot{\sigma}$. Then paths that have obvious ETS are abundant. They are not even restricted to be in a wedges bundle. Each is guided by one potential H_y as before. And CRT holds for the example. A path which after some time $t = t_o$ is *only* involved in special polygon sides with or without ETS is called a *standard path*. In section 7 we prove **Theorem 3**: CRT holds for *standard paths* and for *standard functions*. In fact we show that special side ETS in different one-strata of Σ (with different curves $R_u = 0$) cannot cooperate to give a spiraling result. We call these paths standard because the other paths, that involve general side ETS are very rare. Functions which do have only standard paths are also called standard functions.

General side ETS (rare paths, rare function germs).

Essential trace speed can exist for a general Newton polygon side. This is seen in example II of section 5, where Σ is a great circle ($z = 0$) on S^2 . Necessary conditions for existence of ETS are strong as seen in section (6.4) and we call functions that have such ETS for that reason *rare*[2] . The first necessary condition restricts the pair (r,σ) as before to be in a sharp wedges-bundle. The tangents at the cusps of the wedges are represented in the algebraic *cusp curve* $\tilde{\Sigma} \subset \Sigma \times \mathbb{R}$ with equation $F_\sigma(\mu) = 0$. But now the ETS is represented by vectors along Σ that depend on the wedge that is on $\tilde{\lambda}_i \in \tilde{\Sigma}$, and different wedges at $\lambda \in \Sigma$ *can have opposite essential trace speeds at* λ when the path is in different wedges, near points $\tilde{\lambda}_i$ and $\tilde{\lambda}_2 \in \tilde{\Sigma}$ both covering $\lambda \in \Sigma$. The ETS-vector F at $\tilde{\lambda}_i$ is a "weighted mean" of several gradients along Σ

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[2] Question: Is the algebraic curve $R_u = 0$ on which a general Newton polygon side has ETS, necessarily a great circle (like $z = 0$) ?

of potentials H_{y_1}, H_{y_2} etc. This makes analysis harder. In section 8 we prove CRT for the function in example II, and for similar functions. In section 8.1 **Theorem 4** we reduce CRT for irreducible Σ to an "evident" conjecture for the remaining rare paths and rare functions. For reducible Σ we present some arguments in section 8.2 in favour of CRT, but do not obtain a complete proof.

Notes for four lectures at the Ecole Normale Supérieure, Paris

1. Preparations.

1.1. Formulation of the conjecture of R. Thom, polar coordinates.

The *oriented curves of steepest descent* of a *real analytic function* $f(x)$, $x = (x_1,...,x_n)$, on the space $\mathbb{R}^n$ or some neighborhood U of 0 with euclidean metric $dx^2 = \Sigma_{i=1}^n dx_i^2$, are defined by the differential equation

$$\dot{x} = \frac{dx}{dt} = - \text{grad } f = - *df , \tag{1.1}$$

or in real coordinates

$$\dot{x}_i = \frac{dx_i}{dt} = - \frac{\partial f}{\partial x_i}, \; i = 1,...,n . \tag{1.1}$$

We often call these curves *paths* (for short). Let $0 \in \mathbb{R}^n$ be *critical point* of f *with value* $f(0) = 0$.

Lemma 1. If a solution $x(t)$ through $x = x(0)$, has points $x(t_i)$, $i = 1,2,..$ that converge for $t_i \to \infty$ to the critical point $0 \in \mathbb{R}^n$, then the same holds for any such sequence t_i. *The solution* $x(t)$ *has a unique point as limit*

$$x(\infty) = \lim_{t\to\infty} x(t) = 0 \in \mathbb{R}^n .$$

This follows from the *theorem of Lojaciewicz* [Loj] and [BCR], saying that $c > 0$ and $0 < b < 1$ exist such that near $0 \in \mathbb{R}^n$

$$| \text{grad } f | > c f^{1-b} > 0 .$$

Proof: By (1.1) the arc length $s = s(t)$ along the curve $x(t)$ obeys

$$\frac{ds}{dt} = \sqrt{\Sigma \left(\frac{dx_i}{dt}\right)^2} = |\text{ grad } f\ | .$$

Then

$$-\frac{df}{ds} = -\frac{df}{dt} / \frac{ds}{dt} = -\Sigma_i \frac{\partial f}{\partial x_i}\frac{dx_i}{dt} / |\text{grad } f| = \Sigma_i \left(\frac{dx_i}{dt}\right)^2 / |\text{grad } f| = |\text{grad } f|$$

Hence : $$-\frac{df}{ds} > c\, f^{1-b} \text{ , and } -\frac{df^b}{ds} > bc .$$

By integration between $f = f_1 > 0$ and $f = 0$, we get $bcs_1 < f_1^b$. The *steepest descent curve* is then as short as we please by choice of f_1 . The lemma follows, as "descending to two different points" is then impossible.

We compactify the path $x(t)$ with the endpoint $x(\infty) = 0$ to a compact embedded arc which is analytic for $0 \leq t < \infty$.

Now we can formulate the *conjecture of René Thom* [T]:

CRT : *The steepest descent curve* $x(t)$ *has a tangent at its critical endpoint* $x(\infty)$.

Definitions. Let $0 \in U \subset \mathbb{R}^n$, U open convex, be critical point of the analytic function $f : U \to \mathbb{R}$. The set of points $x \in U$ for which the steepest descent curve has $x(\infty) = 0$ as endpoint is called the *catch-set* (or *in-set*) of $0 \in \mathbb{R}^n$. The complement in U is the *escape domain* .

Example 1. In figure 1a we indicate for the function

$$f = x_1(x_1^2 + 3\, x_2^2) : \mathbb{R}^2 \to \mathbb{R} ,$$

with

$$-\text{ grad } f = (-3(x_1^2 + x_2^2) \text{ , } -6x_1x_2) ,$$

the flow, the catch set and the escape domain. There are three endpoint tangents at $0 \in \mathbb{R}^2$. The tangent on $x_2 = 0$ applies for the whole interior of the catch-set, which is foliated by paths. The same expression

$f : \mathbb{R}^3 \to \mathbb{R}$, but considered as function on $\mathbb{R}^3$ in variables x_1, x_2, x_3 has the same catch set with the same tangents *in the plane* $x_3 = 0$. See figure 1b . The escape domain now contains points that descent to other critical points on the line $x_1 = x_2 = 0$ for which $x_3 \neq 0$ at level $f = 0$.

We will be only concerned with points in the catch-set.

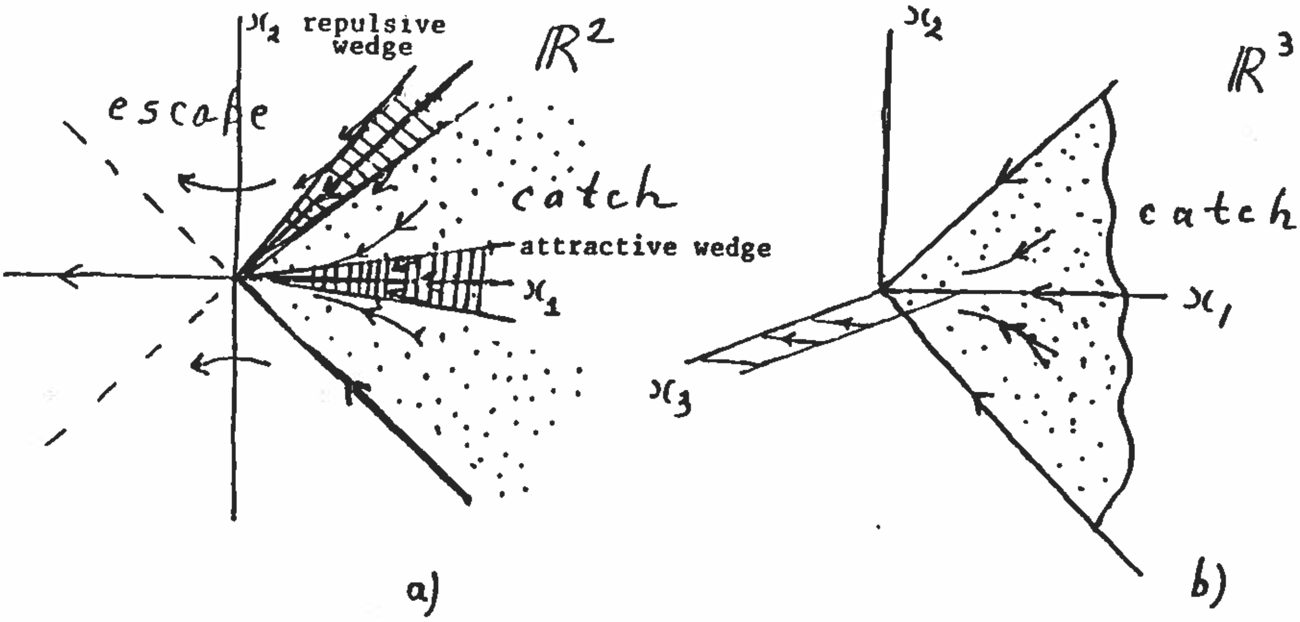


figure 1

$$- \text{grad } f = (-3(x_1^2+x_2^2), - 6x_1x_2)$$

Example 2. $f = x_1^p/p + \alpha\ x_2^q/q$, $p \geq q \geq 2$, $\alpha > 0$. We only consider integers p and q *both even.* The solutions of (1.1),

$$\dot{x}_1 = - x_1^{p-1} \ , \ \dot{x}_2 = - \alpha\ x_2^{q-1} \ ,$$

are as follows.

a) For $p = q = 2$ some solutions of (1.1) parametrized by $c \in \mathbb{R}$ are :

$$x_2 = c\, x_1{}^{\alpha}$$

with *endpoint tangents* $x_2 = c\, x_1$ for $\alpha = 1$, $x_2 = 0$ for $\alpha > 1$. If we add the point $0 \in \mathbb{R}^2$ the curves are analytic for integral $\alpha \geq 1$, of class C^k for $1 \leq k < \alpha < k+1$ at 0 .

b) For $q = 2 < p$ some solutions are:

$$x_2 = c \exp[-(\alpha(p-2)\, x_1^{p-2})^{-1}] = c \exp(-\beta / x_1^{p-2})$$

with common *endpoint tangent* $x_2 = 0$. The curves are of class C^∞ without being analytic for $c \neq 0$ at 0 .

c) For $2 < q < p$, we find

$$\frac{1}{\alpha(q-2)\, x_2^{q-2}} = \frac{1}{(p-2)\, x_1^{p-2}} + c^1 ,$$

$$x_2 = \beta\, x_1{}^{\frac{p-2}{q-2}} \, [1 + c\, x_1{}^{p-2}\, (1+o(x_1))]$$

with common *endpoint tangent* $x_2 = 0$. The curves are of class C^k for $1 \leq k < \frac{p-2}{q-2} < k+1$, analytic for $2 \geq k = \frac{p-2}{q-2}$ at 0 .

The line $x_1 = 0$ covers *endpoint tangents* for isolated solutions in cases b) and c). In all these examples the path is of differentiability class C^1 at 0 . But in general we cannot hope for higher differentiability than C^1 .

1.2. The fundamental differential equations and the limitset $\Omega \subset \Sigma \subset S^{n-1}$.

Write the given real analytic function as a convergent powerseries in homogeneous polynomials $\bar{H}_\ell(x) = \bar{H}_\ell(x_1,\dots,x_n)$ of degree ℓ :

(1.3) $$f = \Sigma_{\ell \geq k} \bar{H}_\ell(x) \quad , \quad \bar{H}_k(x) \not\equiv 0 \ , \ k \geq 2 \ .$$

Introduce in $\mathbb{R}^n\backslash\{0\}$ the *polar coordinates* $r = |x| = \sqrt{\Sigma_j x_j^2} > 0$, and $\omega = (v_1,\ldots,v_n)$ in the unit-sphere S^{n-1} by $x = r\omega$, $x_i = r\,v_i$, $i = 1,\ldots,n$, $\Sigma\, v_i^2 = 1$. The first coordinate r is a real number. The second "coordinate" ω is a point in S^{n-1}. If the path $(r(t), \omega(t))$ is a steepest descent curve then $\omega(t) \subset S^{n-1}$ is called its *celestial trace* (from the "view point" $0 \in \mathbb{R}^n$). If the path converges to the critical point 0, then it *has a tangent* at 0 *if and only if the trace converges to a point, by definition of that tangent* (!).

Denote the restriction of $\bar{H}_\ell$ to S^{n-1} by

(1.4) $$H_\ell = \bar{H}_\ell\,|S^{n-1} .$$

Then in polar coordinates

(1.5) $$f = \Sigma_{\ell \geq k}\, r^\ell\, H_\ell(\omega) \ , \ H_k(\omega) \not\equiv 0 \ .$$

Lemma 2. *The fundamental differential equations for steepest descent curves are given in polar coordinates by :*

(1.6)
$$\dot{r} = - \Sigma_{\ell \geq k}\, \ell r^{\ell-1} H_\ell(\omega)$$
$$\dot{\omega} = - \Sigma_{\ell \geq k}\, r^{\ell-2} \operatorname{grad} H_\ell(\omega) .$$

Here the gradient $\operatorname{grad} = *d$ is with respect to the Riemannian metric in S^{n-1}. These formulas are well known for $\mathbb{R}^2$ with the usual polar coordinates r and θ defined by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, and $\operatorname{grad}\theta = 1$. Intuitively they follow from this case $n = 2$. We give a formal proof for any dimension below. By definition of $\Phi_k(r,\omega)$ and the vector $V_k(r,\omega)$ we write (1.6) as:

(1.7) $$\dot{r} = - kr^{k-1}\,(H_k(\omega) + r\,\Phi_k(r,\omega))$$

$$\dot{\omega} = - r^{k-2}(\text{grad } H_k(\omega) + r\, V_k\,(r,\omega))\,.$$

After a suitable scalar coordinate multiplication $r' = \alpha r$, $\omega' = \omega$ for some $\alpha > 0$, we can assume the convergence of (1.6) for $r \le 1$. We can even assume the inequalities

(1.8) $$|\Phi_k(r,\omega)| < 1\,,\ |V_k\,(r,\omega)| < 1\,,\ \text{for } r \le 1\,,\ \omega \in S^{n-1}\,.$$

Proof. By $2r\dot{r} = \Sigma_j 2x_j\dot{x}_j$, (1,1) and (1,3), we have :

$$\dot{r} = \frac{1}{r}\Sigma_\ell\, \Sigma_j\ -(rv_j)\partial_j \bar{H}_\ell(rv_1,\dots,rv_n))\,.$$

As $\bar{H}_\ell$ is homogeneous of degree ℓ

$$\dot{r} = -\frac{1}{r}\Sigma_\ell\, \ell\ \bar{H}_\ell(rv_1,\dots,rv_n) = -\frac{1}{r}\Sigma_\ell\, \ell r^\ell\ H_\ell(v_1,\dots,v_n)\,.$$

$$\dot{r} = -\ \Sigma_{\ell\ge k}\, \ell r^{\ell-1}\ H_\ell(\omega)\,.$$

Next we have with $v_j = x_j/r$, (1,1) , (1,3) :

$$\dot{v}_i = \frac{\dot{x}_i - \dot{r}v_i}{r} = \frac{1}{r}\Sigma_\ell\,[-\partial_i\bar{H}_\ell(rv_1,\dots,rv_n) + v_i\ \Sigma_j v_j\partial_j\bar{H}_\ell(rv_1,\dots,rv_n)] =$$

$$= -\frac{1}{r}\Sigma_\ell\,[\partial_i\ \bar{H}_\ell(rv_1,\dots,rv_n) - v_i\ \Sigma_j\ v_j\ \partial_j\ \bar{H}_\ell(rv_1,\dots,rv_n)]\,.$$

As these terms are homogeneous of degree ℓ-1 in $(x_1,\dots,x_n) = (rv_1,\dots,rv_n)$ we get

$$\dot{v}_i\ = -\frac{1}{r}\Sigma_\ell\, r^{\ell-1}\,[\partial_i\bar{H}_\ell(v_1,\dots,v_n) - v_i\ \Sigma_j v_j\bar{H}_\ell(v_1,\dots v_n)]\,.$$

Then

$$\dot{\omega}\ =\ (\dot{v}_1,\dots,\dot{v}_n\,) = -\ \Sigma_\ell\ r^{\ell-2}\,[\text{grad } \bar{H}_\ell(\omega) - \langle\ \omega\ ,\ \text{grad } \bar{H}_\ell(\omega)\ \rangle.\omega]$$

$$\dot{\omega} = -\ \Sigma_{\ell\ \ge k}\ r^{\ell-2}\ \text{grad } H_\ell(\omega)\,.$$

This ends the proof.

For a solution $x(t)$ with $x(\infty) = 0$ we put

$$x(t) = (r(x(t)), \omega(x(t)) = (r(t),\omega(t)) \tag{1.9}$$

The endpoint property is expressed by

$$\lim_{t\to\infty} r(t) = 0 . \tag{1.10}$$

Define the *limitset* $\Omega = \omega(\infty) = \lim_{t=\infty}\omega(t) \subset S^{n-1}$ of the trace $\omega(t)$ of $x(t)$, for $x(0) = p$, in the catch set, by

$$\Omega = \Omega(p) = \{\omega \in S^{n-1} : \exists t_i, i = 1,2,\ldots \lim_{i\to\infty} t_i = \infty, \text{ and } \omega = \lim_{i\to\infty}\omega(t_i)\}. \tag{1.11}$$

If $r(t)$ attains a value $r > 0$ then it attains by (1.10) also all smaller values. For any such value r_0 there is a maximal value of t , *denoted* $t_0(r_0)$, for which r_0 is attained :

$$r(t_0(r_0)) = r_0 \text{ and } r(t) < r_0 \text{ for } t > t_0(r_0) . \tag{1.12}$$

Clearly:

Lemma 3a. The limitset Ω is compact and connected. The curve $x(t)$ has a tangent at $x(\infty) = 0$ if and only if Ω consists of one point.

Lemma 3b. The limitset $\Omega \subset S^{n-1}$ is contained in one critical level L of $H_k(\omega) : S^{n-1} \to \mathbb{R}$.

Proof. For any *non critical value* u of $H_k(\omega)$ let $\varepsilon > 0$ be the minimal value of

$$|\operatorname{grad} H_k(\omega)| \quad \text{for} \quad \omega \in L(u) = (H_k)^{-1}(u) .$$

The vector $\dot\omega$ in (1.7) is $H_k(\omega)$ - descending by (1.8) for $H_k(\omega) = u$ and $r < r_0 = \frac{1}{2}\varepsilon$. Therefore if $H_k(\omega(t)) \leq u$ for any $t_1 \geq t_0(r_0)$ then $H_k(\omega(t)) < u$ for all $t > t_1$, as the level $L(u) \subset S^{n-1}$ is a barrier against increasing $H_k(\omega(t))$.

As the same holds for every non critical level $L(u)$, the lemma follows. We can now define $H_k(\Omega) = H_k(\omega)$ for any $\omega \in \Omega(p)$. We conclude from lemmas 3a and 3b:

Lemma 3. *The compact connected limitset* $\Omega = \Omega(p)$ *is contained in one topological component* $\Sigma = \Sigma(p)$ *of the real algebraic critical set of* H_k :

(1.13) $$\Sigma \subset \text{crit } H_k = \{\omega \in S^{n-1} : \text{grad } H_k(\omega) = 0\} ,$$

and

$$H_k(\Omega) = H_k(\Sigma) \geq 0 .$$

Note that Σ covers a topological component of a real algebraic variety in the real projective space of dimension $n - 1$.

Proof: If $H_k(\Omega) < 0$ then take $r_o = -\frac{1}{2}H_k(\Omega) > 0$ and by (1.7) and (1.8) one has $\dot{r} > 0$ for $t \geq t_o(r_o)$. Then $x(t)$ "escapes".

2. Some special cases where CRT holds.

Here we first prove **Theorem 1.** (I understand that in recent years independent unpublished proofs have been given; recently Xing Lin HU wrote a proof of 1a, b <u>and</u> c.)

CRT holds for $n = 2$, it holds for any n in case $H_k \equiv c_1 > 0$ is constant, and in case $f \equiv \bar{H}_k$ is homogeneous.

2.1 Theorem 1a. CRT *holds* if the *leading term* $H_k \equiv c_1$ *is constant*.

Proof. If $c_1 < 0$ then all points near $x = (o,\omega)$ "escape". As $H_k \not\equiv 0$, we can assume $c_1 > 0$. Choose $r \leq r_o \leq \frac{1}{2} c_1$, $t > t_o(\tau_o)$. Then by (1.7), (1.8) :

$$\dot{r} < 0 \, , \, |\dot{r}| > \frac{1}{2} c_1 \, k \, r^{k-1} \, , \, |\dot{\omega}| < r^{k-1} \, .$$

$$\left|\frac{d\omega}{dr}\right| = |\dot{w} \, / \, \dot{r}| < \frac{2}{kc_1} = c_2 \, .$$

As $r(t)$ is *monotone* decreasing, then $\omega(t)$ describes a path in S^{n-1} with length

$$\int_0^{r_o} \left| \frac{d\omega}{dr} \right| dr < c_2 r_o \, .$$

Let $\omega_1 \neq \omega_2$ be different points in the limitset Ω. Then

$$\text{dist}\,(\omega_i, \omega(t_o(r_o))) < c_2 r_o \, , \, i = 0,1 \, ,$$

$$\text{dist}\,(\omega_o, \omega_i) < 2 \, c_2 \, r_o \, .$$

Choose r_o such that $2c_2 \, r_o < \frac{1}{2} \text{dist}\,(\omega_o \, , \omega_1)$, to get a contradiction. Theorem 1a is proved.

Example 3. $f = x_1^2 + x_2^2 + x_2^3$. Here $\Sigma = S^1$.

Then, by (1.6), $\dot{r} = -2r - 3r^2 v_2^3$, $\dot{\omega} = r \text{ grad } v_2^3$.

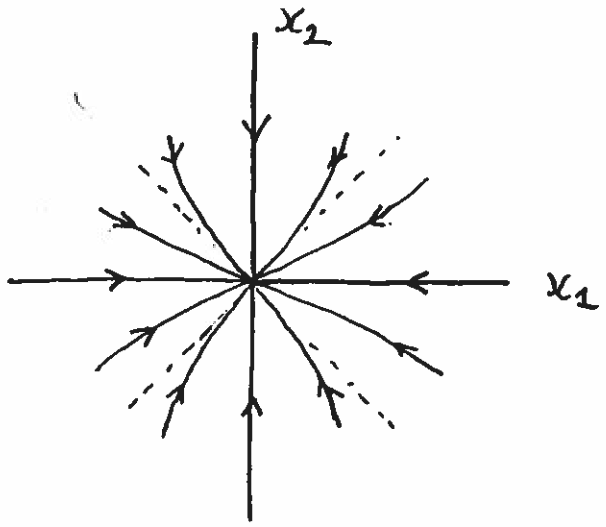


figure 2

$$\left|\frac{d\omega}{dr}\right| = |\dot{\omega}/\dot{r}| = \frac{1}{2}|\text{grad } v_2^3| + o(r)$$

Theorem 1b. Corollary. CRT *holds for* $n = 2$.

Proof. Either $H_k(\omega)$ is constant (see theorem 1a) or not, and then Ω , a connected set, is contained in a proper algebraic subset of S^1 . This is a finite set. Hence $\Omega = \Omega(p)$ is one point for every point p in the catch set.

2.2 Theorem 1c. *CRT holds for a* function (2.3) if it is <u>homogeneous</u> :

$$f \equiv \bar{H}_k, k \geq 2 .$$

Proof. The equations (1.7) are now

(2.1) $$\dot{r} = -kr^{k-1} H_k(\omega) , \dot{\omega} = -r^{k-2} \text{ grad } H_k(\omega) .$$

The integrals of the second equation are the same as those of

(2.2) $$\frac{d\omega}{dt} = -\text{ grad } H_k(\omega) .$$

They have a unique endpoint in a critical point of H_k on S^{n-1} by lemma 1 , generalised to the Riemannian manifold S^{n-1} instead of $\mathbb{R}^n$.

3. Study of the case $H_k(\Omega) > 0$. In particular for $n = 3$.

In this section we prove

Theorem 2. *If* $H_k(\Sigma) > 0$, *and* $\omega_o \in \Omega \subset \Sigma$ *lies in a n-2-stratum of a suitable stratification of* $\Sigma \subset S^{n-1}$, *then* Ω *is one point. For* $n = 3$ *therefore* Ω *is one point whenever* $H_k(\Omega) > 0$.

3.1 An irreducible polynomial equation $\bar{R}(x) = 0$ for an n-2-stratum Λ in case $H_k(\Omega_x) \geq 0$.

Let $\omega \in \Sigma \subset S^{n-1}$ be a critical point of $H_k(\omega)$. We can assume that H_k is not a constant. Such points correspond, one to one, to critical halflines $\{x = r\omega , r > 0\}$ in $\mathbb{R}^n$ of the real function on $\mathbb{R}^n\backslash\{o\}$ defined by

$$\bar{H}_k(x/r) = \bar{H}_k(x)/r^k \quad . \tag{3.1}$$

The critical points of this function on $\mathbb{R}^n\backslash\{o\}$ are given by

$$\partial_i(H_k(x)/r^k) = \frac{r^2\partial_i\bar{H}_k - kx_i\bar{H}_k}{r^{k+2}} = 0$$

$$r^2\partial_i \bar{H}_k - kx_i\bar{H}_k = 0 \quad . \tag{3.2}$$

As we have $H_k(\Omega) = H_k(\Sigma) = c \geq 0$, then $\Sigma \subset S^{n-1}$ obeys the homogeneous polynomial equation

(3.3 even) $\quad \Phi = \bar{H}_k(x) - c\, r^k = 0$ for k even, as well as for $c = 0$,

or

(3.3 odd) $\quad \Phi = \bar{H}_k^2(x) - c^2 r^{2k} = 0$ for k odd.

Let Λ be an *open* n-2-stratum of a stratification of $\Sigma \subset S^{n-1}$, of codimension one in S^{n-1} . Let $\bar{R}(x)$ be an irreducible factor of Φ that vanishes on Λ , and denote by p the highest integer for which near a point ω of Λ

(3.4) $$\Phi = \bar{R}^p . \bar{V} .$$

We consider the case k *even* , and the case $c = 0$. As $\bar{R}$ is irreducible then $d\bar{R} = (\partial_1 \bar{R},\dots,\partial_n \bar{R})$ vanishes at most in a proper algebraic subset of Λ . We include that set as well as any point for which $\bar{V} = 0$ in a lower dimensional stratum of our stratification of Σ_x . Then if $p = 1$ the equations (3.2) and (3.3 even) concerning points $x = r\omega$, cannot be satisfied for all $\omega \in \Lambda$. By contradiction therefore

(3.5) $$p \geq 2 .$$

For odd k the same conclusion can be deduced.

We summarize our observations in

Lemma 4. Let Σ be the component of crit $H_k(\omega) \subset S^{n-1}$ that contains Ω , and Λ a n-2-stratum of a suitable stratification of Σ . Then there is a homogeneous irreducible polynomial $\bar{R}$, a maximal integer $p \geq 2$, and a homogeneous polynomial $\bar{V}$, such that near any $\lambda \in \Lambda$

(3.6) $$H_k(\omega) - c = e\, R^p(\omega)|V(\omega)|$$

where $$H_k(\omega) = \bar{H}_k(x/r) = \bar{H}_k(x)/r^k$$

$$R(\omega) = \bar{R}(x/r) = \bar{R}(x)/r^m \text{ , m the degree of R}$$

$$V(\omega) \text{ is analytic and } V(\omega) \neq 0 \text{ for } \omega \in \Lambda$$

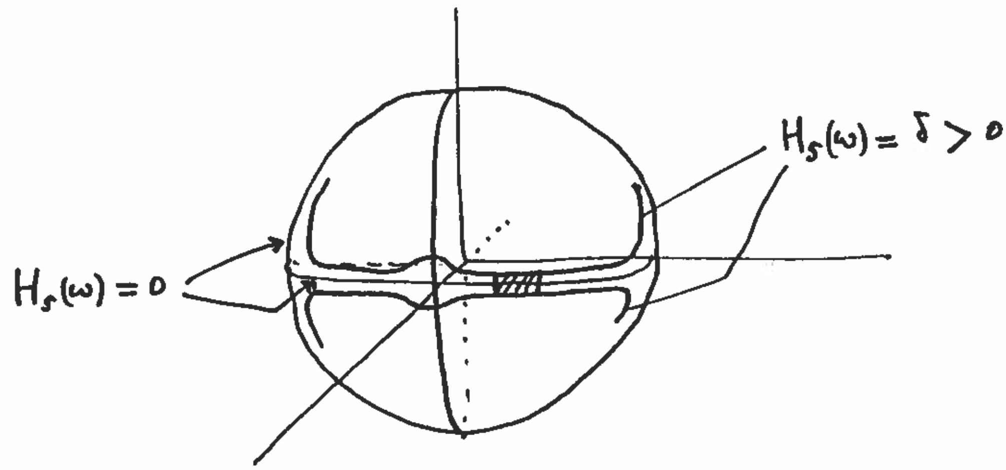
$$e = V(\omega)/|V(\omega)| = +1 \text{ or } -1$$

For $\omega \in \Lambda$ one has

(3.6) $$d\, H_k(\omega) = 0,\ R(\omega) = 0,\ dR(\omega) \neq 0 .$$

The restrictions on our stratification were made to assure that the function H_k behaves near any $\lambda \in \Lambda$ like a p-th power of a good coordinate. The danger is illustrated in

Example 4. Let $f = z^2(x^2 + z^2)\, y + y^6 + z^7 = \bar{H}_5(x,y,z) + y^6 + z^7 \in \mathbb{R}^3$. With respect to the critical point $0 \in \mathbb{R}^3$, the function $H_5(x,y,z)$ on S^2 is positive for $z \neq 0$, $y > 0$, negative for $z \neq 0$, $y < 0$, zero for $yz = 0$. The function $H_5(\omega)$ has levels which are near the points $y = 0$, and near the points $x = 0$, not "parallel" to the critical curve $z = 0$ in S^2. All levelcurves different from $z = 0$ have near the latter points a *bump or they move away*. The stratification of Σ will be chosen so that all such exceptional points, and some more to come, are in the 0-strata. See figure 3.



box: $\omega = (\sigma,\lambda) \subset [-\delta,\delta] \times D$

figure 3

3.2 Orthogonal analytic coordinates on S^{n-1} near $\omega_0 \in \Lambda$.

Near a point $\omega_0 \in \Omega \cap \Lambda$ we introduce the following analytic "coordinates" $(\sigma(\omega),\lambda(\omega)) \in \mathbb{R} \times D$.

(i) (3.7) $$\sigma(\omega) = R(\omega) \cdot \sqrt[p]{|V(\omega)|}, \; |\sigma|^p = |H_k(\omega)-c| .$$

This real function σ has the same level hypersurfaces in S^{n-1} as $H_k(\omega)$. But $\sigma(\omega)$ *is nonsingular near* ω_o, also on Λ ! Note that $V(\omega) \neq 0$ near ω_o .

(ii) To define the second "coordinate" of ω , the point $\lambda(\omega)$, we consider the orthogonal trajectory (gradient curve) of the function σ (or H_k) , and denote the intersection with Λ by $\lambda = \lambda(\omega)$. See figure 4.

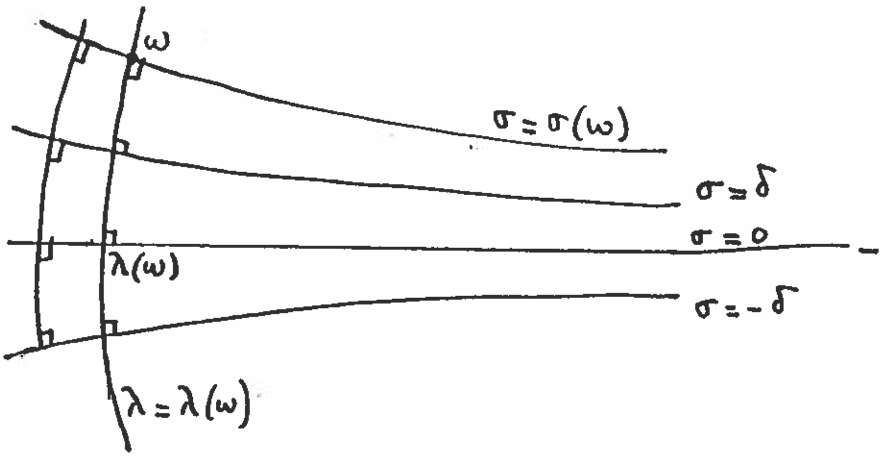


figure 4

Note that the coordinate λ takes values in the (n-2)-stratum Λ . For $n = 3$ this is a curve, which can be parametrised by arc length s with a choice of orientation, that is of increasing s .

We next see the three orthogonal "coordinates" in $\mathbb{R}^n$ in a *coordinate box*

(3.8) $$B = B(\delta,D) = \{(r,\sigma,\lambda)\} : 0 \leq r \leq \delta, \ |\sigma| \leq \delta \ , \ \lambda \in D\}$$

D is a small compact (n-2) disc in Λ , around $\omega_o = \lambda(\omega_o)$, $\delta > 0$.

Now and later again we will define conditions on ω_o and $\delta > 0$ for given D .

I. We choose $\delta > 0$ so small that c is the only critical value of $H_k(\omega)$ in the interval $[c-\delta^p, c+\delta^p]$.

II. The map and coordinate λ sends each level $\sigma = \varepsilon$ in box B (3.8), into Λ that is onto level $\sigma = 0$. For $\varepsilon \to 0$ this converges C^∞ to the isometry that is the identity map in Λ .

Consider the orthogonal decomposition

$$\dot{\omega} = \frac{d\omega(t)}{dt} = \frac{d\omega(t)}{dt}\bigg|_\Lambda + \frac{d\omega(t)}{dt}\bigg|_\sigma$$

in a component tangent to the "constant" σ-level and the orthogonal component respectively. Then for small $\delta > 0$ we have for $|\sigma| \le \delta$:

$$\text{(3.9)} \qquad \frac{1}{2}\left|\frac{d\lambda(t)}{dt}\right| = \frac{1}{2}\left|\frac{d\lambda(\omega(t))}{dt}\right| \le \left|\frac{d(\omega(t))}{dt}\bigg|_\Lambda\right| \le 2\left|\frac{d\lambda(\omega(t))}{dt}\right| .$$

Note that

$$\text{(3.10)} \qquad \frac{d\omega(t)}{dt}\bigg|_\sigma = \frac{d\sigma(\omega(t))}{dt} \cdot \text{grad}\,\sigma\,(\omega(t)) .$$

Now we go back to our steepest descent curve $x(t) = (r(t),\sigma(t),\lambda(t))$. Let t_1 be so large that the distance in S^{n-1} between $\omega(t)$ and Σ_x is smaller than half the distance between $(\delta \times D) \cup (\delta \times D)$ and Σ_x for all $t \ge t_1$. Let $r(t_1) = r_o$ and take t_1 so large that moreover $r(t) \ge r_o = \delta$ implies $t = t_1$. So $t_1 = t_o(r_o)$ in the sense of (1.12).

If $x = x(t_1)$ lies in the *coordinate box* , then the curve $x(t)$ will for $t > t_1$ not meet the *top* $\sigma = \delta$, or the *bottom* $\sigma = -\delta$, or the *side* $r = \delta$, nor the "side $r = 0$" except for $t = \infty$. We call the union of these four parts of the boundary of the box, the "*barrier*" after time t_1 *of Box* B .

If $n = 3$ and $H_k(\omega)$ is not constant, then Σ is a component of an algebraic variety of dimension ≤ 1 . Its suitable stratification consists of open 1-strata and points. If whenever $\omega_o \in \Omega \cap \Lambda$ for a one-stratum implies $\Omega = \omega_o$, then we are done !

3.3 Proof of theorem 2.

The equations (1.6), (1.7), (3.7), (3.9) yield for our case $H_k(\omega) = c > 0$.

$$\frac{dr}{dt} = - kr^{k-1} \; [(c+e\,\sigma^p) + r\,\Phi_k(r,\omega)]$$

(3.13)

$$\frac{d\omega}{dt} = - p\,e\,r^{k-2}\,\sigma^{p-1}\;\mathrm{grad}\;\sigma \;-\; r^{k-1}\;V_k\;(r,\omega)$$

and

$$\left.\frac{d\omega}{dt}\right|_\Lambda = - r^{k-1}\;\left.V_k(r,\omega)\right|_\Lambda$$

(3.14)

$$\left.\frac{d\omega}{dt}\right|_\sigma = - p\,e\,r^{k-2}\,\sigma^{p-1}\;|\,\mathrm{grad}\;\sigma\,| \;-\; r^{k-1}\;\left.V_k(r,\omega)\right|_\sigma \;.$$

For a suitable box $B(\delta,D)$ at ω_o we have in view of (1.7) with constants c_1, c_2, c_3 ,

$$\frac{dr}{dt} < 0 \;,\; \left|\frac{dr}{dt}\right| > \frac{1}{2}\,c\,k\,r^{k-1} \;,\; \left|\left.\frac{d\omega}{dt}\right|_\Lambda\right| < c_1 r^{\,k-1} \,,$$

and $\left|\frac{d\lambda}{dt}\right| < c_2 r^{k-1}$ by (3.11), and we have a *cone inequality*

$$\left|\frac{d\lambda(\omega(t))}{dr}\right| < c_3 \,. \tag{3.15}$$

Then if $(r(t_1),\omega(t_1)) \in B(\delta,D)$ is near the center of $0 \times D$, $r(t_1)$ goes monotonically to 0 for t_1 going to ∞ and $|\sigma(t)| < \delta$. By (3.15) we have a nested family of "wedges" with parameter t_1 ,

$$|\sigma| < \delta \ , \ |\lambda-\lambda(\omega(t_1))| \le c_3|r-r(t_1)| \ ,$$

whose intersection with $\Lambda(r=\sigma=0)$ in $B(\delta,D)$ converges to one point $\Omega = \omega$, the limit of the trace. This ends the proof of theorem 2.

If f has $0 \in \mathbb{R}^3$ as an isolated critical point at a minimum of f , then we know that some neighborhood of $0 \in \mathbb{R}^3$ belongs completely to the catch-set. Then starting on a point sufficiently near to the λ-axis inside some $B(\delta,D)$, its path remaining in the wedge must end on Σ , hence on the λ-axis. Then the points of Σ (of dimension one) reached by celestial traces of paths are everywhere dense on Σ as well as locally open, so we have

Theorem 2A. If $n = 3$, $H_k(\Sigma) > 0$, and f has 0 as an isolated minimal critical point, then every point of Σ is limit of some trace, and the corresponding unit vector at 0 in $\mathbb{R}^3$ is tangent vector at $x(\infty)$ of some path $x(t)$.

4. An example of standard essential trace speed (ETS) with potential $H_6|\Sigma$.

Essential trace speed (ETS) will be defined before lemma 8 of section 5. In the examples we will use the expression in a loose way.

The function

$$(4.1) \qquad f = (4z^2-r^2)^2 + (8x^2+r^2)r^4 = r^4H_4 + r^6H_6 \geq 0$$

has one critical point $0 \in \mathbb{R}^3$ where $f(0) = 0$ is minimal. All steepest descent paths converge therefore to $0 \in \mathbb{R}^3$ and the *catch-set* is $\mathbb{R}^3$.

Near the critical component Σ , with equation $z = \frac{1}{2} r$ on S^2 , we introduce the *coordinate* $\sigma = (4z^2-r^2)/r^2$. See figure 5a). Then

$$(4.1) \qquad f = r^4\sigma^2 + r^6H_6 ,$$

and the fundamental equations (1.6) for (r,ω) are

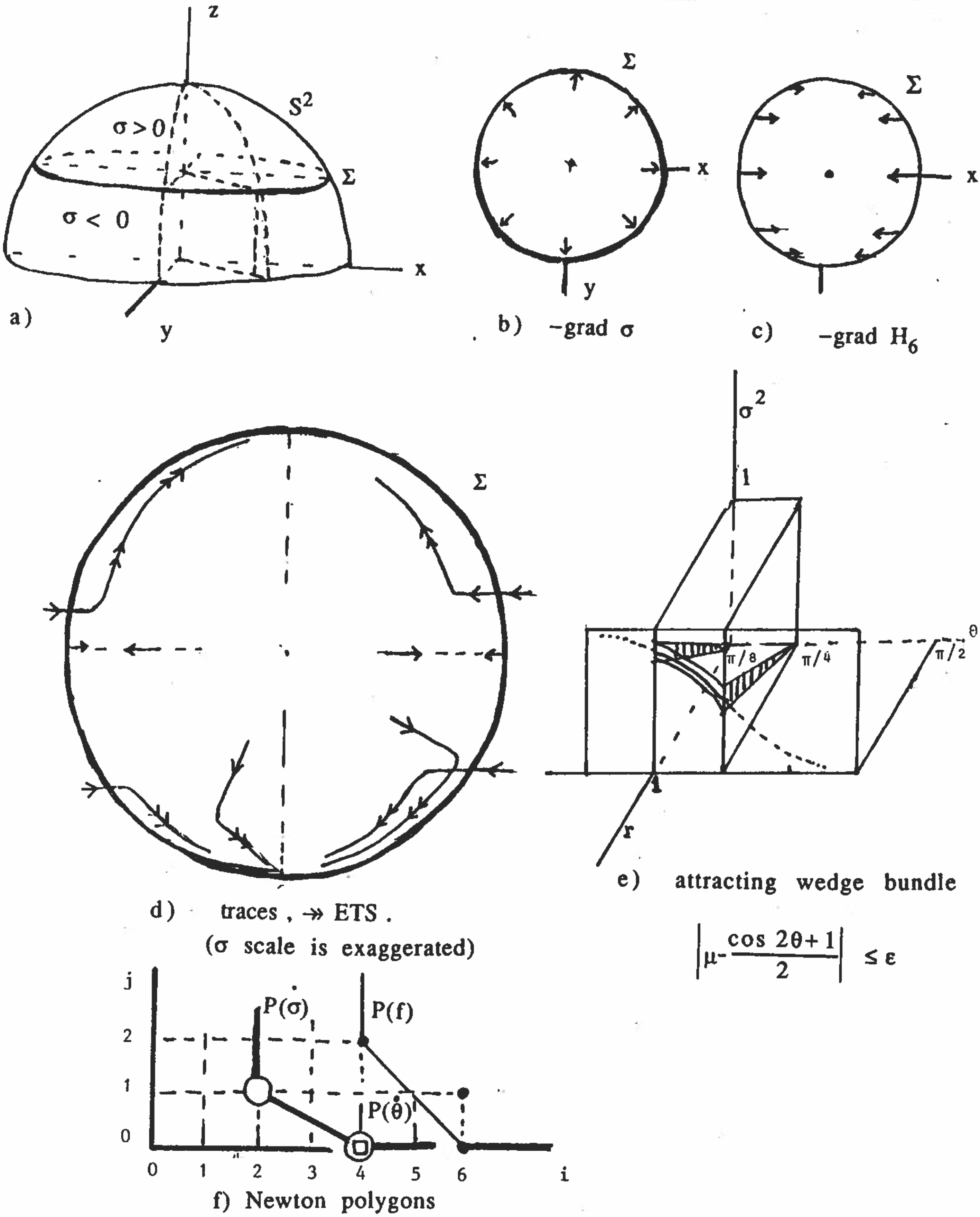
$$(4.2) \qquad \begin{aligned} -\dot{r} &= 4r^3\sigma^2 + 6r^5H_6 \quad > 0 , \\ -\dot{\omega} &= 2r^2\sigma \text{ grad } \sigma + r^4 \text{ grad } H_6 . \end{aligned}$$

We introduce as second *orthogonal* (polar) *coordinate* in S^2 , the angle θ by the equations

$$x = \frac{1}{2} r \cos \theta \, . \sqrt{(3-\sigma)} \, , \; y = \frac{1}{2} r \sin \theta \, . \sqrt{(3-\sigma)} \; .$$

Then we calculate with $r^2 - z^2 = (3-\sigma)r^2/4$,

$$(4.3) \qquad H_6 = 8x^2/r^2 + 1 = 2(3-\sigma) \cos^2 \theta + 1 = (3 \cos 2\theta+4) - \sigma(\cos 2\theta+1).$$



$$f = (4z^2 - r^2)^2 + (8x^2 + r^2)r^4 = r^4 H_4 + r^6 H_6\ ,\ \sigma = \frac{4z^2 - r^2}{r^2}\ .$$

figure 5 , Standard ETS

In figures 5b) and 5c) we suggest on the disc $\sigma > -\varepsilon$ of S^2 , and for small $\sigma > 0$ the vector fields $-\text{grad}\ \sigma$ and $-\text{grad}\ H_6$ orthogonal to the levels of the functions σ and H_6 . We now discuss points for which $0 < \theta < \pi/2$. For $\sigma = 0$ a trace point transverses Σ with σ increasing, but small to begin with, and then moves away from Σ as long as σ/r^2 is small. For large $\sigma/r^2 > 0$ however the first term of $\dot{\omega}$ dominates and the trace point moves straight in the direction of Σ . *These two "forces" compensate near values of* σ/r^2 *for which* $-\dot{\sigma}$ *vanishes*, and *only then* $-\dot{\theta}$ can dominate strongly over $-\dot{\sigma}$. The formulas are obtained by differentiation of H_6 with respect to the orthogonal coordinates σ and θ . This gives

$$\text{grad}\ H_6 = \partial_\sigma H_6 \cdot \text{grad}\ \sigma + \partial_\theta H_6 \cdot \text{grad}\ \theta =$$

$$= -(\cos 2\theta + 1)\ \text{grad}\ \sigma + 6 \sin 2\theta\ \text{grad}\ \theta + \text{neglect}\ .$$

We obtain as components of $-\dot{\omega}$

$$\begin{cases} -\dot{\sigma} = [2r^2\sigma - r^4(\cos 2\theta+1)]|\text{grad}\ \sigma| \\ -\dot{\theta} = \qquad 6r^4 \sin 2\theta . |\text{grad}\ \theta| \end{cases}$$

As $|\text{grad}\ \sigma|$ and $|\text{grad}\ \theta|$ are $\text{const.}(1+0(|\sigma|))$ near Σ , we can neglect these factors for small σ . The trace will have essential trace speed for small $\sigma > 0$ and $r > 0$ inside an attracting wedge bundle over Σ , which is locally $\left|\mu - \frac{\cos 2\theta + 1}{2}\right| < \varepsilon,\ \mu = \sigma/r^2,\ \varepsilon > 0$. See figure 5e) for a part of a wedge bundle.

The ETS is guided by the component of $-\text{grad}\ H_6$ in the θ-direction. It is guided by (decreasing value of) $H_6|\Sigma$. This continues until the minimal value of H_6 is approached for $\theta = \pi/2$. For $\theta = \pi/2$ the path descends to $0 \in \mathbb{R}^3$ in the plane $x = 0$.

For $\sigma < 0$, $|\sigma|$small, the two "forces" cooperate and the trace point moves to Σ , transverses it at finite time $t_1 < \infty$ and enters the domain just discussed.

The plane $y = 0$ is invariant. It presents a standard 2-dimensional case.

In figure 5d) we suggest several traces with their ETS part marked by $\rightarrow\rightarrow$. Traces in S^2 may intersect each other at points where one of them has not yet reached the ETS part. Note that we have exaggerated the scale of σ in order to see more details of the traces.

Conclusion: In the example ETS exists and is guided by a potential $H_6|\Sigma$ in Σ . $\Omega \subset \Sigma$ cannot be different from one point. The only points we find on Σ are $\theta = \pm\, \pi/2$ where almost all traces end, and $\theta = 0$ or π in the invariant plane $y = 0$. Those points represent the tangent at the corresponding paths to $0 \in \mathbb{R}^3$.

Exercise. Describe the flow of steepest descent completely, with all celestial traces. In particular those in the invariant plane $z = 0$.

5. Study of the case $H_k(\Sigma) = 0$ for $f : \mathbb{R}^3 \to \mathbb{R}$. Two nice examples with essential trace speed.

5.1. Introduction. Newton sets and Newton polygons.

Let $(r(t),\omega(t))$ be a steepest descent path converging to the critical point 0 , for an analytic function $f : \mathbb{R}^3 \to \mathbb{R}$,

(5.1) $$f = \overline{H}_k + \Sigma_{\ell>k} \overline{H}_\ell = \Sigma_{i\geq k} \overline{H}_i = \Sigma\, r^i H_i \, .$$

Let the limitset Ω of the celestial trace $\omega(t)$ be contained in the topological critical component $\Sigma \supset \Omega$ of the algebraic critical set

$$\text{crit } H_k = \{\omega \in S^2 : \text{grad } H_k(\omega) = 0 \} \quad \text{and} \quad H_k(\Omega) = 0 \, .$$

A path of steepest descent for a function g , is a curve orthogonal to the levels of g hence it is also orthogonal to the levels of $f = g^2$. Any steepest descent curve of g appears therefore as steepest descent curve of $f = g^2$. This proves part 3) of the following

Lemma 5. Given Σ there is no restriction in the arguments in favour of CRT, if the following (convenient) restrictions are made.

1) $\overline{H}_k = \overline{R}_1^{\,p_1} \ldots \overline{R}_v^{\,p_v}\, \overline{W}$ is an algebraic factorisation of $\overline{H}_k$ with $\overline{R}_1, \ldots \overline{R}_v$ irreducible, and $p_1 , \ldots p_v$ maximal exponents.

2) $R_1,\ldots R_v$ are exactly the irreducible factors of H_k that vanish on at least one of the one-strata of the connected real algebraic variety component $\Sigma \subset S^2$.

3) $f = g^2$ is a square of an analytic function. In particular $k,p_1,\ldots p_v$ *are even*, $\overline{W}$ *is a square*, $\overline{H}_k$ *and* $H_k \geq 0$ *are squares and not* identically zero. If $v = 1$ then Σ is called *simple* (not composite), otherwise it is called *composite* .

Proof. 1) and 2) are clear. Taking into account lemma 6, the change of parameter (t) does not hurt the conclusions for CRT.

As $(r(t),\omega(t))$ in the catch-set converges to $(0,\Sigma)$ there exists for any $\delta > 0$, and any *neighborhood* U *with boundary* ∂U *of* Σ in S^2, a "time" t_1, such that for $t \geq t_1$ we have

(5.2) $$0 < r(t) < \delta \, , \omega(t) \in U \, , \omega(t) \notin \partial U \, .$$

f *need not be a square* for this conclusion.

Therefore:

Lemma 6. For any (δ,U) it suffices to study the fundamental differential equations (1.6) of lemma 2 for $0 \leq r \leq \delta$, $\omega \in U$, for any open tubular neighborhood U with boundary ∂U of Σ.

For a one-stratum Λ in Σ on one of the irreducible curves $R = R_u$ of lemma 5, local orthogonal coordinates σ and λ are again defined, but there is now no need to follow the old definition. For local analysis on Σ, the choice $\sigma = R$ is often the best. See however the example "variant" in section 7.1.

In this paragraph we will concentrate our attention on one stratum of the critical component Σ, and for one factor $R = R_u$. We will use a *coordinate box* B for the fundamental differential equations (1.6),

$$B = B(\delta,\Gamma) = B(\delta,\gamma,\lambda_0) = \{(r,\sigma,\lambda) : 0 \leq r < \delta \, , |\sigma| \leq \delta \, , \lambda \in \Gamma\} \, ,$$

$$\Gamma = \{ \lambda : \lambda_0 - \gamma \leq \lambda \leq \lambda_0 + \gamma \} \, ,$$

and also a "positive box" B^+.

(5.3) $$B^+ = B^+(\delta,\Gamma) = \{(r,\sigma,\lambda_0) \in B : \sigma \geq 0 \} \subset B \, .$$

The box B^+ is in particular useful in case $f = g^2$ is a square as we will see.

The faces defined by $\lambda = \lambda_0 + \gamma$ and $\lambda = \lambda_0 - \gamma$ are called the *ends of the* box B^+ (resp. B). The function f is then expressed as follows:

(5.4) $$f = \Sigma_{i\geq k} \bar{H}_i = \Sigma_{i\geq k} r^i H_i = \Sigma \, A_{ij} r^i \sigma^j \, ,$$

where $A_{ij} = A_{ij}(\lambda)$ is analytic and $\Sigma = \Sigma_{i\geq k, j\geq o}$. By differentiation with respect to r we get

$$(5.5) \qquad - \dot{r} = \Sigma\, iA_{ij}r^{i-1}\sigma^j .$$

As σ and λ are orthogonal coordinates on S^2 the tangent vector $-\dot{\omega}$ decomposes in two orthogonal component vectors, multiples of the vectors grad σ and grad λ with norms $|\text{grad}\ \sigma|$ and $|\text{grad}\ \lambda|$. If λ is arclength on Σ then for points on $\sigma = 0$ one has $|\text{grad}\ \lambda| = 1$. Both σ and λ are close to constant on a small box B . We find in B by differentiation with respect to σ and λ :

$$(5.6) \qquad - \dot{\sigma} = \{\Sigma_{i\geq k, j\geq 1}\ jA_{ij}r^{i-2}\sigma^{j-1}\}\ |\text{grad}\ \sigma|$$

as grad σ points in the direction of increasing σ , and

$$(5.7) \qquad - \dot{\lambda} = \{\ \Sigma_{i\geq k, j\geq o}\ \partial_\lambda\ A_{ij}\ r^{i-2}\sigma^j\}\ |\text{grad}\ \lambda|$$

as grad λ points in the direction of increasing λ .

Definition. The *Newton set* $N(f) = N(f,R)$ of the powerseries f in (5.4) is the union of the set $\{(i,j) : A_{ij}$ is not identically zero $\} \subset \mathbb{R} \times \mathbb{R}$, and the points $(+\infty, 0)$ and $(0, +\infty)$. The *Newton polygon* $P(f)$ is the boundary $\partial \mathcal{H} N(f)$ of the convex hull $\mathcal{H} N(f)$ of the Newton set $N(f)$. Each declining side of $P(f)$ represents a quasi-homogeneous polynomial of f (see below).

Lemma 7. If $f = g^2$ then the terms of the powerseries development of f that correspond to the vertices respectively the sides of $P(f)$ are squares (but not identically zero) respectively squares of quasi homogeneous polynomials.

Proof. As $f = g^2$, $P(f)$ is obtained from $P(g)$ by a (geometrical) multiplication with factor two of $\mathbb{R}^2$ from the origin. This multiplication corresponds to a squaring of vertex terms and quasi homogeneous polynomials respectively.

From the equations (5.4-7) we read the following relations between Newton sets and polygons:

$$N(-\dot{r}) = N(r^{-1}f) \quad , \quad P(-\dot{r}) = P(r^{-1}f)$$

$$N(-\dot{\lambda}) \subset N(r^{-2}f) \quad , \quad P(-\dot{\lambda}) \subset \mathcal{H} N r^{-2}(f) \ .$$

Therefore $P(-\dot{r})$, which equals of course $P(\dot{r})$, is obtained from $P(f)$ by a translation over one unit to the left, $N(-\dot{\lambda})$ is obtained from $N(f)$ by a translation over two units to the left, and then deleting those points for which $A_{ij}(\lambda) = \text{constant} \neq 0$, that is $\partial_\lambda A_{ij} = 0$. For examples of Newton sets and polygons see figures 5 f), 6, 7 and 8.

$N(\dot{\sigma})$ is obtained from $N(r^{-2}f)$ by a translation one unit down, but for the fact that we must delete points which fall then into $\mathbb{R} \times (-1)$. A declining side of $P(f)$ which has a vertex on $\mathbb{R} \times 0$ is called *special*. Also the vertex of such a side on $\mathbb{R} \times 0$ is called special. Other declining sides of $P(f)$ are called *general*. If f has a homogeneous factor $\overline{R}^{\,\ell}$ with $\ell \geq 1$ then all sides of $P(f)$ are *general*. If f has no factor $\overline{R}$, then *one* side of $P(f)$ is *special*. The definitions are slightly modified in section 7.

We now make some further *remarks concerning a general side*. A general side of $P(f)$ with slope m/n of $P(f) = \partial \mathcal{H} N(f)$ is suggested in figure 6. Also $P(-\dot{r})$, $P(-\dot{\sigma})$ and $P(-\dot{\lambda})$ are indicated. The terms in the powerseries (5.3) for f can be ordered in different ways. One way is by the degree of r. A different way is as follows. For $\sigma \geq 0$, we restrict to B^+ and order terms by the *total degree* in $\bar{\sigma} = \sigma$ and $\bar{r} = r^{n/m}$, corresponding to the declining side with slope $\alpha = m/n$ of $P(f)$, and for $\sigma \geq 0$, $r \geq 0$. That is we substitute $r = \bar{r}^{\,m/n}$, $\sigma = \bar{\sigma}$ and then order. The leading terms of f are as follows.

$$f = \Sigma \overline{H}_i = \Sigma r^i H_i = \Sigma_{\ell=0}^{s} A_\ell r^{N+\ell n} \sigma^{M-\ell m} \ , \tag{5.8}$$

$$A_\ell = A_\ell(\lambda) .$$

Here we write "f =" short for "the leading terms of f are".

The leading terms form a so-called quasi-homogeneous polynomial in r and σ . The *constant weighted degree* in $\bar{r}$ and $\bar{\sigma}$ is the rational number, $(N+\ell n)m/n + (M-\ell m) = Nm/n + M$. Every line parallel to the side of $P(f)$ contains points of constant weighted degree. We see in figures 6, 7, 8 the parallel sides for $P(-\dot{r})$, $P(-\dot{\sigma})$ and $P(-\dot{\lambda})$ (the lowest possible position of points in $N(-\dot{\lambda})$ is supposed to be realized here). Many points of $P(-\dot{\sigma})$ *are under the line of the side for* $P(-\dot{\lambda})$. They represent terms which *from the point of view of the coordinates* $\bar{\sigma}$ and $\bar{r}$ may dominate $(-\dot{\lambda})$. From this viewpoint there seems to be little chance that λ during descent will be able to move essentially with respect to σ and r . This is what we will examine. Independent of $P(f)$ we can introduce *another view* by a *slope* $\alpha > 0$ (like m/n) of any declining line. Then we order terms by total degree in $\bar{\sigma} = \sigma$ and $\bar{r} = r^{(1/\alpha)}$, obtained by substitution $\sigma = \bar{\sigma}$ and $r = \bar{r}^\alpha$ in the powerseries for f , $-\dot{r}$, $-\dot{\sigma}$, $-\dot{\lambda}$.

In figure 6 we show the Newton set $N(f)$ and Newton polygons $P(f)$, $P(-\dot{r})$, $P(-\dot{\sigma})$, $P(-\dot{\lambda})$ for a *nice* function f . It is nice because there are only few terms in $-\dot{\sigma}$ in competition with $-\dot{\lambda}$, for *general sides* of $P(f)$. This follows from the fact that here $\sigma = z/r$, that is Σ is a great circle (geodesic) on S^2 . In fact essential trace speed (defined below) will be realised in example II. In figure 7 we show $N(f)$, $P(f)$, etc. for a "usual function" . The reader may have in mind a function f for which

$$\overline{H}_k = r^{40} \sigma^{40} , \sigma = (4z^2 - r^2)/r^2 .$$

We only show the part of the diagram near to one general side. Many more terms of $-\dot{\sigma}$ are in overwhelming competition with $-\dot{\lambda}$ in fig. 7.

We now concentrate our attention on a general side of P(f), $f = g^2$, and rewrite the leading terms in (4.9) as follows:

$$f = \Sigma\, A_\ell r^{N+sn}\, \sigma^{M-sm} \,.\, (\sigma^{(s-\ell)m}/r^{(s-\ell)n})$$

(5.9) $$= r^{N+sn}\sigma^{M-sm}\, \Sigma_{\ell=0}^{s}\, A_\ell \mu^{s-\ell}$$

where $\mu = \nu^m = \sigma^m/r^n$ is a new variable and so is $\nu = \mu^{1/m} = \bar{\sigma}/\bar{r}$.

In the same view we find the leading terms for $-\dot{r}$,

(5.10) $$-\dot{r} = r^{N-1+sn}\, \sigma^{M-sm}\, \Sigma(N+\ell n)\, A_\ell\, \mu^{s-\ell}.$$

(If $f = g^2$ then N, M and s are even numbers.)

As $M - sm \geq 1$ for a general side of P(f), we also have

(5.11) $$-\dot{\sigma} = \{r^{N-2+sn}\, \sigma^{M-1-sm}\, \Sigma(M-\ell m)\, A_\ell \mu^{s-\ell}\}\, |\mathrm{grad}\, \sigma|$$

(5.12) $$-\dot{\lambda} = \{\, r^{N-2+sn}\, \sigma^{M-sm}\, \Sigma(\partial_\lambda A_\ell)\, \mu^{s-\ell}\}\, |\mathrm{grad}\, \lambda|.$$

We can divide by a common factor $r^{N-2+sn}\, \sigma^{M-1-sm}$ by choice of a different parameter t. We can also neglect the factors $|\mathrm{grad}\, \sigma|$ and $|\mathrm{grad}\, \lambda|$ which are uniformly bounded away from 0 in box B^+.

Then we find for $\sigma > 0$

$$f = r^2\sigma\, \Sigma_{\ell=0}^{s}\, A_\ell \mu^{s-\ell} = r^2\sigma\, F_f$$

$$-\dot{r} = r\sigma\,.\, \Sigma_{\ell=0}^{s}(N+\ell n)\, A_\ell\, \mu^{s-\ell} = r\sigma\, F_r,$$

(5.13) $$-\dot{\sigma} = \Sigma(M-\ell m)\, A_\ell\, \mu^{s-\ell} = F_\sigma,$$

$$-\dot{\lambda} = \sigma\,.\, \Sigma\partial_\lambda A_\ell\, \mu^{s-\ell} = \sigma F,$$

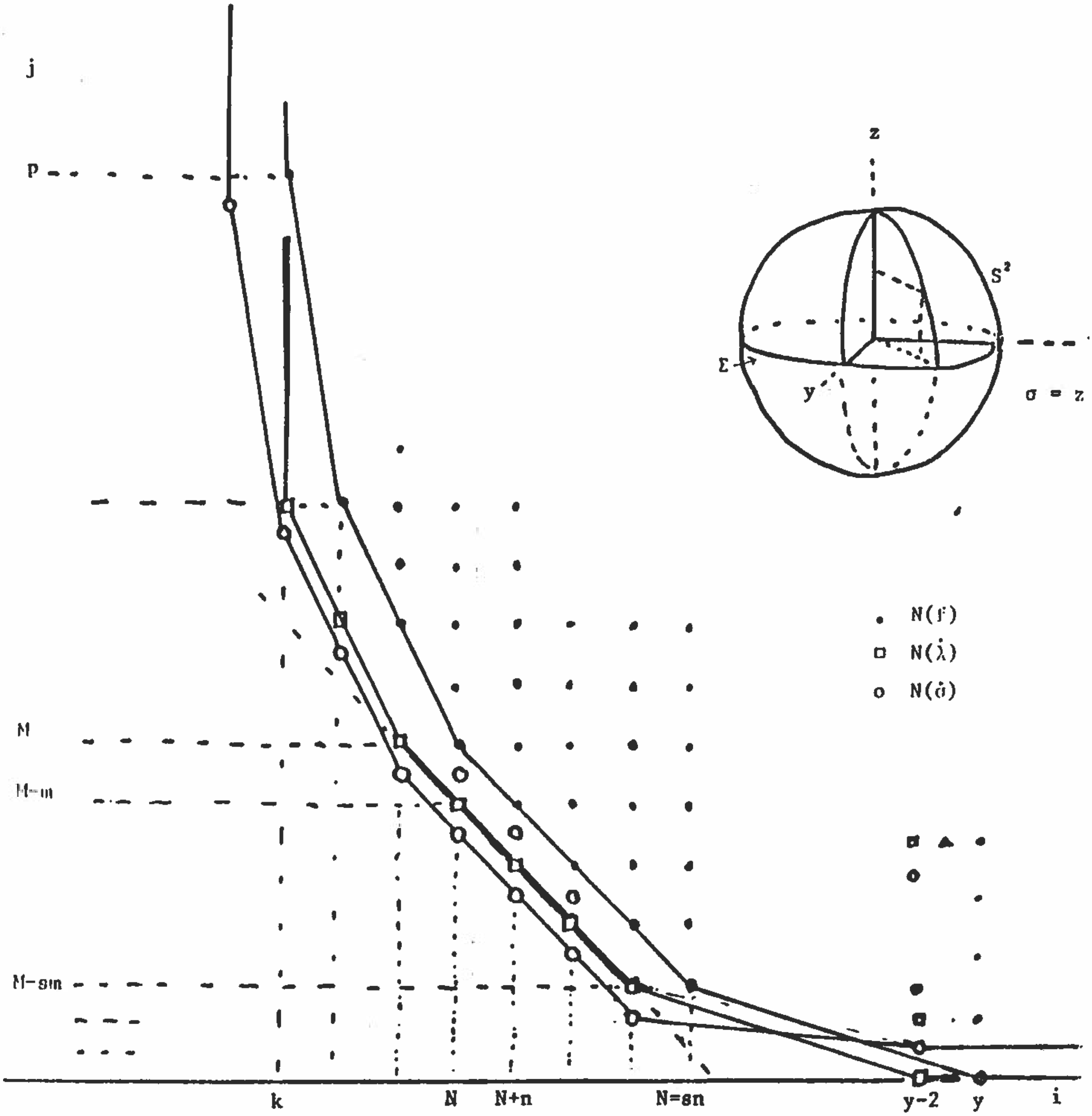


figure 6

$$f = r^k \sigma^p + \dots + \Sigma_{\ell=o}^{s} A_\ell(\lambda)\, r^{N+\ell n}\, \sigma^{M-\ell m} + \dots + A_y(\lambda)\, r^y + \dots \quad ; \; m = n = 2 \; .$$

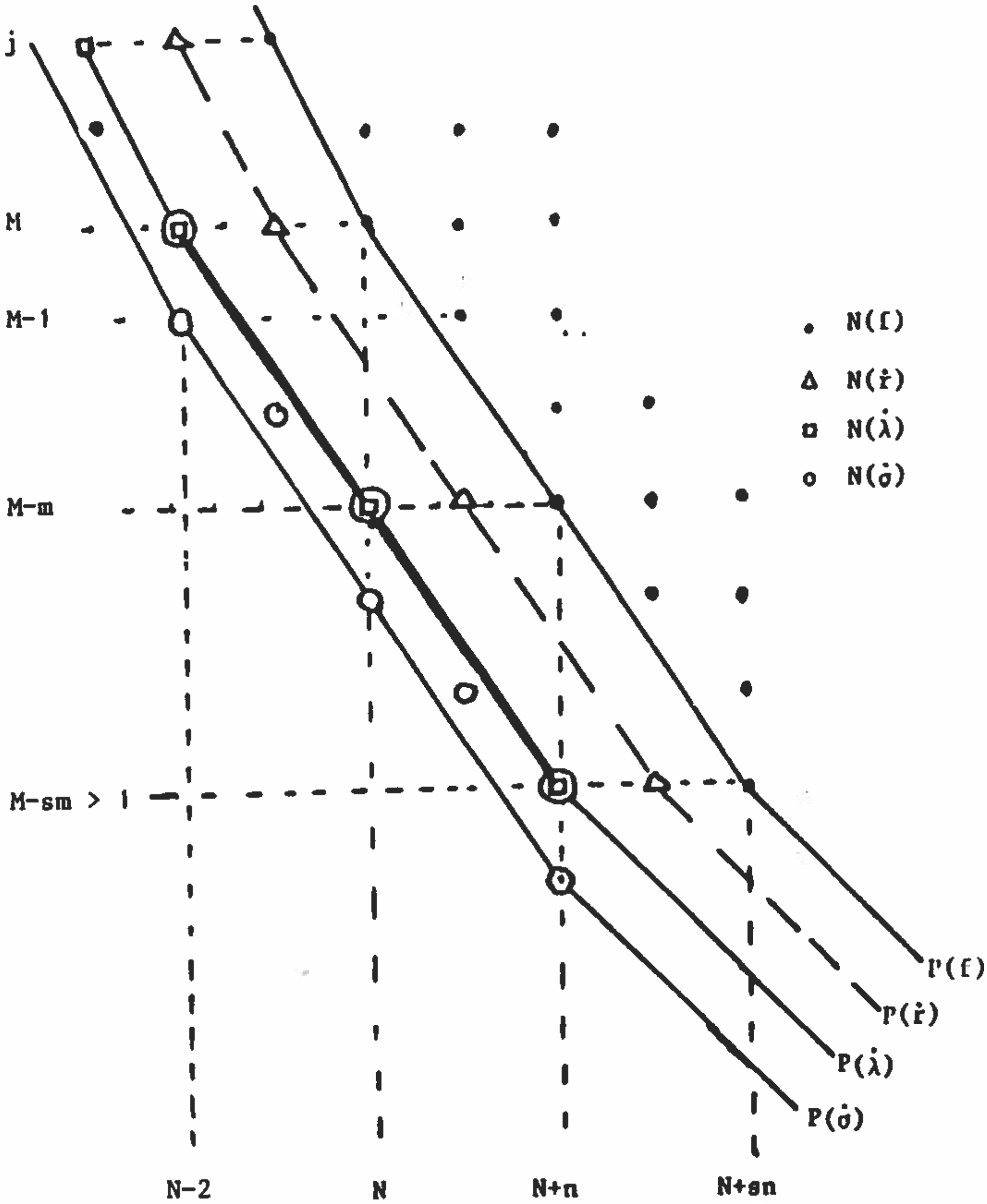


figure 7

Too much competition for $-\lambda$.

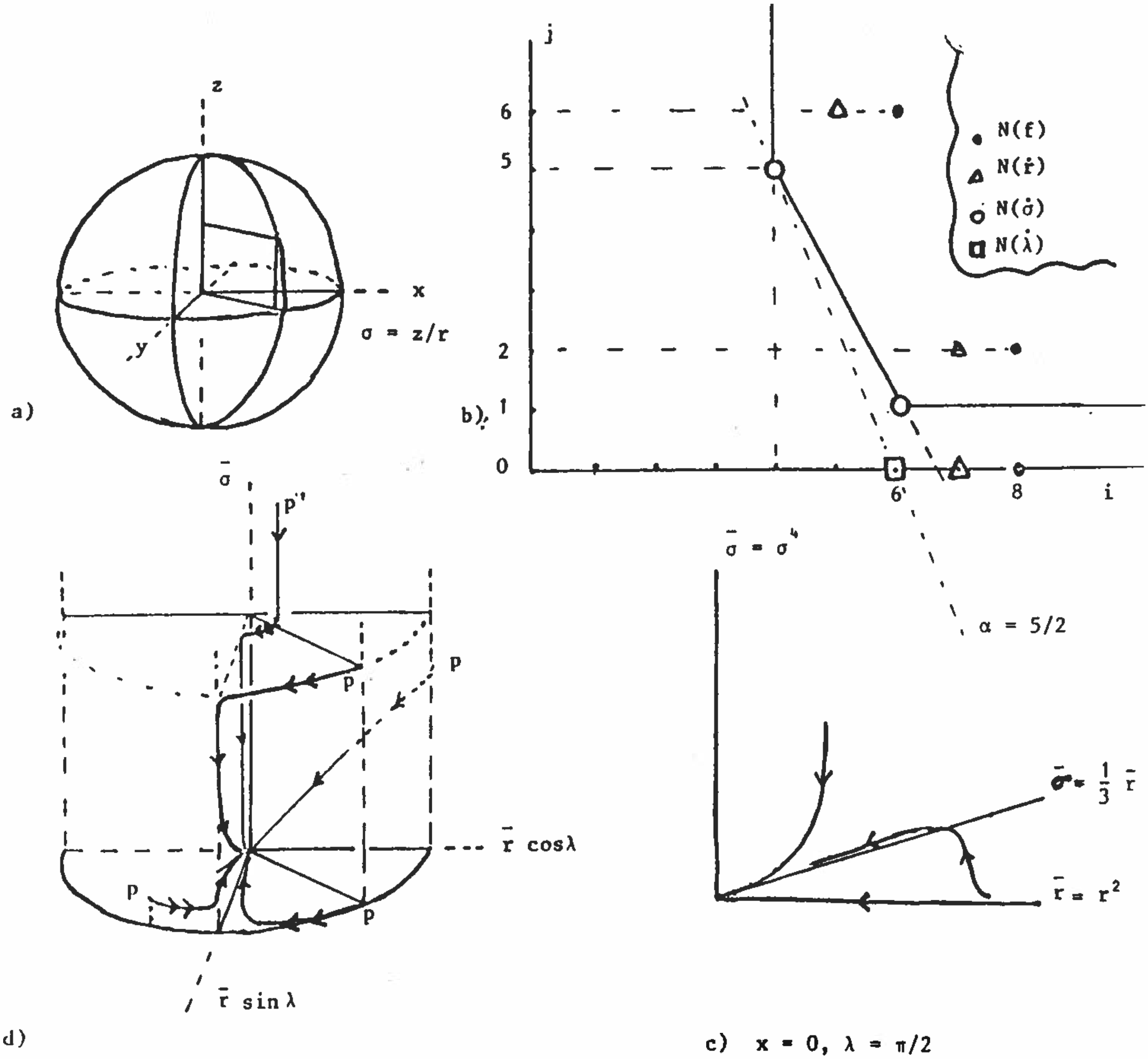


figure 8

Essential trace speed →>

where the polynomials F_r, F_σ and F, are defined by the equal signs.

In the *scales* (new variables) $\bar{\sigma}, \bar{r}, \lambda$ the equations are

$$(5.14)\qquad \begin{aligned} -\dot{\bar{r}} &= (n/m)\ \bar{r}\bar{\sigma}\ F_r \\ -\dot{\bar{\sigma}} &= F_\sigma \\ -\dot{\lambda} &= \bar{\sigma}\ F\ . \end{aligned}$$

These formulas confirm our above observations concerning the competition between leading terms of $-\dot{r}$, $-\dot{\sigma}$ and $-\dot{\lambda}$ respectively.

Definition. *The point* $(0,\omega_0) = (0,0,\lambda_0) \in B^+$ is said to have *essential trace speed* (ETS) in B^+ (resp. B), if for any $\varepsilon > 0$ there is an arc of a descent path to $0 \in \mathbb{R}^3$ $(r(t), \sigma(t), \lambda(t))$ with $o < r(t_1) < \varepsilon$, $o < \sigma(t) < \varepsilon$, $\lambda(t_1) = \lambda_0$, which meets the boundary of B^+ for the first time, after t_1 , in one of the ends $\lambda = \lambda_0 + \gamma$ or $\lambda = \lambda_0 - \gamma$ of box B^+ (resp. B). Such arcs are then "realisations of the essential trace speed".

Lemma 8. If the limitset $\Omega \subset \Sigma$ is not one point and $\omega_0 \in$ Interior $\Omega \subset \Sigma$, then ω_0 has essential trace speed.

Proof. This is obvious because the point $\omega(t)$ on the trace has to come as near as we please to every point of Ω (like ω_0) after any time t_2 , with both $\sigma(t)$ and $r(t)$ tending to zero.

MAIN ARGUMENT

Consequently essential trace speeds are *necessary* for the existence of a (counter) example where Ω is not one point. We will describe in the next paragraphs all possible essential trace speeds.

In the following sections of this paragraph we describe *essential trace speed* in two examples. To keep the examples simple we did not insist that f be a square. In both examples Σ is a great circle $z = 0$. The second example is "very singular". It shows unexpected "dramatic" behaviour of a gradient flow.

The examples are: I) A *nice* function $f := \overline{H}_6 + \overline{H}_8$ with a very dominant essential trace speed for a *special side* of P(f) , and II) A *nice* function with essential trace motion concerning a general side of P(f) . In example I it will be seen that the essential motion is guided by one obvious global algebraic potential X (it is H_8) along the critical component Σ of $H_k(f)$. In example II it is seen that essential trace speed can only happen within one of two "sharp wedge bundles" and in each it is guided (in general) by a different vector-field on Σ . The two vector-fields can be combined in one vector-field F on a graph $\tilde{\Sigma}$ which in the example is a double covering of Σ . See section 6.

5.2. Example I. A nice example with primitive essential trace speed. (See figure 8)

The example is the function

$$f \;=\; z^6 + r^6(3x^2+y^2) = \overline{H}_6 + \overline{H}_8 = r^6H_6 + r^8H_8 .$$

In this example all paths with minimal value at the (isolated) critical point $0 \in \mathbb{R}^3$ descent to $0 \in \mathbb{R}^3$. The catch-set is $\mathbb{R}^3$. The equator $z = 0$ on S^2 is the critical one-dimensional component Σ of H_6 on S^2 . Put $\sigma = z/r$, $x^2 + y^2 = r^2 - z^2 = r^2(1-\sigma^2)$, and let λ , arclength on the equator, be defined by

$$x = r\,(1-\sigma^2)^{1/2} \cos\lambda\,,\quad y = r\,(1-\sigma^2)^{1/2} \sin\lambda\,.$$

The coordinates σ and λ are orthogonal. See figure 8a). By the symmetry $f(x,y,z) = f(x,y,-z)$, it suffices to consider $\sigma \geq 0$. We find

$$f = r^6\sigma^6 + r^8H_8 = r^6\sigma^6 + r^8(1-\sigma^2)\,(2+\cos 2\lambda)\,,$$

and the fundamental differential equations (5.5,6,7) :

$$- \dot{r} = 6r^5\sigma^6 + 8r^7(1-\sigma^2)\ (2+\cos\ 2\lambda) \ > \ 0$$

(5.15) $$- \dot{\sigma} = \{6r^4\sigma^5 - 2r^6\ \sigma(2+\cos\ 2\lambda)\} \times |\text{grad}\ \sigma|$$

$$- \dot{\lambda} = \qquad - 2r^6\ \sin\ 2\lambda \times |\text{grad}\ \lambda|\ .$$

Let us fix a value λ_o, $o < \lambda_o < \pi/2$. As $|\text{grad}\ \sigma| - 1 = 0(\sigma)$ and $|\text{grad}\ \lambda| - 1 = 0(\sigma)$, we can approximate (5.15) without hurting the qualitative properties of the solutions, after dividing by r^4 , in a small positive coordinate box B^+ around λ_o , by

$$- \dot{r} = 6r^6\sigma^6 + 8r^3(2+\cos\ 2\lambda_o) \ > \ 0$$

(5.16) $$- \dot{\sigma} = 6\sigma^5 - 2r^2\ \sigma(2+\cos\ 2\lambda_o)$$

$$- \dot{\lambda} = \qquad - 2r^2\ \sin\ 2\lambda_o \ .$$

Now see the Newton sets and polygons in figure 8 b). The Newton polygons $P(\dot{\sigma})$ and $P(\dot{\lambda})$ have a common supporting line (a common tangent) with slope $\alpha = 5/2$. Any view defined by a slope α , can be used to introduce a scaling namely $\bar{\sigma} = \sigma$ and $\bar{r} = r^{1/\alpha}$. Define $\mu = \bar{\sigma}/\bar{r} = \sigma/r^{1/\alpha}$, or $\sigma = \mu r^{1/\alpha}$. Consider a part of the path where μ remains bounded away from 0 and from ∞ . Then leading terms with this scaling give for very small $r > 0$ and $\sigma > 0$, the estimate

$$\left|\frac{d\bar{\sigma}}{d\lambda}\right| = \left|\frac{d\sigma}{d\lambda}\right| = \left|\frac{\dot{\sigma}}{\dot{\lambda}}\right| = \frac{6\sigma^5}{2r^2\ \sin\ 2\lambda_o} = \frac{6\mu^5 r^{5/\alpha} r^{-2}}{2\ \sin\ 2\lambda_o}$$

(5.17) $$\frac{d\bar{\sigma}}{d\lambda} = c.\mu^5.r^{(2/\alpha).(5/2-\alpha)} \ , \ c \ \text{a constant}\ .$$

For $\alpha > 5/2$, $\dot{\lambda}$ can be neglected with respect to $\dot{\sigma}$, and certainly there is no essential trace speed in box B^+ . However $\bar{\sigma}$ will decrease until in a different view $\dot{\lambda}$ will strongly dominate. Note that if μ goes to 0 or to ∞ , we do need a different view and slope to see the next part of the trace. For $0 < \alpha < 5/2$, σ can be neglected and there is essential trace speed. In particular the derivative $\dot{\lambda}$ of the coordinate λ is guided by (decreasing value of) the potential function $H_8|\Sigma$. Therefore Ω cannot be different from one point. It is a point. In figure 8 d) traces are suggested and essential trace speeds are indicated. The point on the trace will move, after ETS is attained, "parallel" to Σ until the potential $H_8|\Sigma$ approaches its minimal value.

We call the trace speed *primitive* because the potential involves only one homogeneous part of f namely $\overline{H_8}$. For the minimum at value $\lambda = \pi/2$, the differential equations (4.15) reduce to two equations for $\sigma \geq 0$, $r \geq 0$ with $\mu = \sigma^4/r^2$

$$-\dot{r} = 6r^5\sigma^6 + 8r^7(1-\sigma^2)$$

$$-\dot{\sigma} = 6r^6\sigma(\mu-1/3) \ .$$

Such equations are well understood. Here σ increases for $\mu < 1/3$, decreases for $\mu > 1/3$, and (σ,r) converges to $(0,0)$ with asymptotic direction at $(0,0)$ given by $\bar{\sigma} = \bar{r}/3$ in the scales $\bar{\sigma} = \sigma$ and $\bar{r} = r^{2/4}$ as suggested in figure 8c). The corresponding Puisseux-branch in the plane $x = 0$ is given by

$$\mu = \sigma^4/r^2 = 1/3 \ , \qquad \sigma = z/r = cr^{1/2} \ , \qquad z = cr^{3/2} \ , \qquad c = 3^{-1/4} \ .$$

As $r^2 = y^2 + z^2$, for $x = 0$, we find the *Puisseux-branch* expressed in coordinates y and z by

(5.18) $$z = c\, y^{3/2} \ , \ y \geq 0 \ , \text{ in the plane } x = 0 \ .$$

We mention that all paths outside the invariant planes $x = 0$, $y = 0$, $z = 0$ and in $y > 0$, $z > 0$ approach $0 \in \mathbb{R}^2$ with this Puisseux-branch asymptote. They have a common tangent $x = z = 0$ at 0, and each is in $\mathbb{R}^3$ of differentiality class C^1. See figure 8d) for a qualitative picture in the coordinates $\bar{\sigma}, \bar{r}$ and λ. The paths in the invariant plane $y = 0$, $z > 0$ ($\lambda \in 0$ and $\lambda = \pi$) end with the asymptote $\mu = \bar{\sigma}/\bar{r} = 1$, with Puisseux-branch $z = x^2$ in the plane $y = 0$.

Exercise. Describe the flow of descent completely, with all celestial traces, in particular in the invariant plane $z = 0$.

5.3 Example II. A nice example of steepest descent with essential trace speeds that are guided by a weighted mean of gradients of potentials !

The example is the function

$$f = z^7 + \Sigma_{\ell=0}^{2} \Phi_\ell (x,y)\, r^{2+2\ell} z^{3-\ell} = \overline{H_7} + \overline{H_9} + \overline{H_{10}} + \overline{H_{11}},$$

Φ_ℓ is homogeneous of degree 4 in x and y.

We apply the same substitutions as in example I,

$$\sigma = z/r\ ,\ x = r(1-\sigma^2)^{1/2} \cos \lambda\ ,\ y = r(1-\sigma^2)^{1/2} \sin \lambda\ ,$$

and find

$$f = r^7\sigma^7 + r^9\sigma\, (\varphi_o\sigma^2 + \varphi_1\sigma r + \varphi_2 r^2)\, (1-\sigma^2)^2\ ,$$

with $\varphi_\ell = \varphi_\ell(\lambda) = \Phi_\ell$ (cos λ, sin λ).

The relevant critical component Σ of H_7 on S^2 is again the equator $z = 0$ $(\sigma=0)$.

The fundamental differential equations in a box B (or B^+), are

$$-\dot{r} = 7r^6\sigma^7 + (9\varphi_o r^8\sigma^3 + 10\,\varphi_1 r^9\sigma^2 + 11\,\varphi_2 r^{10}\sigma)\,(1-\sigma^2)^2$$

$$-\dot{\sigma} = \{7r^5\sigma^6 + [3\varphi_o r^7\sigma^2 + 2\varphi_1 r^8\sigma + \varphi_2 r^9 + \text{h.o}]\}\,(1-\sigma^2)^2 \times |\text{grad}\ \sigma|$$

$$-\dot{\lambda} = \{\partial_\lambda \varphi_o r^7 \sigma^3 + \partial_\lambda \varphi_1 r^8 \sigma^2 + \partial_\lambda \varphi_2 r^9 \sigma\} (1-\sigma^2)^2 \, |\text{grad } \lambda|$$

where some higher order terms have been neglected. The *complete* Newton set $N(f)$, and the Newton polygons $P(f)$, $P(-\dot{r})$, $P(-\dot{\sigma})$ and $P(-\dot{\lambda})$ are seen in figure 9a). Note that points on the 45°-declining side of $P(-\dot{\sigma})$ are the only points of $N(-\dot{\sigma})$ that are not above the parallel side of $P(-\dot{\lambda})$. Therefore $-\dot{\lambda}$ has not very much competition from terms of $-\dot{\sigma}$. Put $\mu = \sigma/r$, $(\sigma = \mu r)$. Then leading terms in the differential equations, keeping μ bounded by an upper bound $\bar{\mu}$, are, for $0 < \sigma < \varepsilon$, $0 < r < \varepsilon$, ε small depending on $\bar{\mu}$, given by (5.14) :

$$\begin{aligned} -\dot{r} &= r\sigma(9\,\varphi_o\mu^2 + 10\,\varphi_1\mu + 11\varphi_2) = r\sigma F_r \\ (5.19) \qquad -\dot{\sigma} &= (3\varphi_o\mu^2 + 2\varphi_1\mu + \varphi_2) = F_\sigma \\ -\dot{\lambda} &= \sigma(\partial_\lambda\varphi_o\mu^2 + \partial_\lambda\varphi_1\mu + \partial_\lambda\varphi_2) = \sigma F . \end{aligned}$$

We also replaced $|\text{grad } \sigma|$ and $|\text{grad } \lambda|$ by 1 as we did in example I.

A *very simple* set of equations is obtained with a *choice* of Φ_ℓ such that φ_o, φ_1 and φ_2 are constants:

$$(5.20) \qquad F_\sigma = (2\mu\text{-}1)(\mu\text{-}1) \equiv 2\,\mu^2 - 3\mu + 1 \;,\; 3\varphi_o = 2 \,,\, 2\varphi_1 = -3 \,,\, \varphi_2 = 1 \,.$$

But in this case there is no trace speed at all, $\dot{\lambda} = 0$.

Then $F = 0$, and F_r is positive definite, whereas

- $\dot{\sigma} = 0$ for roots $\mu_1 = 1/2$ and $\mu_2 = 1$

- $\dot{\sigma} < 0$ for $1/2 < \mu < 1$

- $\dot{\sigma} > 0$ for $\mu > 1$ and for $\mu < 1/2$.

Moreover, $-\dot{r}$ as well as f are negative for $\sigma < 0$, so that points with $\sigma < 0$ "escape". The function f and the gradient vectorfield are symmetric around the z-axis and the problem is essentially two-dimensional. In the (r,σ)-plane with $r > 0$ we find a well understood vectorfield (see figure 9b). The line $\sigma = 1/2\ r$ is asymptotic to a *separatrix* that separates the escape domain from an open set of paths that have the line $\sigma = r$ as asymptotic tangent. By rotation about the z-axis we get a *separatrix surface* that is asymptotic to a cone $\sigma = 1/2\ r$ in the coordinates. It separates a 3-dimensional escape domain from a domain of paths that are asymptotic at $(\sigma,r,\lambda) = (0,0,\lambda)$ to the cone $\sigma = r$. In (x,y,z) coordinates this invariant asymptote cone has equation $z = 1/2\ r^2$. The vector field is *repulsive near this separatrix surface*. The vector field is *asymptotically attracting to the cone* $\sigma = r$ (that is to $z = r^2$) for paths in the remaining open domain, except for the invariant z-axis.

We next choose $\varphi_o, \varphi_1, \varphi_2$ not constant but near to the values 2/3, $-3/2$ and 1 in (5.20). Under such a perturbation some qualitative properties remain invariant, others like essential trace speed change.

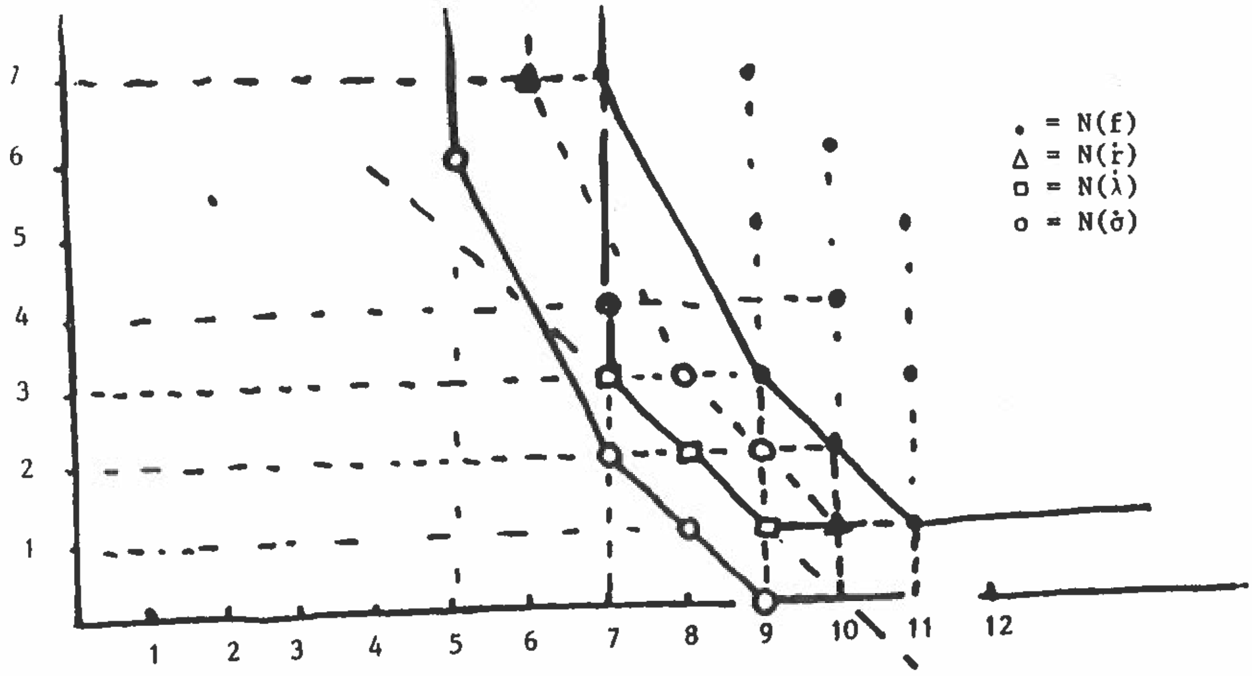


figure 9a)

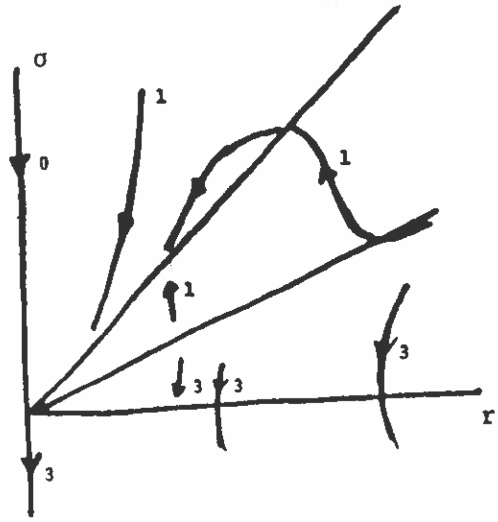


figure 9 b)

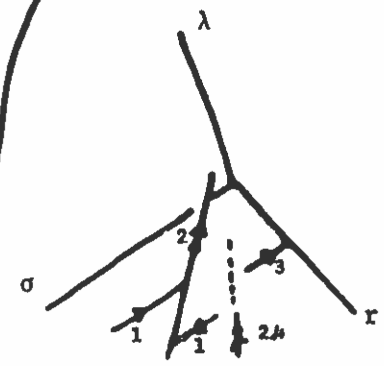


figure 10 a)

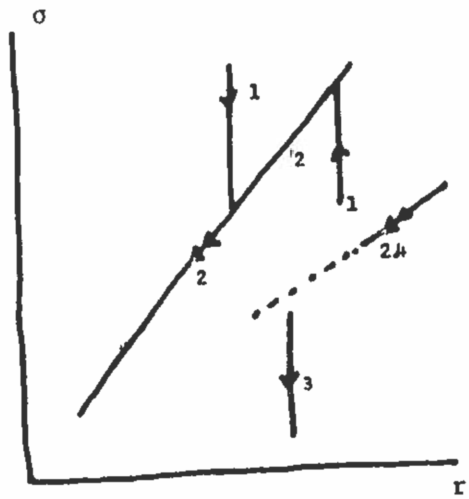


figure 10 b)

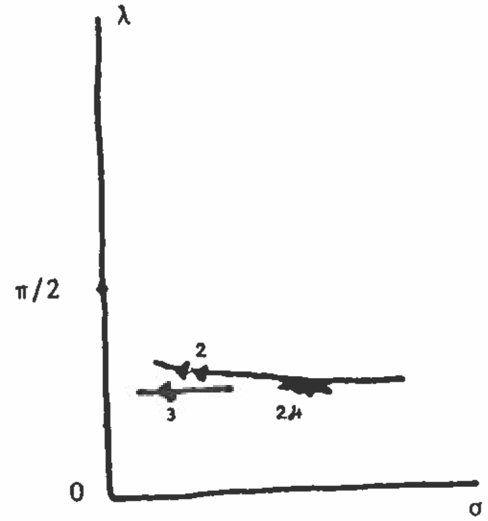


figure 10 c)

figure 10
A computed example (Ben Hinkle's program)

1) $F_\sigma = 0$ has two roots $\mu_i(\lambda)$, $0 < \mu_1(\lambda) < \mu_2(\lambda)$, near to 1/2 and 1 respectively. The values $\mu_i(\lambda)$ form a graph $\tilde{\Sigma} \to \Sigma$, a double covering over the critical component Σ.

2) From (4.19) we see that $-\dot{\sigma}$ dominates over $\dot{\lambda}$ and $\dot{r}$ as long as F_σ is bounded away from the roots μ_1 and μ_2.

3) There is a *separatrix surface*, asymptotic to the cone $\sigma = \mu_1(\lambda) . r$, and an *attracting asymptotic cone* asymptotic to the cone $\sigma = \mu_2(\lambda) . r$.

Later we prove that for essential trace speed in our case it is *necessary* that $F_\sigma(\mu) = 0(\sigma^\alpha)$ for some $\alpha > 0$. This means that for essential trace speed, μ must be very close to one of the roots $\mu_i(\lambda)$ of $F_\sigma(\mu) = 0$. The essential trace speed is (if it exists) given by

$$\Sigma_{\ell=0}^{2}(\partial_\lambda A_\ell(\lambda))\, \mu_i^{2-\ell}(\lambda) \qquad\qquad i = 1,2$$

that is by potentials $A_\ell(\lambda)$, whose gradients along Σ are weighted by coefficients $\mu_i^{2-\ell}(\lambda)$.

That the essential trace speed in fact is realised was seen for a special case with a computer program of Ben Hinkle from Cornell University and with his help as follows. But in this computed example there is only one effective potential namely φ_2, so chosen because the computer-time became already very long.

With the choice

$$3\,\varphi_o = 2 \quad , \quad 2\varphi_1 = -3 \quad , \quad \varphi_2 = 1 + 0.04 \cos 2\lambda \quad ,$$

the equations can be reduced to

$$-\dot{r} = r\,\sigma.\rho^2$$

$$-\dot{\sigma} = 2\sigma^2 - 3\sigma r + r^2(1+0.04 \cos 2\lambda)$$

$$-\dot{\lambda} = -2\varepsilon r^2 \sin 2\lambda .$$

Some computed paths are seen in figures 9 and 10. Essential trace speeds are indicated by a double arrow. The numbers mean the following.

- $\boxed{1}$- , the path moves with decreasing or increasing σ , and $-\dot{\sigma}$ dominating, to a situation 2 ; -$\boxed{2}$-, essential trace speed, with increasing λ as seen in figure 10. The path goes to the asymptotic Puisseux-branch $\sigma = r$, for $\lambda = \pi/2$; -$\boxed{3}$-, escape; -$\boxed{2.4}$-, path in the separatrix surface with again essential trace speed and converging to the Puisseux branch $\sigma = \frac{1}{2}r$, for $\lambda = \pi/2$. This (unstable!) path *was approximately computed* by following reversed time starting at a suitable point. The computed picture did show a slight increase of λ , that means essential trace speed.

5.4. <u>A more general example (again) with the equator</u> $z = 0$ <u>of</u> S^2 <u>as (critical) component</u> Σ <u>of</u> Crit H_k .

In figure 6 Newton-sets and Newton-polygons are shown for a case where $H_k = \sigma^p$, $\sigma = z/r$, p large. The views defined by slopes $\alpha < (M\text{-}sm)/(y\text{-}N\text{-}sn)$, smaller than the slope of the special side of P(f) , can reproduce for suitable H_{N+sn} and H_y the qualitative behaviour, and essential trace speeds guided by H_y , as in example I.

The views defined by slopes of general sides of P(f) can reproduce behaviour of paths and essential trace speeds as in example II. *These trace speeds realize in disjoint wedge-bundles, some attractive, some repulsive,* in directions that can be parallel or opposite to each other near any non-exceptional point ω of Σ . Some slopes may give no

ETS at all. So there can be different behaviour of the trace in different wedge bundles at ω .

The curve $\tilde{\Sigma}$ can be a finite covering of Σ for suitable coefficients, and for each general P(f)-side. This was the case in the special example II. We prove CRT for this case and similar cases in section 8.

6. Local theory along Σ^+ for $f = g^2$, $H_k(\Sigma) = 0$.

6.1. One way traffic (down) on the two boundary surfaces of certain local wedge bundles $W(\mu,\bar{\mu},\Gamma^1,m/n)$ for a general Newton polygon side.

We introduce, also for the case that Σ is composite, some new notations, apart from those already given in lemma 5. The intersection of the connected *critical graph* $\Sigma \subset S^2$ with the algebraic curve R_u , given by $R_u = 0$, is denoted $\Sigma_u = \Sigma \cap R_u$. Some (any) open component $\mathring{D}$ of $S^2\backslash\Sigma$ is selected, and one of its boundary components, a topological circle, is denoted $\Sigma^+ \subset \Sigma$. It has a one-sided tubular neighborhood U^+ inside $\mathring{D}$. See figure 11. Denote $\Sigma_u^+ = \Sigma^+ \cap \Sigma_u$. The factors $\bar{R}_u$ in $\bar{H}_k = \bar{R}_1^{p_1} \ldots \bar{R}_v^{p_v} \overline{W}$ are chosen such that $R_u > 0$ for $\omega \in \mathring{D}$.

We reserve the indices $u = 1,\ldots v$ for those irreducible factors R_u that involve sides of Σ^+ , and absorb the other factors in W . Recall lemmas 5 and 7 for consequences of the assumption $f = g^2$. We concentrate our attention on one side $\Sigma_u^+ = \Sigma_u \cap \Sigma^+$ of Σ^+ and use as a first coordinate near a point $\omega_0 \in \Sigma_u^+$ the function $\sigma = \sigma_u = R_u$ and as a second orthogonal coordinate λ . These coordinates are used in a coordinate box

$$B^+ = B^+(\delta,\Gamma) \, , \, \Gamma = \{\lambda : \lambda_0 - \gamma \le \lambda \le \lambda_0 + \gamma\} \, ,$$

$$0 < r \le \delta \, , \; 0 \le \sigma \le \delta \, , \lambda \text{ arc length.}$$

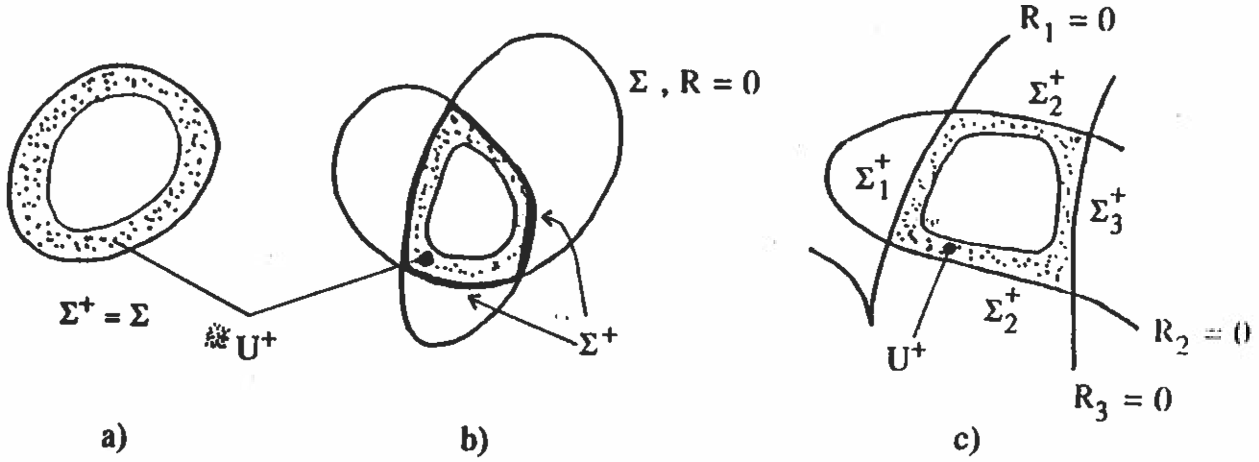
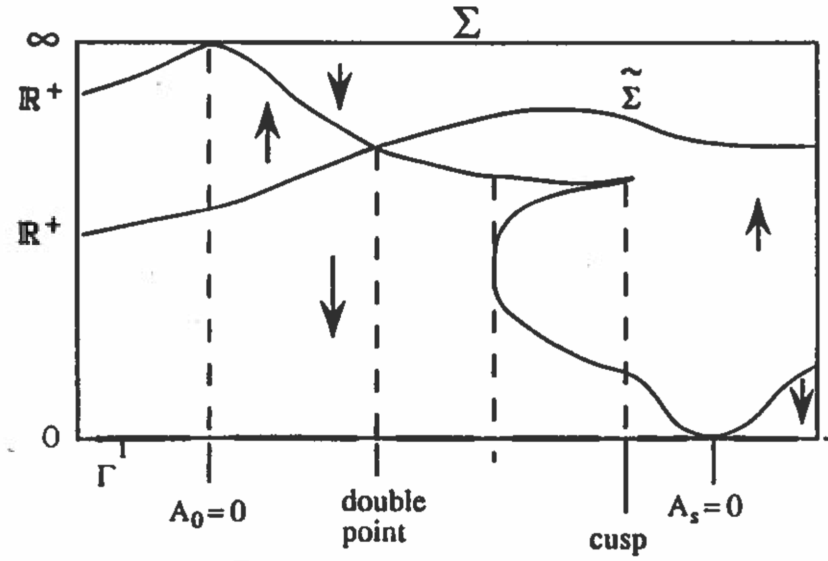


figure 11



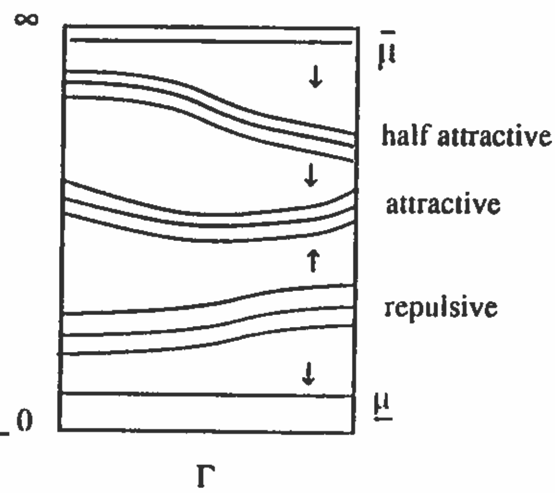


figure 12 a) $\tilde{\Sigma} \subset \Sigma \times \mathbb{R}^+$ figure 12 b) $W(\tau,\Gamma) \subset W(\mu,\bar{\mu},\Gamma)$

Intermezzo on special Newton polygon sides. If a declining side of $P(-\dot{\sigma})$ concerning R_u is *not parallel* to a corresponding side of $P(f)$, then $P(f)$ contains at the lower end of that *special* side a point $(i,j) = (y,0)$ with minimal y and H_y is a square but different from the constant zero. *The local* ETS-*theory* for such a special side of $P(f)$ *is easy* (see section 7). Simple cases are as follows.

a) The function $(H_y|\Sigma)$ on Σ is *not constant*, that is $(y-2,0) \in P(-\dot{\lambda})$, and also $(y-2,0) \in P(-\dot{\sigma})$ (the standard case; compare the example in section 4).

Then ETS for points on R_u may exist, and if so the trace is guided by (decrease of) the potential H_y . The ETS is realised by paths moving in a wedge bundle.

b) $(y-2,0) \in P(-\dot{\lambda})$, but $(y-2,0) \notin P(-\dot{\sigma})$ (the nice case: compare example I in § 5). Then ETS for points on R_u exists, and it is guided by the potential $H_y|R_u$.

c) If $H_y|\Sigma$ is *constant*, then the only case that needs attention is $(y+1,0) \in N(f)$ with $(y-1,0) \in P(\dot{\lambda})$. But as $(y-1,0) \in P(-\dot{r})$ as well, then $-\dot{\lambda}$ cannot dominate $-\dot{r}$, and ETS is excluded.

End of Intermezzo.

We now concentrate on the view defined by the slope $\alpha = m/n$ of one general side $P(f)$. That means $M-sm \geq 1$ in (6.2) below. To simplify the notation we discuss only the case of one simple closed curve $v = 1$, $\Sigma^+ = \Sigma_1^+ = \Sigma$, $\sigma = \sigma_1 = R = R_1$, as the local properties on a one-stratum of a composite Σ are the same. Leading terms concerning the view $\alpha = m/n$ are again expressed by (5.9-12) and shorter by (5.13) and (5.14). Convenient orthonormal scales for a coordinate box B^+ are $\bar{\sigma} = \sigma = R$, $\bar{r} = r^{m/n}$, λ . We call the $\bar{\sigma}$-coordinate axis *vertical.*

The restrictions of the real functions $A_\ell = A_\ell(\lambda)$ to Σ are algebraic functions.

They are :

(6.1) $$[(H_{N+\ell n}/\sigma^{M-\ell m}) \mid \Sigma] .$$

Note that the arc length λ is not algebraic on Σ , but if needed for proofs we can replace it by an algebraic coordinate on Σ . A function on S^2 is algebraic if it is induced from the double covering of S^2 over the projective plane $P(\mathbb{R},2)$ with homogeneous coordinates $x,y,z,$ from an algebraic function on that projective plane.

Definition: The cusp curve $\tilde{\Sigma}$ *is the algebraic curve with equation* $F_\sigma = 0$ *in the open annulus* $\Sigma \times \mathbb{R}^+ \subset \Sigma \times P$. See (6.2).

The projective line $P = P(\mathbb{R},1)$ is a one-point-compactification of $\mathbb{R}$ with its non homogeneous coordinate μ . This coordinate μ determines the projective and algebraic structure on P . The differential equation $-\dot{\sigma} = F_\sigma$ in (5.13) determines the polynomial equation of even degree s in μ :

(6.2) $$F_\sigma = \Sigma_0^s\,(M-\ell m)\ A_\ell\,\mu^{s-\ell} = 0\ ,\ A_\ell = A_\ell(\lambda)\,,\,\lambda \in \Sigma\,,\,\mu > 0\ .$$

The positive roots μ_i of this equation form a real part $\tilde{\Sigma} \subset \Sigma \times \mathbb{R}^+$, which can be viewed as a subset of real points in the complex algebraic extension, an algebraic curve $\tilde{\Sigma}(\mathbb{C}) \subset \Sigma(\mathbb{C}) \times P(\mathbb{C},1)$. The set $\tilde{\Sigma}$ is empty if there are no positive roots. The projection $\tilde{\Sigma} \to \Sigma$ can be a double covering as we saw in example II. A point of $\tilde{\Sigma}$ can cover a point of Σ multiple, but not more than s-tuple, as the degree in μ of (6.2) is s . The real algebraic curve $\tilde{\Sigma}$ can be union of more than one irreducible components, some of which can have multiplicity greater than one over their projection in Σ .

By lemma 7 we know that the algebraic functions A_0 and A_s on Σ are squares and not identically zero. Therefore as $F_\sigma = 0$ has a root

$\mu = 0$ if and only if A_s takes the value zero, that value $\mu = 0$ is attained only for a finite number of exceptional points on the algebraic curve Σ . The equation $F_\sigma = 0$ has a root $\mu = \infty$ only if A_0 takes the value zero. This again happens only for a finite number of exceptional points on Σ . Other exceptional points on Σ are those at which $dR = d\sigma = 0$ (see §3 where we discussed bad behaviour of a coordinate σ). These points are singularities of the curve $\Sigma \subset S^2$, like double points and cusps, and they are also finite in number. The multiplicities of the roots of (6.2) are also functions of $\lambda \in \Sigma$. They are locally constant except for changes at a finite number of points in Σ . Those of these points at which $\tilde{\Sigma}$ is smooth but tangent to a line $\lambda =$ constant, have not yet been included in the above set of exceptional points. They are now also added to the exceptional points. The complement in Σ of all exceptional points is a 1-manifold with a finite number of open components. We add the other points of Σ and $\tilde{\Sigma}$ and obtain stratified spaces. Topologically they are graphs. See figure 12 a) and 12 b) for a survey of exceptional points on Σ with respect to the slope $\alpha = m/n$.

Let $\Gamma^1 \subset \Sigma$ be the compact submanifold of points at arclength distance at least ι from any exceptional points in Σ and $\tilde{\Gamma}^1 \subset \tilde{\Sigma}$ the covering of Γ^1 under projection. With every component Γ of Γ^1 , covered by $\tilde{\Gamma} \subset \tilde{\Gamma}^1$, there is a coordinate box $B^+(\delta,\Gamma)$. All of these have the same Newton Polygon $P(f)$ with one general side of slope $\alpha = m/n$. We denote their union with a common δ by $B^+(\delta,\Gamma^1)$. As A_0 and A_s are uniformly bounded away from 0 and ∞ on Γ^1 , the positive roots of the equation (5.2) $F_\sigma = 0$ on the compact curve Γ^1 are bounded away from 0 and ∞ . So there exist positive constants $\underline{\mu}$ and $\bar{\mu}$ from which positive roots $\mu_i(\lambda)$, $\lambda \in \Gamma^1$ are uniformly bounded away as follows :

(6.3) $$0 < \underline{\mu} < 2\underline{\mu} < \mu_i(\lambda) < \frac{1}{2}\,\bar{\mu} < \bar{\mu} < \infty .$$

Then we see from (5.14) that given any $\xi > 0$ there exists $\varepsilon > 0$ such that if $0 < \sigma < \varepsilon$, $0 < r < \varepsilon$, $\mu = \underline{\mu}$ or $\mu = \bar{\mu}$ then

$$|\dot{\lambda}/\dot{\sigma}| < \xi, \ |\dot{r}/\dot{\sigma}| < \xi .$$

In the scales $\bar{\sigma} = \sigma$, $\bar{r} = r^{n/m}$, λ, the tangent vector $(-\dot{r}, -\dot{\sigma}, -\dot{\lambda})$ of a path is as near as we please to vertical ("*as vertical as we please*"). The tangent vector of a path at a point of $\sigma^m/r^n = \mu = \underline{\mu}$ or $\mu = \bar{\mu}$ points almost vertically in the indicated scales. In particular the path is transversal, and pointing down, at the two boundary surfaces of the wedge $W(\underline{\mu},\bar{\mu},\Gamma^1)$ defined by $\underline{\mu} < \mu < \bar{\mu}$. We conclude that, for δ small, the boundary surfaces of $W(\underline{\mu},\bar{\mu},\Gamma^1)$ allow only one-way-traffic (down) inside $B^+(\delta,\Gamma^1)$ as announced in the title of this section 6.1. Note that $\iota > 0$ could be chosen as small as we please. So every non-exceptional point on Γ can be included for some $\iota > 0$.

6.2. <u>Essential trace speed in</u> $W(\underline{\mu}, \bar{\mu}, \Gamma^1)$ <u>is excluded outside sharp wedges bundles</u> $W(\tau\,\sigma^\beta, \Gamma) \subset W(\underline{\mu}, \bar{\mu}, \Gamma) \subset B^+(\delta,\Gamma)$.

We consider the product bundle $\tilde{\Gamma} \to \Gamma$ for a component Γ of $\Gamma^1 \subset \Sigma$. Take $0 < \tau \le \tau_0$ and $\tau_0 > 0$ so small that the tubular neighborhoods $|\mu(\lambda)-\mu_i(\lambda)| \le \tau_0$ in $\Gamma \times \mathbb{R}^+$ are disjoint. The "wedges bundle" consisting of several wedges for each $\lambda \in \Gamma$, $\Gamma \subset \Gamma^1$ in $B^+(\delta,\Gamma)$.

$$W(\tau,\Gamma) = \{(r,\sigma,\lambda) : |\mu-\mu_i| \le \tau, \ \underline{\mu} \le \mu \le \bar{\mu}\} \subset W(\underline{\mu},\bar{\mu},\Gamma) \subset B^+(\partial,\Gamma),$$

has compact boundary surfaces in

$\partial_+W(\tau,\Gamma)$ defined by $\mu = \mu_i + \tau$, and in $\partial_-W(\tau,\Gamma)$ defined by $\mu = \mu_i - \tau$.

See figure 12 b) for intersections of extensions of wedges with $\bar{r} = 1$. There $\mu = \bar{\sigma}/\bar{r} = \bar{\sigma} = \sigma$.

As $A_0 \ne 0$, $A_s \ne 0$ for $\lambda \subset \Gamma$, the polynomial $F_\sigma(\mu)$ of degree s in μ is uniformly bounded away from the polynomial *zero*. The set of the first s derivatives of F_σ with respect to μ at points of $\tilde{\Sigma}$ (that is for $\mu = \mu_i$) is therefore uniformly bounded away from $(0,0,...0)$. We conclude that $c_1 > 0$ exists such that

$$(6.4) \qquad |F_\sigma(\mu_i+\tau)| > \frac{1}{2} c_1 \tau^s , \quad \text{and} \quad |F_\sigma(\mu_i-\tau)| > \frac{1}{2} c_1 \tau^s ,$$

for small $\tau > 0$. Thus $|F_\sigma|$ has a lower bound $\frac{1}{2} c_1 \tau^s$ on the boundary surfaces of $W(\tau,\Gamma)$. Then $-\dot{\sigma}$ dominates over $-\dot{\lambda}$ and $-\dot{r}$ by (5.14) at these boundary surfaces. The vector $(-\dot{r}, -\dot{\sigma}, -\dot{\lambda})$ is there almost vertical for $0 < \sigma < \delta$, $0 < r < \delta$, $\delta > 0$ small, in the scales $\bar{\sigma} = \sigma$, $\bar{r} = r^{m/n}$, λ for $B^+(\delta,\Gamma)$. Any path meeting a boundary surface meets it transversally.

Stronger, c_1 and τ_o can be chosen so that (6.4) holds for all points not inside the wedges bundle

$$W(\tau,\Gamma) \cap B^+(\delta,\Gamma)$$

for some $\delta > 0$ and $0 < \tau \le \tau_o$. Therefore *essential trace speed in* $W(\underline{\mu},\bar{\mu},\Gamma) \subset B^+(\delta,\Gamma)$ is excluded *unless* (necessary!) the (moving point on the) *path is inside the wedges bundle* $W(\tau,\Gamma) \subset W(\underline{\mu},\bar{\mu},\Gamma)$. The same holds for every component Γ of Γ^1 .

We next replace the wedges bundle $W(\tau,\Gamma)$ as follows by a *sharp wedges bundle* $W(\tau\, \sigma^\beta,\Gamma)$ for some *small* $\beta \ge 0$. By (6.4) we know

$$(6.5) \qquad |F_\sigma(\mu_i+\tau\,\sigma^\beta)| > \frac{1}{2} c_1 \tau^s \sigma^{\beta s} \quad \text{and} \quad |F_\sigma(\mu_i-\tau\,\sigma^\beta)| > \frac{1}{2} c_1 \tau^s \sigma^{\beta s} .$$

We substitute this in (5.13) and (5.14). Then for small $\beta s > 0$ the leading terms for the view defined by the slope $\alpha = m/n$, given by (5.14), determine again dominance of $\dot{\sigma}$ over $\dot{r}$ and $\dot{\lambda}$, but now for all points outside the *sharp wedges bundle* over Γ ,

$$W(\tau\sigma^\beta, \Gamma) = \{(\sigma,\bar{r},\lambda) \in B^+(\delta,\Gamma) : \underline{\mu} < \mu < \bar{\mu},\ \bar{\sigma} = \sigma ,\ |\mu-\mu_i| \le \tau\,\sigma^\beta\} .$$

The same holds for every component Γ of Γ^1 . We conclude as follows.

Lemma 9. *Essential trace speed* inside the wedge bundle $W(\mu,\bar{\mu},\Gamma^1) \subset B^+(\delta,\Gamma^1)$ is excluded *unless* (necessary) *the* (moving point on the) *path is inside the sharp wedges bundle* $W(\tau\sigma^\beta,\Gamma)$ for some component Γ of Γ^1.

Remark 1. Any component of the sharp wedges bundle $W(\tau\sigma^\beta,\Gamma)$ is either an *attractive wedge bundle* over Γ or a *repulsive wedge bundle* or a *half attractive wedge bundle.* This depends on the direction in which the paths transverse the two boundary surfaces of the wedge bundle under consideration. See figure 12 b). (Compare the steepest descent near 0 for the functions of one variable $x : f = x^2 , - x^2$ and x^3 respectively, where an interval around $x = 0$ is attractive, repulsive, or half-attractive respectively). The corresponding parts of $\tilde{\Gamma} \subset \tilde{\Sigma}$ will also be called attractive, repulsive and half-attractive respectively. If a path of a point in an attractive wedge bundle realizes ETS , then so does every path of points in some small neighborhood of the first point. ETS is a *stable* property for such points (and for such paths) in $B^+(\delta,\Gamma)$.

The two curves bounding a wedge at a point $\omega \in \Sigma$, have a common tangent at their common endpoint. They form the cusp whose tangent in a $(\bar{\sigma} ,\bar{r})$ plane is represented by roots of $F_\sigma = 0$. For that reason we called $\tilde{\Sigma} \subset \Sigma \times \mathbb{R}^+$ the cusp curve.

Remark 2. The introduced notions can be combined for all general sides (with slopes $\alpha_i = m_i/n_i > \alpha_{i+1} = m_{i+1}/m_{i+1} > 0$) of P(f) , concerning a simple closed curve $\Sigma = \Sigma^+$ (resp. $\Sigma_u^+ \subset \Sigma$ for a non simple irreducible curve Σ , or a composite curve Σ). We get a *pile*, namely the union $\cup(\tilde{\Sigma})$, *of cusp curves* $\tilde{\Sigma} \subset \Sigma \times \mathbb{R}^+$ each as in figure 12 a). The case i is placed *above* the case i + 1). We also get a pile of wedges bundles over compact intervals $\Gamma \subset \Sigma$.

Observe that the arrows in figure 12 b) imply that for *small* μ a path point in $B(\delta ,\Gamma)$ concerning case i will move "vertically", and with decreasing μ , and decreasing σ , to the underlying case i + 1 , that is the view defined by $\alpha_{i+1} = m_{i+1}/n_{i+1}$. Also for *large value* μ a path in $B(\delta,\Gamma)$ will have decreasing μ and σ , until it approaches and enters the highest (and therefore attracting or half-attracting) sharp wedge bundle in $W(\mu,\bar{\mu},\Gamma)$.

Going down by views defined by decreasing slopes $\alpha_i = m_i/n_i$ it is *only at the slope* of *a special side of* $P(f)$ that $\mu = 0$ and $\sigma = 0$ can be attained at a finite time $t < \infty$, and the path goes to the other Box $B^-(\delta,\Gamma)$ with $\sigma < 0$. Moving up from the view α_{i+1} to the view α_i could at most happen near exceptional points "$A_o = 0$" and "$A_s = 0$" .

Problem 1: Could it happen an infinite number of times (for $t \to \infty$) on a path that returns again and again in the course of time? This is interesting for existence of paths with Ω different from a point. See section 8.

6.3. The vector-field F along $\tilde{\Sigma} \to \Sigma$ concerning $\alpha = m/n$, for a simple closed curve $\Sigma = \Sigma^+$ on S^2 .

We start from the formula (5.13), $-\dot{\lambda} = \sigma F$, where F is a sum of derivatives $\partial_\lambda A_\ell$ of potential functions A_ℓ with coefficients $\mu^{s-\ell} > 0$ as weights. The potentials restricted to $\Sigma \subset S^2$ are algebraic functions. The coefficients $\mu^{s-\ell}$ need to be represented, for essential trace speed (ETS), in sharp wedges bundles. Asymptotically these coefficients are $(\mu_i(\lambda))^{s-\ell}$ where μ_i is root of the equation (6.2) : $F_\sigma = 0$.

We lift the arclength (and metric) λ on Σ , to an arclength (and metric) $\tilde{\lambda}$ on $\tilde{\Sigma}$ by the projection $\tilde{\Sigma} \to \Sigma$. Then we extend the vector-field $F = \dot{\lambda}/\sigma$, for $\sigma > 0$ and $\sigma \to 0$ to a vector-field on the metric graph $\tilde{\Sigma}$ above Σ . It is the *limit*:

$$F = \Sigma_\ell(\partial_\lambda A_\ell)\,\mu_i^{s-\ell} \quad \text{on} \quad \tilde{\Sigma} \to \Sigma . \tag{6.6}$$

F is well defined at all points of $\tilde{\Sigma}$ that are not exceptional. (Note that some points $\tilde{\lambda} \in \tilde{\Sigma}$ covering a given exceptional point $\lambda \in \Sigma$ need not be called exceptional on $\tilde{\Sigma}$. This is easily seen in figure 12 a)). If some non-exceptional point $\tilde{\lambda}$ has value $F \neq 0$ in (6.6) then so has every non-exceptional point of $\tilde{\Sigma}$ by analytic continuation, except for a finite number of *new* exceptional points where the vector F vanishes. For example it may change sign like descent for a maximum or a minimum of a real function.

We can now conclude to

Lemma 10. *If* ETS *exists at* $\tilde{\lambda} = (\mu_i,\lambda) \in \tilde{\Sigma} \subset \mathbb{R}^+ \times \Sigma$ *a non exceptional point of* $\tilde{\Sigma}$ *, then it is defined up to a positive factor by the vector-field* F *on* $\to \Sigma$:

(6.7) $$F(\tilde{\lambda}) = \Sigma\, \partial_\lambda A_\ell(\lambda)\, \{\mu_i(\lambda)\}^{s-\ell} = \Sigma \partial_\lambda A_\ell \mu_i^{s-\ell} .$$

Then it *also occurs at points near to any given initial non-exceptional point* $\tilde{\lambda} = \tilde{\lambda}_o \in \tilde{\Sigma}$. At different points $\tilde{\lambda}$ covering one and the same point $\lambda \in \Sigma$, the vector-field can project into different vector-fields on Σ and for example with opposite directions.

Remark. The same local conclusions hold for all general sides of P(f) concerning any side Σ_u^+ of $\Sigma^+ \subset S^2$, as defined at the beginning of § 6.1.

6.4. Local ETS is very rare for a general side of P(f) . Necessary conditions for ETS .

In figure 7 we see the common slope $\alpha = m/n$ of a general side of polygon P(f) with equation

$$j - M = \alpha(i-N) ,$$

and the side of polygon $P(-\dot{\sigma})$ on a line with equation

(6.8) $$j - (M-1) = \alpha(i-(N-2) .$$

This line is called "*the first line*" in the i-j-plane. *The "last line"* is the parallel line

(6.9) $$(j-M) = \alpha(i-(N-2)) \ .$$

It contains points of $P(-\dot{\lambda})$ (for example on a whole side of $P(-\dot{\lambda})$) if there is any hope for ETS . It may contain points of $N(-\dot{\sigma})$ and does so in figure 7. If $m \neq n$ then there are parallel *lines between* the first and the last line for rational numbers q , $o < q < 1$ such that

(6.10) $$(j-M-q) = \alpha(i-(N-2)) \ ,$$

contains lattice points of $\mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$, some of which may be in $N(-\dot{\sigma})$ as seen in figure 7.

The points of the first line give the known contribution and leading term

$$-\dot{\sigma} = F_\sigma = \Sigma_{\ell=o}^{s}(M-\ell m)\ A_\ell \mu^{s-\ell} \ .$$

If $\dot{\lambda}$ has any chance to dominate over $\dot{\sigma}$ and yield ETS then μ must be (by lemma 9) near to a root μ_i of the equation

(6.11) $$F_\sigma(\mu_i) = \Sigma_{\ell=o}^{s}(M-\ell m)\ A_\ell(\lambda)\mu_i(\lambda)^{s-\ell} = 0 \ .$$

For each in-between-line we can collect the terms of the powerseries in μ for $-\dot{\sigma}$, represented on that line (6.9) and obtain some function $F_{\sigma,q}(\mu)$. It must identically vanish by substitution of $\mu = \mu_i$, because otherwise it would dominate over the leading part (6.6) of $\tilde{F}$, and no ETS would be possible. So refering to a neighborhood of a point $\tilde{\lambda} = (\mu_i,\lambda) \in \tilde{\Sigma}$ in $\tilde{\Sigma}$, we have the first set of *necessary analytic conditions* for ETS :

(6.12) $$F_{\sigma,q}(\mu_i) \equiv 0 \ , \quad 0 < q < 1 \ .$$

The "*last line*" gives an analogous contribution in the powerseries of $-\dot{\sigma}$ which we denote $F_{\sigma,1}$. If this contribution is not identically zero then the quotient $|-\dot{\lambda}/-\dot{\sigma}|$ is bounded at almost all values λ for small δ , $0 < r \leq \delta$, $0 < \sigma \leq \delta$ and ETS is excluded even inside the concerned wedge bundles! So we have the second *necessary condition* for ETS .

(6.13) $$F_{\sigma,1}(\mu_i) \equiv 0 \ , \ (q = 1) \ .$$

It is the only necessary condition if $m/n = 1$. It seems hard to find examples where the necessary local conditions, in particular (6.13) are fulfilled. The only examples I know are simple generalisations of example II in section 5. They have the following properties: Σ is a great circle on S^2 , say given by $z = 0$; $\lambda = \theta$ is equivalent to an algebraic function namely $x/y = \text{cotg}\ \theta$, not only on Σ but on S^2 . So these are two *algebraic local orthogonal coordinates* σ *and* cotg θ . Moreover if $(i,j) \in P(f) \cap N(f)$ then $(i,j+1) \notin N(f)$. The last property implies (roughly) the condition $F_{\sigma,1} \equiv 0$.

Problem 2: Find other examples for which the necessary conditions for ETS are fulfilled or prove that there are no other examples.

Remark on the choice of σ . If (σ,λ) , $\sigma = R$, is chosen and we are interested only in $\sigma \geq 0$, then good coordinate systems of orthogonal coordinates different from (R,λ) are $(-R,\lambda)$, (R^2,λ) and (R^3,λ) . It also might happen (very unlikely!) that there exists a homogeneous polynomial function $\tilde{T}$, so that the leading part of $f = \Sigma A_\ell(\lambda)\, r^{N+\ell m} R^{M-\ell m}$ is expressed in new orthogonal coordinates $R' = RT$ and λ' , with $\tilde{R}T$ a homogeneous polynomial, by $f = \Sigma A'_\ell(\lambda')r^{N+\ell n}(R')^{M-\ell m}$. It is difficult to see what then happens with all the necessary conditions for ETS. The first condition (cusp curve) does not change in quality as long as $T \neq 0$ on the relevant stratum of Σ .

Note that it is easy to make many specific classes of functions f for which the conditions are not fulfilled so that ETS does not exist for general P(f) sides. Then only special sides can produce ETS .

Definitions. A function f with critical point $0 \in \mathbb{R}^3$ is called *rare* or *very singular* at $\tilde{\lambda} = (\lambda,\mu_i) \in \tilde{\Sigma}^+_u$ with respect to a general side $\alpha =$ m/n of P(f) , if the conditions (6.11, 12, 13) are fulfilled for some choice of σ . The function f is called *nowhere very singular*, if there are no such very singular points $\tilde{\lambda}$ (for any general slope $\alpha = m/n$ and *any* $\Sigma^+_u \subset \Sigma^+ \subset \Sigma \subset \text{crit } H_k$, with $H_k(\Sigma) = 0$) .

Some comments.

1. At some non-exceptional point $\lambda \in \Sigma$ and on a non-exceptional $\Gamma \in \Sigma$, $\lambda \in \Gamma$, we may find different coverings $\tilde{\Gamma}^a$, $\tilde{\Gamma}^b$ and $\tilde{\Gamma}^c$ such that there is no ETS in case a), there is ETS defined by a vector-field F in case b), and there is ETS defined by a vector-field F with opposite direction in case c). All are "realised" in different wedge bundles over Γ that can be attractive or repulsive. For example they can all be attractive. Then in case a) a path in Box $B^+(\delta,\Gamma)$ will go straight to a nearby point on Γ , in case b) it will escape from $B^+(\delta,\Gamma)$ in one end, and in case c) it will escape from $B^+(\delta,\Gamma)$ in the other end. The projected traces occur all together along Γ . They are observed by the celestial observer at $0 \in \mathbb{R}^3$ with mixed feelings.

2. Problem 3 (jump of trace): A path in a repulsive wedge bundle may at any moment leave this wedge bundle and "jump" to an attractive wedge bundle. It seems unlikely that a path will do so at approximately one λ-value an infinite number of times for $t \to \infty$ in one global path. This could have interesting consequences for Thom's conjecture. See conjecture CNOS in section 8.

7. Standard ETS.

A *path* is called *standard* if it has ETS at most and *only at special sides* of polygons $P(f,R_u)$. A *function* is called *standard*, if all its paths are standard. In this section we prove (not assuming that f is a square):

Theorem 3: *Ω is a point for every standard path.* CRT *holds for every standard function.*

For the proof we only need the technical

Lemma: Let $x(t)$ be a path for the analytic function f with $\omega(\infty) = \Omega \in \Sigma$, $H_k(\Sigma) = 0$, and suppose $x(t)$ has for no general side of any Newton polygon $P(f) = P(f,R_u)$ essential trace speed. Then Ω is one point.

Theorem 3 applies if f is "*nowhere very singular*". In § 7.1 we illustrate the theorem in an example *with no general* P(f) sides at all. In § 7.2 we prove the theorem for Σ irreducible, in § 7.3 for Σ reducible.

7.1 An example with reducible Σ .

We study the function

$$f = x^2y^2(z-x-y)^2 + r^{50}\,xy(x+y) + r^{54}\,x(x-y) + r^{56}\,(y-x) =$$

$$= \overline{H}_6 + \overline{H}_{53} + \overline{H}_{56} + \overline{H}_{57} = r^6 H_6 + r^{53} H_{53} + r^{56} H_{56} + r^{57} H_{57} .$$

We concentrate our attention on *one selected domain* D defined on S^2 by $x > 0$, $y > 0$, $z - x - y < 0$, with boundary the triangle Σ^+ . As coordinates inside D we use $\sigma = -(z-x-y)/r$ near the side $z-x-y = 0$, $\sigma = y/r$ near the side $y = 0$ and $\sigma = x/r$ near the side $x = 0$ of $\Sigma^+ \subset S^2$. The gradients of the function H_6 in D but near Σ^+ are suggested in figure 13 b). Those in points on Σ^+ of the functions H_{53} , H_{56} and H_{57} are suggested in figure 13 c). Very near each side of Σ^+ , we obtain a balance (as seen before) in wedge bundles one for each side of Σ^+ . We observe essential trace speeds ETS inside D along each side of Σ^+ ,

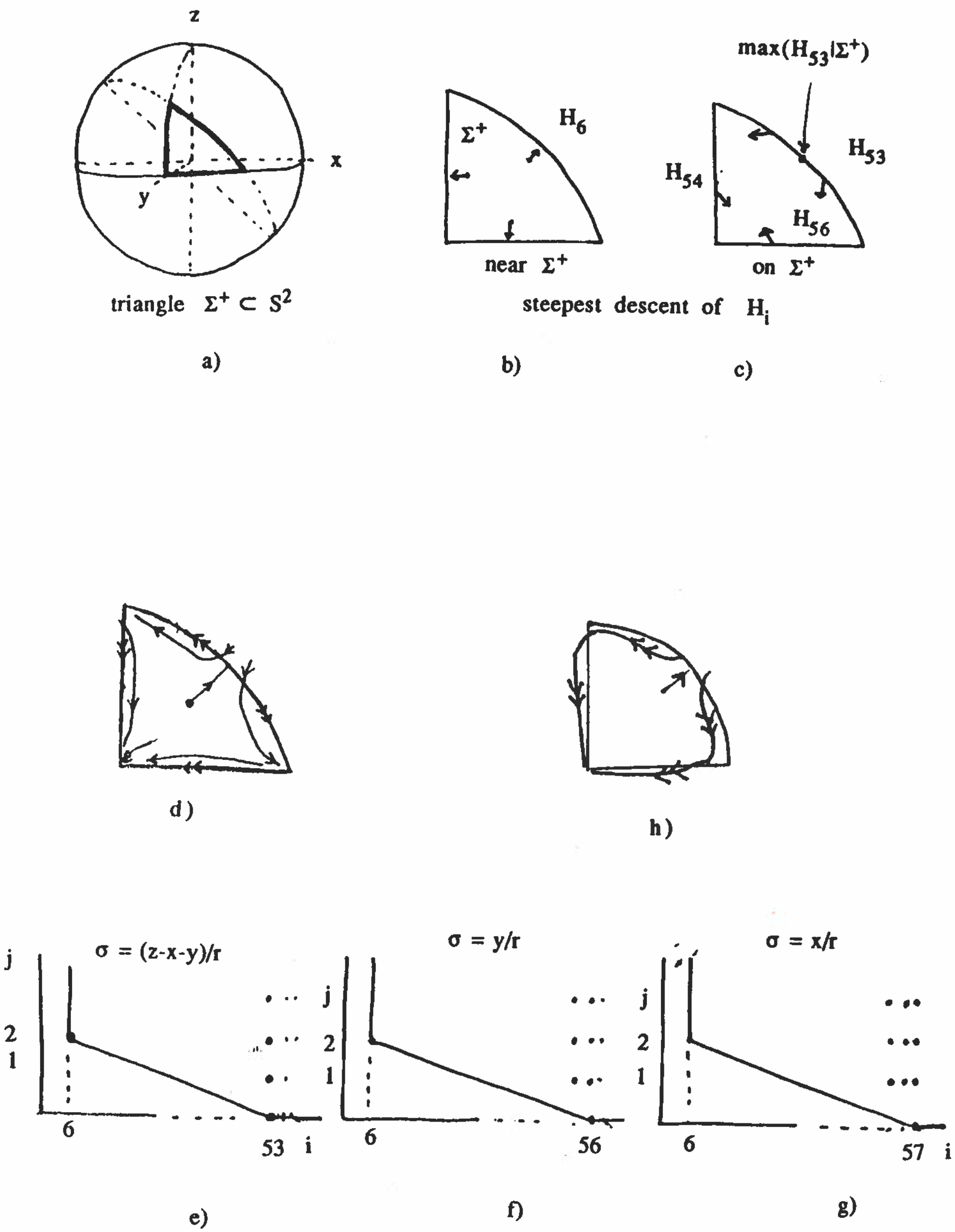

z
x
y
triangle $\Sigma^+ \subset S^2$
a)
Σ^+
H_6
near Σ^+
b)
$\max(H_{53}|\Sigma^+)$
H_{53}
H_{54}
H_{56}
on Σ^+
c)
steepest descent of H_i
d)
h)
$\sigma = (z-x-y)/r$
j
2
1
6
53
i
e)
$\sigma = y/r$
j
2
1
6
56
f)
$\sigma = x/r$
j
2
1
6
57
i
g)

figure 13

guided by (decrease of) different potential functions H_{53}, H_{56} and H_{57} on these sides, as seen in figure 13 d).

In figure 13 e), f) and g) we show the Poincaré polygons P(f) for each of the sides of Σ^+ . Each Newton polygon $P(f) = P(f,R_u)$ has only one declining side, which is a special side. There are no "general sides" at all. So our example is "nowhere very singular". Although the different potentials that guide ETS for each side of Σ^+ seem rather independent, they do not cooperate in this case to produce a spiraling effect. This is so in particular because $H_{53} = xy(x+y)$ vanishes at the ends of the corresponding side of Σ^+ . In § 7.3 we prove that such cooperation is never possible in case all ETS occur only on special P(f) sides. Note that in <u>this</u> example the wedge bundles cannot be made to fit at any corner of Σ^+ . No trace curve follows more than one side of D in this example.

<u>Variant</u>. This is different for the function $f = x^2y^2(z-x-y)^2 + r^{52}(x+y)$ as seen by taking $\sigma = -xy(z-x-y)$ as local coordinate on each of the edges. Then $H_{53}|\Sigma^+$ guides ETS along each edge of Σ^+ on one side of this edge. See figure 13 h). But for some edges ETS is on the "outside" of Σ^+ . We see curves that follow first one edge, $(z-x-y) = 0$, with ETS, and then another edge, $(x=0)$, with ETS.

7.2 <u>Proof for irreducible</u> Σ .

Let $y > k$ be the smallest integer i for which A_{io} in (5.4) is not identically zero. The point $(y,o) \in \mathbb{R}^2$ is the endpoint of the last declining side of P(f) , a *special* side with slope α_2 say. The algebraic function A_{yo} on Σ is a square, and either a non-negative function with a finite number of zeros or a positive constant $c > 0$. Let α_1 be the slope of the next (general) side of $P(f) : \alpha_1 > \alpha_2$.

We study various slopes $\alpha > 0$ and therefore various views on the fundamental differential equations in Boxes $B^+(\delta, \Gamma)$. For $\alpha > \alpha_2$ we have the views related to general sides of P(f) . Also for $\alpha = \alpha_2$ we have a situation like that with a general side. We include this case in our definition of *very singular* if the conditions (6.11) and (6.12) are satisfied. Then we need not discuss them.

All remaining cases of ETS are as follows:

Case 1. *The standard case* A_{yo} *not constant,* A_{y1} *not identically zero.* See the example of section 4 and figure 14 a) for Newton sets and polygons. The point $(i,j) = (y-2,0)$ is in $N(-\dot{\sigma}) \subset \mathbb{R}^2$ with powerseries value A_{y1}. It is also in $N(-\dot{\lambda})$ with powerseries value $\partial_\lambda A_{yo}$. Let α_3 be the slope of the common "tangent" of $P(-\dot{\sigma})$ and $P(-\dot{\lambda})$.

For $\alpha < \alpha_3$, there is no dominance of $-\dot{\lambda}$ over $-\dot{\sigma}$ because $|\dot{\lambda}/\dot{\sigma}| = |\partial_\lambda A_{yo}/A_{y1}|$ is bounded except near zeros of A_{y1}. So there is no essential trace speed. For $\alpha = \alpha_3$ of the common tangentline of $P(\dot{\sigma})$ and $P(\dot{\lambda})$ we can however have essential trace speed. As in section 6, § 1 and 2, we can define an algebraic curve $\tilde{\Gamma} \subset \Gamma \times \mathbb{R}^+$ over Σ with equation $F_\sigma = 0$. If F_σ is a polynomial of degree $q \geq 1$ in μ, then there can be q wedge bundles in $B^+(\delta,\Gamma)$, attractive, half-attractive or repulsive. For each of them the essential trace speed is guided however by the same pure potential $A_{yo} = H_y|\Sigma$. An *example* with $q = 2$ is given (without details) in figure 14 a). The relevant coefficients like A_{yo} (positive), A_{y1}, $A_{y-1,2}$ and $A_{y-2,3}$ must be suitable to get indeed wedge bundles and essential trace speeds on $\Sigma = \Sigma^+$.

Case 2. A_{yo} *not constant,* $A_{y1} \equiv 0$. Then $(y-2,0) \notin N(-\dot{\sigma})$. This is the nice case. Compare example I in section 5 and figure 14 b). Let α_3 be the slope of the common tangent of $P(-\dot{\sigma})$ and $P(-\dot{\lambda})$, see figures 8 and 14 b). If $\alpha < \alpha_3$, then $-\dot{\lambda}$ dominates over $-\dot{\sigma}$ and $-\dot{r}$. This means that $|\dot{\sigma}/\dot{\lambda}|$ and $|\dot{r}/\dot{\lambda}|$ are as small as we please, for σ and r small, in case $\partial_\lambda A_o \neq 0$. Then $-\dot{\lambda}$ dominates and we do have essential trace speed like in example I, but now guided by the potential A_{yo}.

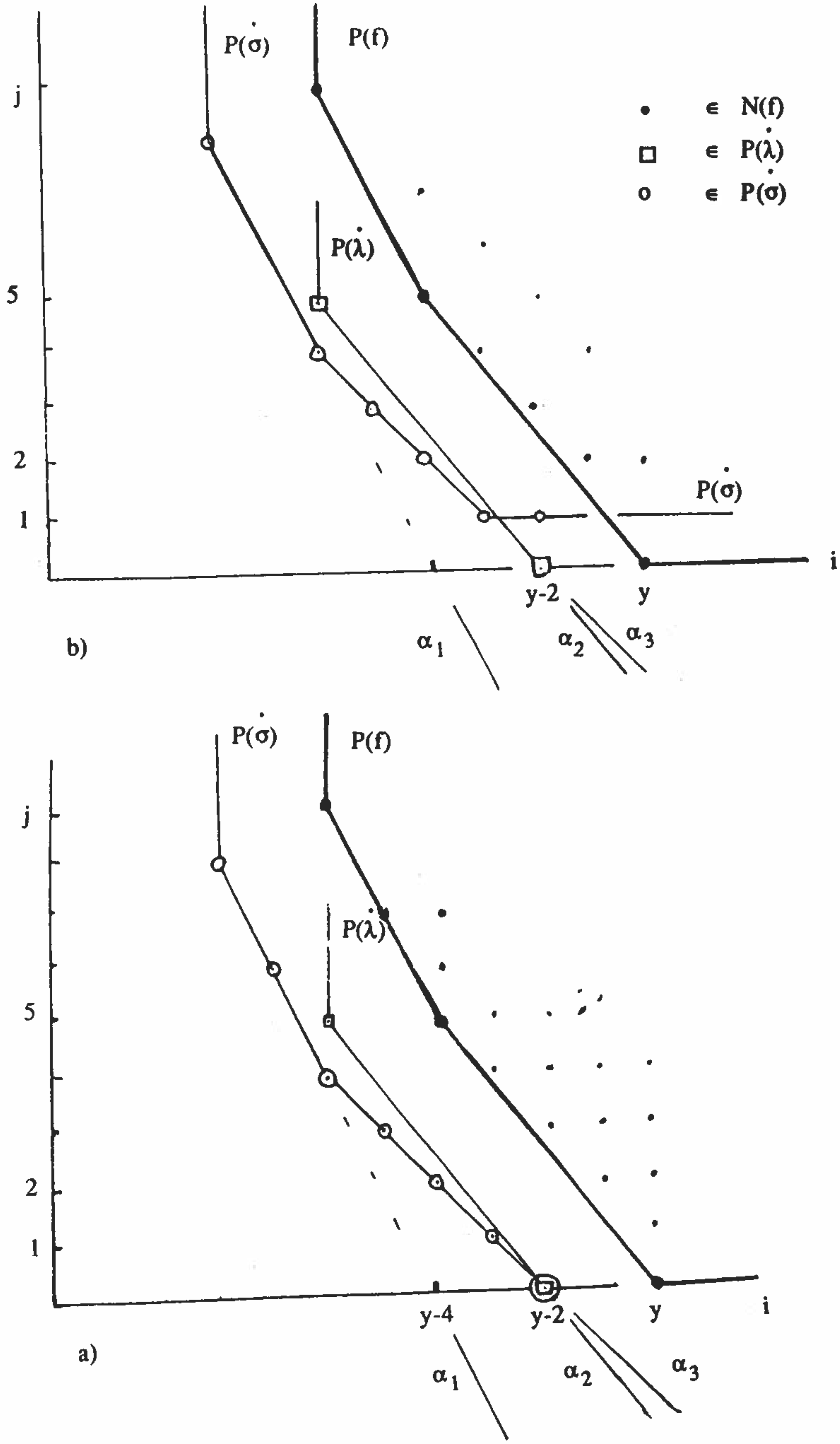


figure 14 ETS for special sides of P(f)

The common tangent with slope $\alpha = \alpha_3$ meets $P(-\dot{\lambda})$ in one point and $P(-\dot{\sigma})$ in either one point or a whole side. In the latter case we obtain for the view $\alpha = \alpha_3$ again possibly a real algebraic curve $\tilde{\Sigma} \to \Sigma$ and wedges bundles. Then ETS may arise, but here it only realises in certain wedge bundles, attractive, half-attractive or repulsive. In all cases ETS is guided by (decrease of) $A_{yo} = H_y|\Sigma$. In figure 14 b) Newton polygons for this case are seen.

Case 3. $A_{yo} > 0$ is *constant*. Suppose $A_{y+1,0}$ is not constant, then $(y\text{-}1,0) \in N(-\dot{r})$ with powerseries coefficient $A_{yo} = c > 0$ and $(y\text{-}1, 0) \in N(-\dot{\lambda})$ with powerseries coefficient $\partial_\lambda A_{y+1,0}$. Essential trace speed is impossible as $-\dot{\lambda}$ cannot dominate $-\dot{r}$, by their Newton sets.

We conclude, that in all possible cases $H_y|\Sigma$ guides ETS in case Σ is on an irreducible curve. If there is ETS then near to Σ it follows decrease of a unique potential, and the trace point converges to some point on Σ where this potential on Σ has a minimum or critical value. Then Ω is one point and CRT holds.

7.3 Proof for reducible Σ .

Let $\Sigma \subset S^2$, $H_k(\Sigma) = 0$, be a topological component of crit H_k for the analytic function f . And let $R_u = 0$, $1 \le u \le v_1$, be the irreducible algebraic curves in S^2 that cover the one-dimensional strata of Σ . Suppose $R_u = 0$, $1 \le u \le v_2 \le v_1$, are those components that do have ETS either on one side or on the other side of the curve $R_u = 0$. The ETS is guided along R_u by one potential $H_{y_u} | R_u$, and a path with ETS is in an attractive, repulsive or half-attractive wedge-bundle, or possibly not in a wedge bundle as in the case of the nice examples. The path can jump from a repulsive wedge to an attractive wedge, but if so then the guiding potential remains the same along R_u namely H_{y_u} . Now suppose a not exceptional point $\lambda \in \Sigma$ is contained in the interior of a limitset Ω of a trace. So Ω is not one point. Then some path for small r

> 0 and small $\sigma > 0$ moves in a box $B(\partial,\Gamma)$ and comes back later there again guided by the same H_{y_u} hence moving in the same direction along Σ . There is a corresponding closed immersed circle S in Σ . Let $\mathfrak{A} \subset \Sigma$ be the subarc of the path, which embeds near $S \subset \Sigma \subset S^2$. As S is an immersed circle, it must be oriented by the local ETS that is guided by potentials H_u . Suppose (after reordering) exactly the factors R_u , $1 \leq u \leq v_3 \leq v_2$, are involved in $\mathfrak{A}$.

We next consider the smallest value y of all y_u $1 \leq u \leq v_3$. Clearly, for components H_j , $j < y$, that do not vanish, we conclude that H_j has R_u as a factor for $1 \leq u \leq v_3$. For example of course H_k (the leading part of the development of f) has all these factors. By definition H_y has at least one among R_u , $1 \leq u \leq v_3$, not as a factor. Suppose $1 \leq v_4 \leq v_3$ and H_y has exactly no factors R_u for $1 \leq u \leq v_4 \leq v_3$. Then $H_y|R_u$ will guide ETS on $S \cap R_u$ for $u \leq v_4$ and it will vanish on $S \cap R_u$ for $v_4 < u \leq v_3$. The arcs $S \cap R_u$, $v_4 < u \leq v_3$ form maximal connected oriented large arcs on S where H_y guides ETS, but H_y will vanish on each end of *these* large arcs. Therefore H_y has a maximum or minimum on each of these arcs and no trace can follow that arc from its beginning to its end. (See the example in § 7.1). This contradicts the needed orientation in S . Then Ω is one point and every trace of a path in the catch set with $\Omega \subset \Sigma$ ends in one point. This completes the proof of theorem 3. Note that we had no need to specify the wedges bundles for special Newton polygon sides for our theorem 3.

8. The remaining rare paths and rare functions.

We now know that CRT is at most false for rare paths and rare functions.

8.1. Reduction of CRT to an "evident" conjecture (CNOS) in case Σ is irreducible.

Let $x(t)$ be a path with $x(\infty) = 0 \in \mathbb{R}^3$ for a function $f = g^2 \geq 0$. Suppose $\Omega = \lim_{t=\infty} \omega(t) \subset \Sigma$ *is not one point*. Then by theorem 2, $M_k(\Omega) = 0$, and by theorem 3, $x(t)$ is a rare path and must have ETS for some value of u on some general side of $N(f) = N(f,R_u)$.

Let $\sigma = R_u$ be chosen positive near the polygon $\Sigma^+ \subset \Sigma$ on the inside of Σ^+ as defined and seen in figure 11 section 6.1 p. 43. Let this choice of σ give ETS and contribute to Ω. Then $\sigma(t) > 0$ remains positive in the view determined by the general side of $N(f)$ with slope say m/n. Suppose also Σ is *irreducible* and denote $R_u = R$. The leading terms with this view are given in (5.8) - (5.13) for any compact arc Γ_1 in an open one-stratum of the subgraph $\Sigma^+ \subset \Sigma$ and t sufficiently large. These leading terms are:

$$
\begin{aligned}
f &= r^2\sigma \quad \Sigma_{\ell=0}^{s} A_\ell \quad r^{N+\ell n-2}\,\sigma^{M-\ell m-1} \\
-\dot{r} &= r\sigma \quad \Sigma(N+\ell n)\,A_\ell \; r^{N+\ell n-2}\,\sigma^{M-\ell m-1} \\
-\dot{\sigma} &= \quad \Sigma(M-\ell m)\,A_\ell\; r^{N+\ell n-2}\,\sigma^{M-\ell m-1} \qquad (8.1)\\
-\dot{\lambda} &= \sigma \quad \Sigma\ \partial_\lambda A_\ell \quad r^{N+\ell n-2}\,\sigma^{M-\ell m-1}.
\end{aligned}
$$

Note that N, M, and S are even as $f = g^2$; also $A_0 \geq 0$ and $A_s \geq 0$ are squares, and not identically zero by the definition of the Newton polygon $N(f)$ squares, and not identically zero F or large t we have an asymptotic approximation the first necessary condition for ETS (see section 6.4) : $-\dot{\sigma} = 0$. We write this in the forms

(8.2) $0 = \Sigma(M-\ell m)\ A_\ell\ \mu^{s-\ell} = \Sigma(M-\ell m)\ A_\ell\ \nu^{(s-\ell)m};\ \Sigma(M-\ell m)\ A_\ell \nu^{M-\ell m} = 0$

$$\text{for } \mu = \sigma^m/r^n = \nu^m\ ,\ \nu = \sigma/r^{n/m}\ .$$

The solutions form the *cusp curve* $\tilde{\Sigma}$ of points $\tilde{\lambda} = (\lambda,\mu) \in \tilde{\Sigma}$, as illustrated in figure 12 p. 43. The path $x(t)$ moves for large t during ETS inside wedge bundles, and $\tilde{\lambda}(t) = (\lambda(t),\mu(t))$ is then near $\tilde{\Sigma}^+$, above the arc $\Gamma_1 \subset \Sigma$, hence near an open stratum $\tilde{\Gamma}_1$ of $\tilde{\Sigma}^+ \subset \tilde{\Sigma}$. It follows that the limit set Ω has a lift $\tilde{\Omega}$ which contains, for ETS, parts of curves immersed in $\tilde{\Sigma}^+$. They must determine a unique orientation by the vector $-\dot{\lambda}$, on each one-stratum segment of $\tilde{\Omega}$. It may (might) happen that $\tilde{\Omega}$ has a point $\tilde{\lambda}_0$ not on $\tilde{\Sigma}^+$. This can occur, in principle, if a path moves for a while inside a repulsive wedge bundle and then leaves it and jumps to an attractive wedge bundle, passing near to $\tilde{\lambda}_0$. As $\tilde{\lambda}_0 \in \tilde{\Omega}$ this *is repeated* for $t \to \infty$ *an infinite number of times* and $\tilde{\lambda}_0$ is contained in a limit curve in $\tilde{\Omega}$ connecting a point on a repulsive segment of $\tilde{\Sigma}^+$ to a point on an attractive segment of $\tilde{\Sigma}^+$. This curve is vertical as seen in (6.1) and figure 12 b).

Proof of a special case. We now *first prove* CRT *for any path in a case where the above complication does not occur*, namely example II and similar cases. More precisely, let $\Sigma = \Sigma^+$ be a smooth circle and $\tilde{\Sigma} \to \Sigma$ a 2q-fold covering of alternating attractive and repulsive circles whose union is $\tilde{\Sigma}$. If the path is in a repulsive wedge bundle and leaves it, then it goes to an attractive wedge bundle which it will never leave again. So then it will not return to the repulsive wedge bundle. So $\tilde{\Omega}$ must be one of the components say $\tilde{\Omega} = \tilde{\Sigma}_1$ of $\tilde{\Sigma}$. From the equations (8.1) (8.2) we deduce for large t concerning the asymptotic approximation of a path $x(f)$ in $\tilde{\Sigma}_1$:

$$F_f^{\dagger} \overset{\text{def}}{=} \Sigma_{\ell=0}^{s} A_\ell\ \nu^{M-\ell m} \geq 0 \ \ (\text{as } f \geq 0)\ . \tag{8.3}$$

Observe that $A_\ell = A_\ell(\lambda)$, F_f is periodic in the arc length λ , and $\nu = \nu(\lambda)$ is by (8.2) a root of

(8.4) $$F_\sigma^\dagger \stackrel{\text{def}}{=} \Sigma\, (M-\ell m)\, A_\ell\, \nu^{M-\ell m-1} = 0\,.$$

Differentiation of F_f with respect to λ gives for every one stratum on $\tilde{\Sigma}_1$:

$$\partial_\lambda F_f^\dagger = \Sigma_{\ell=o}^{s}\, (\partial_\lambda A_\ell)\, \nu^{M-\ell m} + (\partial_\lambda \nu)\, \Sigma_{\ell=o}^{s}\, (M-\ell m)\, A_\ell \nu^{M-\ell m-1} \stackrel{\text{def}}{=} F^\dagger + 0\,.$$

The second part is 0 by (8.4).

Here $F^\dagger$ is proportional to $-\dot{\lambda}$, which must be of constant sign (> 0 or < 0) for all open strata of the graph on $\tilde{\Sigma}_1 = \tilde{\Omega}$ by ETS, but could be zero at vertices. Hence

$$\partial_\lambda F_f \geq 0 \quad , \text{ not } \equiv 0\,.$$

By integration F_f cannot be periodic in λ , a contradiction. Then Ω is a point and CRT holds for this path. (Acknowledgement: for this argument I got essential help from Jean Bourgain).

We now *return to the general situation.* Suppose $\tilde{\Omega}$ contains a repulsive point $\tilde{\lambda}_o \subset \tilde{\Sigma}$ with ETS. A path point $(\lambda(t_o), \mu(t_o))$ near to $\tilde{\lambda}_o$ must come from path points $(\lambda(t), \mu(t)) \subset \Gamma \times \mathbb{R}^+$ for some $t_1 < t < t_o$ inside the repulsive wedge, as seen by inverting time. $\tilde{\Omega}$ then contains a $\tilde{\lambda}$-interval $\tilde{\Gamma} \subset \tilde{\Sigma}$, covering $\Gamma \subset \Omega \subset \Sigma$, with ends in $\lambda(t_1)$ and $\lambda(t_o)$. The path segment between t_1 and t_o is near to $\tilde{\Gamma}$ and it must return there, but after some time nearer to $\tilde{\Gamma}$, *and* again and again, an infinite number of times for $t \to \infty$. The wedge is then called superrepulsive for the path $x(t)$ and the path is called *superselected* at $\tilde{\Gamma}$ (called so because it should survive the infinitely repeated repulsive sharp wedge influences). This seems to be impossible (no proof !). So we now propose the

Conjecture (CNOS). *No path can be superselected for any arc* $\tilde{\Gamma}$ *covering an arc* $\Gamma \subset \Omega$ *in an open one-stratum of* Σ .

Theorem 4. If the conjecture CNOS is true then CRT *holds for irreducible* Σ .

Proof. By CNOS for any general side of N(f) we find that Ω contains no repulsive parts of $\tilde{\Sigma}$ and therefore $\tilde{\Omega}$ lies in $\tilde{\Sigma}^{att}$ the union of the closures of the attractive one-strata of $\tilde{\Sigma}$. In particular for some t_o and $t \geq t_o$ the path x(t) will not go down from the view on one general side with slope m/n (necessarily through a repulsive wedge bundle !) to a view for a side of N(f) with the next smaller slope $m_1/n_1 < m/n$. So $\Omega \subset \tilde{\Sigma}^{att} \subset \tilde{\Sigma}$. Take any path x(t) which follows for $t \geq t_o$ large, one round of a circle immersed in $\tilde{\Omega} \subset \tilde{\Sigma}^+$. We only discuss the open one-strata of $\tilde{\Sigma}^+$ and neglect in the argument what might happen at vertices of the graph $\tilde{\Sigma}^+$. Then the arguments and calculation above for the case of example II apply and can be repeated, and this leads to the same contradiction. Then CRT holds for every path.

8.2 Discussion of the remaining rare paths, for reducible Σ.

As in section 8.1 we assume the conjecture CNOS. Let x(t) be a path with $\Omega = \lim_{t=\infty} \omega(t)$ not one point. By theorem 4 Ω must contain at least two consecutive parts on two different irreducible curves $R_u = 0$ and $R_v = 0$ of Σ. Suppose R_u and R_v meet at a vertex of $\Omega \subset \Sigma^+$, and ETS is due in both irreducible components to general sides of Newton polygons $N(f,R_u)$ and $N(f,R_v)$. The path x(t) must eventually, and repeating so an infinite number of times, appear and disappear inside some piece of wedge bundle at the vertex.

This should imply that the Newton polygons have parallel general sides, and that the cusp-curves for R_u and R_v connect for well chosen coordinates σ_u and σ_v. Then one is led to believe that there must be a common coordinate like

$$\sigma = \sigma_u = \sigma_v = R_u^{p_u} R_v^{p_v}$$

such that leading terms for f would be of the kind

$$f = \Sigma_{\ell=0}^{s} A_\ell \, r^{N+\ell n} \sigma^{M-\ell m}$$

$$= \Sigma_{\ell=o}^{s} A_{\ell}\, r^{N+\ell n}(R_u^{p_u} R_v^{p_v})^{M-\ell m} .$$

All necessary equations for ETS must be fulfilled along $R_u = 0$ *and* along $R_v = 0$. That seems too much for existence of ETS along each of the two sides. And this existence would be needed before a counterexample to Thom's conjecture has a chance to exist. I am confident that CRT also holds for rare paths, but I did not succeed in elaborating the above comments and others and obtain a proof for all cases.

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