Journal of Mathematical Sciences, Vol. 195, No. 2, November, 2013

CRITICAL CONFIGURATIONS OF PLANAR MULTIPLE PENDULI

G. Khimshiashvili and D. Siersma UDC 512.152

Abstract. We consider the oriented area function A on the moduli space $M(P)$ of mechanical linkage P representing a planar multiple pendulum. For generic lengths of the sides of P, it is proved that A is a Morse function on $M(P)$ and its critical points are given by the cyclic configurations of P satisfying an additional geometric condition. For triple penduli, the main result is complemented by a rather comprehensive analysis of the structure of critical configurations. Moreover, we discuss the critical configurations of another natural function on the moduli space of a planar multiple pendulum and the image of the cross-ratio mapping into the plane. A number of related results and open problems are also presented.

CONTENTS

1. Cyclic Configurations of Planar Penduli 199
2. Cyclic Configurations of Triple Penduli 202
3. Spread of Configuration as a Morse Function 206
4. Cross-Ratio Map and Darboux Transformation of a Planar Triple Pendulum 207
5. Concluding Remarks 209
References 210

Introduction

We deal with certain *planar mechanical linkages* [3] which we call *planar multiple penduli*. Recall that a planar mechanical linkage is defined as a mechanism built up from rigid bars (sides) consecutively joined at flexible links (vertices) in such a way that each bar may freely rotate around any of its endpoints (pin-joints) and the whole system is confined to stay in a certain plane (reference plane) [3]. A linkage can move and make various configurations so that the lengths of segments representing bars should remain unchanged but may intersect and vertices may coincide. If it is required that the last vertex coincides with the first one, then we speak of a *polygonal linkage* [3]. If there is no such requirement, in order to distinguish it from the preceding case, one usually speaks of a *planar kinematic chain* [3], but we prefer to think of it as a *planar multiple pendulum*, which seems more intuitive and adequate to the setting accepted in this paper.

Such linkages are also considered as useful models for a robot arm or mechanical manipulator and therefore often called *mechanical n-arms* [14]. However, we will exclusively use the term *planar multiple pendulum* (PMP) to avoid possible misleading interpretations and associations. Actually, we will basically work with the moduli space of a PMP, which appears as a particular case of the general concept of moduli space of mechanical linkage [3].

Moduli spaces of mechanical linkages of various types were actively studied in the last few decades (see, e.g., [12, 18, 31]). In particular, critical points of various functions on moduli spaces have been discussed in [12, 18, 25]. Along these lines, we consider two natural functions on the moduli space of PMP, identify their critical points, and obtain a few results in the spirit of the Morse theory.

First, we discuss the oriented (signed) area considered as a function A on moduli space of a PMP and show that its critical points are given by the cyclic configurations satisfying some additional conditions

Translated from Sovremennaya Matematika i Ee Prilozheniya (Contemporary Mathematics and Its Applications), Vol. 81, Complex Analysis and Topology, 2012.

 1072–3374/13/1952–0198

of orthogonality. As usual, by a cyclic configuration we understand a configuration of linkage such that all the vertices lie on the same circle.

The study of cyclic polygons has a long history starting with elementary classical results such as the Ptolemy theorem and the Brahmagupta formula (see, e.g., [5]). This topic continues to attract considerable interest (see, e.g., [11, 32]), in particular, due to the results and conjectures of D. Robbins concerned with computing the area of cyclic polygon [30]. One of the main aims of this paper is to show that cyclic configurations also arise in the study of planar multiple penduli in a quite natural way.

In Sec. 2 we obtain a number of results on the Morse theory of A following the general paradigm suggested in [23]. In particular, we prove that, for a generic P3P, A is a Morse function (Theorem 2.1), give a simple formula for its Morse index (Theorem 2.2), and show that its critical values can be effectively calculated using results of [30] (Theorem 2.3).

In Sec. 3, we discuss the critical configurations and Morse theory of another natural function on the moduli space of PnP defined as the distance between the first and the last vertices of a configuration (the so-called *spread* of configuration). This function and its square were considered in several papers as a tool for investigating the topology of moduli spaces of *polygonal linkages* (closed polygonal chains) [9, 19]. In particular, the Morse theory of spread was developed in [9, 19]. We present some of those results in Theorem 3.1 and compare them with the parallel results for the oriented area. An essential difference revealed by this comparison is that spread is generically an exact Morse function while the oriented area does not have this property for $n \geq 4$.

In the same section we present two remarks on the *tip-map* (our shorthand for the *end-point mapping* [14]) from the moduli space of PMP to the plane. Namely, for a generic PMP, a tip-map is a stable mapping into the plane, and we describe its image for a generic triple pendulum (Proposition 3.3). These observations may be of interest in comparison with some results in the next section.

In Sec. 4, we introduce the *cross-ratio map* of a generic triple pendulum PMP and establish some of its properties (Theorem 4.1). In the same section we define an analog of the Darboux transformation for a generic P3P and use some recent results from [7] to obtain interesting information on its periodic points (Theorem 4.2).

The critical configurations of PMP play an essential role in all these developments and may be considered as the main object of study in this paper. We tried to make the exposition (reasonably) self-contained and readable. To this end, in the first section we present various auxiliary results and comments about the cyclic configurations of multiple penduli as well as our main result (Theorem 1.1), which shows that certain cyclic configurations are the critical points of the oriented area function. This result suggests a number of natural questions which are discussed in the framework of a general paradigm from [23]. Detailed results are only presented for *triple* penduli despite the fact that most of the statements and arguments admit quite straightforward generalizations for generic PnP's with arbitrary n. However, proving all results in full generality would require much more space. So only the key result (Theorem 1.1) is formulated and proved for multiple n-penduli with arbitrary n, while all other considerations are performed for triple penduli.

It should be noted that many of the considerations and results of the first two sections have been presented in the 2009 preprint of the authors [27]. The discussion in Secs. 3 and 4 is essentially new. Most of the results established for triple penduli seem to have natural generalizations for planar penduli with arbitrary number of links and may also serve as a paradigm for future research in this direction. With this in mind, in the last section we briefly discuss open problems and research perspectives.

1. Cyclic Configurations of Planar Penduli

We begin by giving a rigorous definition of moduli space of a *planar multiple pendulum* (PMP). Let $l = (l_1, \ldots, l_n) \in \mathbb{R}^n$ be a collection of positive real numbers. A planar n-pendulum (PnP) $P_n(l)$ is defined as a linkage consisting of n-sides and $n-1$ pin-joints. For $i = 1, \ldots, n$, the ith side $v_{i-1}v_i$ has length l_i and is pin-joint with the $(i+1)$th side at their common endpoint v_i, which is the $(i+1)$th vertex v_i of the linkage $P_n(l)$. The last side has length l_n and can freely rotate about the vertex v_{n-1}. Notice that the first vertex is denoted by v_0, and l is called the *sidelength vector* of P.

The definition of the moduli space of a planar multiple pendulum $P = P(l)$ runs as follows. One first introduces the set $C_2(P)$ defined as the collection of all n-tuples of points v_i in the Euclidean plane $\mathbb{R}^2$ such that the distance between v_{i-1} and v_i is equal to l_i, where $i = 1, \ldots, n$. Each such collection V of points, as well as the chain of line segments joining the consecutive vertices, is called a configuration of P. We assume that the corresponding (piecewise linear) curve is oriented by the given ordering of vertices. A configuration is called *cyclic* if all vertices lie on a certain circle and *aligned* if all vertices lie on the same straight line. Obviously, the latter type of configuration is a sort of limiting case of the former.

Factoring the configuration space $C_2(P)$ by the natural diagonal action of the group of orientation preserving isometries $Iso_+(2)$ of the plane $\mathbb{R}^2$, one obtains the *moduli space* $M(P)$ of a given PMP [18]. Moduli spaces are endowed with the natural topologies induced by Euclidean metric. It is easy to see that the moduli space $M(P)$ can be naturally identified with the subset of configurations such that $v_0 = (0,0), v_1 = (l_1, 0)$. Configurations satisfying these conditions will be called *normalized*. Thus $M(P)$ is homeomorphic to the set of normalized configurations and can be considered as embedded in $\mathbb{R}^{2n-2}$. This embedding endows it with a differentiable space structure so that one can speak of smooth mappings and diffeomorphisms in this context [2].

After these preparations, it is obvious that the moduli space of any planar n-pendulum is diffeomorphic to $(n-1)$-torus T^{n-1}. In the sequel we will encounter certain subsets of the moduli space. In order to describe these subsets, we first make some comments about the cyclic configurations of points in the Euclidean plane.

Lemma 1.1. *If four points $v_i = (x_i, y_i)$ lie on the same circle, then the following determinant vanishes*:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ x_1^2 + y_1^2 & x_2^2 + y_2^2 & x_3^2 + y_3^2 & x_4^2 + y_4^2 \end{vmatrix} = 0. \tag{1.1}$$

Notice that condition (1.1) is necessary but not sufficient for four points to form a cyclic configuration because this determinant also vanishes in the case where they lie on a straight line. For this reason we will say that a configuration of four points $v_i \in \mathbb{R}^2$ is *quasicyclic* if the above determinant vanishes. Such configurations are relevant for our considerations, and so it seems natural to consider the set of all quasicyclic configurations $QC(P)$ of a given PMP P. In the sequel, we will describe its geometric structure for planar triple penduli. Let us add that many of the results in the sequel are formulated for *generic* PMP or PMP having *generic sidelength vector* l. As usual, this means that the corresponding statements are true for all vectors l in a certain (unspecified) open dense subset of the parameter space $\mathbb{R}^n_+$ (cf. [13, 17]).

Since the moduli space of a PMP is a smooth manifold (the $(n-1)$-dimensional torus) it is natural to consider various geometrically meaningful functions on the moduli space and study critical points of those functions. In particular, taking into account the aforementioned embedding of $M(P)$ into $\mathbb{R}^{2n-2}$, we can consider restrictions to $M(P)$ of polynomial functions on $\mathbb{R}^{2n-2}$. If a function $f : M(P) \to \mathbb{R}$ arises as a restriction to $M(P)$ of a certain smooth function F on $\mathbb{R}^{2n-4}$, then the critical points of f can be found by the Lagrange method as the points $V \in M(P)$ such that $grad\, F$ is orthogonal to the tangent space $T_V(M(P))$ (see [1]). The main aim of this paper is to develop critical point theory for *oriented area* (see [5]) considered as a function on $M(P)$. Alternatively, since the torus is smooth, we can use parametrization by angular coordinates.

To this end, recall that for any configuration V of P with vertices $v_i = (x_i, y_i)$, $i = 0, \ldots, n$, its oriented (signed) area $A(V)$ is defined by

$$2A(V) = (x_0 y_1 - x_1 y_0) + \ldots + (x_n y_0 - x_0 y_n). \tag{1.2}$$

In other words, we add the “connecting” side v_nv_0 turning a given configuration V in a $(n+1)$-gon $\overline{V}$ and compute the oriented area of the latter. Obviously, formula (1.2) defines a smooth function on $\mathbb{R}^{2n-2}$ and also on the moduli space $M(P)$ of any PMP P. Thus we can consider its critical points. Since the Lusternik–Schnirelmann category of an n-torus is equal to $n+1$ [17], from the Lusternik–Schnirelmann theory it follows that A certainly has critical points different from maxima and minima. Thus one may wish to find their amount and describe the behavior of A near its critical points using standard paradigms of singularity theory [1].

Our main result (Theorem 1.1) states that critical points of A in $M(P)$ are given by certain cyclic configurations of P. For triple penduli, we will also show that, in fact, A is a Morse function on generic moduli space, and so one may deal with it in the framework of Morse theory [17]. This suggest a number of natural problems for PnPs with arbitrary n, some of which are investigated in Section 3 for triple penduli. Analogous settings and results have earlier been discussed for polygonal linkages [22], [25].

Before presenting this result, we add a few remarks about the signed area which will appear useful in the sequel. With a given n-pendulum P we associate a $2n$-gon linkage Φ with the sidelength vector $(l_1, \cdots, l_n, l_1, \cdots, l_n)$. Then each configuration V of P defines a “doubled” configuration W of Φ by adding the result of reflection of V in the midpoint of the connecting side v_0v_n. By the additivity of the area function, we get $A(W) = 2A(V)$. By cutting W along other diagonals and connecting opposite points, we get several PMP's with the same area as V. The components of their sidelength vectors are any cyclic permutation of $(l_1, \cdots, l_n)$. The moduli space is again the $(n-1)$-torus and the area function is the same. So it makes no difference which of those PMP's we study, e.g., we can (and often will) suppose that the first arm has the biggest length, etc. Moreover the “opposite” pendulum with sidelength vector $(l_n, \cdots, l_1)$ from v_n to v_0 and with vertices in the opposite order has area function $-A(V)$.

To formulate the main result, we need one more definition. Let us say that a configuration V of a planar n-pendulum P is *diacyclic* if it is cyclic and the “connecting side” v_nv_0 is a diameter of the circumscribed circle (“diacyclic” is a sort of shorthand for “diametrally cyclic”). In other words, the “connecting” side v_nv_0 passes through the center of the circumscribed circle or, equivalently, each interval v_0v_k is orthogonal to the interval v_kv_n (whenever the last phrase is meaningful).

Theorem 1.1. *For any sidelength vector $l \in \mathbb{R}^n_+$, the critical points of A on $M_2(P(l))$ are given by the diacyclic configurations of $P(l)$.*

Proof of Theorem 1.1. As above, we assume that $v_0 = (0, 0)$, $v_1 = (l_1, 0)$. For a configuration $V = (v_0, \ldots, v_n)$ we set $e_i = v_i - v_{i-1}$, $i = 1, \ldots, n$. Obviously, $v_i = e_1 + \ldots + e_i$ and $e_i = l_i(\cos\beta_i, \sin\beta_i)$. Denote by $a \times b$ the signed area of the parallelogram spanned by vectors a and b (i.e., we take the third coordinate of their vector product). The differentiation of vectors e_i with respect to angular coordinates β_j will be denoted by upper dots (i.e., there will appear terms of the form $\dot{e_i}$).

With these assumptions and notation, we can write

$$A = \sum_{j=1}^{n} v_{j-1} \times v_j = \sum_{j=2}^{n} (e_1 + \cdots + e_{j-1}) \times e_j = \sum_{1 \le i < j \le n} e_i \times e_j.$$

Taking partial derivatives with respect to β_k, $k = 2, \ldots, n$, we get

$$\frac{\partial A}{\partial \beta_k} = \sum_{i=1}^{k-1} e_i \times \dot{e_k} + \sum_{i=k+1}^{n} e_k \times \dot{e_i}.$$

Note now the identities

$$\dot{e_i} \times e_j = e_i \cdot e_j = -e_i \times \dot{e_j}.$$

Eventually we get

$$\frac{\partial A}{\partial \beta_k} = -\sum_{i=1}^{k-1} e_k \cdot e_i + \sum_{i=k+1}^{n} e_k \cdot e_i = \left(-\sum_{i=1}^{k-1} e_i + \sum_{i=k+1}^{n} e_i \right) \cdot e_k.$$

Consider now the equations $\partial A/\partial \beta_k = 0$, $k = 2, \ldots, n$, defining the critical set of A. By taking appropriate linear combinations of equations, this system of $n-1$ equations is easily seen to be equivalent to the system of equations

$$\left(\sum_{i=1}^{k-1} e_i \right) \cdot \left(\sum_{i=k}^{n} e_j \right) = 0, \quad k = 2, \ldots, n.$$

In geometric terms, this means that the intervals $v_0 v_{k-1}$ and $v_{k-1} v_n$ are orthogonal for $k = 2, \ldots, n$. It remains to refer to the Thales theorem (see [5]) to conclude that the points $v_0, \ldots, v_n$ lie on a circle with diameter $v_0 v_n$. The proof is complete. □

This theorem reveals the role of (dia)cyclic configurations and serves as a starting point for our further considerations. Notice the analogy with the case of polygonal linkages considered in [8, 22, 25].

Remark 1.1. It is worth noting that diacyclic configurations of PnP can be identified with the set of all real solutions to a system of quadratic equations in 2(n-1) variables. Indeed, taking the coordinates of $n-1$ "movable" vertices in standardized configuration of PnP as variables, conditions on the squared distances as well as orthogonality conditions are quadratic. In this way one obtains a system of $2(n-1)$ quadratic equations depending on $n-1$ parameters (sidelengths l_i). Notice that each parameter enters in exactly one equation as a "free term," so the whole system appears to be a "free term" deformation [1] of a fixed quadratic system with constant coefficients. As is well known, for a polynomial system with parameters, one can investigate the structure of its solutions and their dependence on parameters using a standard method of singularity theory based on consideration of the so-called *bifurcation diagram* [1]. When the number of links n is small, this can be done effectively. Having identified the bifurcation diagram and components of its complement, one can use the Ehresmann theorem [13] to solve various problems concerned with the number and geometry of diacyclic configurations. Concrete examples of application of this strategy will be given in further publications.

2. Cyclic Configurations of Triple Penduli

In this section, we put $n = 3$ and deal exclusively with planar 3-penduli (P3P). We will complement Theorem 1.1 by obtaining a rather complete description of critical points and critical values of the signed area function A on the moduli space of a P3P. It is again convenient to use the term "generic sidelength vector" in the aforementioned sense. Namely, we say that a statement holds for generic sidelength vector l if it holds for each collection l from an open dense subset of the parameter space $\mathbb{R}^3_+$. In the sequel we freely use a number of standard concepts of Morse theory and singularity theory which can be found in [1] and [17].

For convenience, in the P3P case we modify the notation a bit. Vertices v_0, v_1, v_2, v_3 will be denoted by O, A, B, C, respectively. The sidelength vector is written as $(l_1, l_2, l_3) = (a, b, c)$, while side vectors are denoted by $\vec{e_1} = \vec{a}$, $\vec{e_2} = \vec{b}$, and $\vec{e_3} = \vec{c}$. Thus we have $\vec{a} = (a, 0)$, $\vec{b} = b(\cos\beta, \sin\beta)$, $\vec{c} = c(\cos\gamma, \sin\gamma)$. A certain number of our computations were done (and can be verified) using the Mathematica package, and for this reason in a few places below we (re)denote $\beta = x$, $\gamma = y$.

Now the signed area function (modulo a constant factor) can be written in the form

$$A = \vec{a} \times (\vec{a} + \vec{b}) + (\vec{a} + \vec{b}) \times (\vec{a} + \vec{b} + \vec{c})$$
$$= \vec{a} \times \vec{b} + (\vec{a} + \vec{b}) \times \vec{c} = ab\sin[x] + ac\sin[y] + bc\sin[y - x]. \quad (2.3)$$

Below we make an essential use of a well-known paradigm of singularity theory, which is referred to as the "parametric transversality paradigm" (PTP). There are several different formulations of parametric transversality paradigm (see, e.g., [13]). We cannot dwell upon the PTP here and just mention that for our purposes it is sufficient to use a rather elementary version of PTP described in [1]. Now we are ready to formulate and prove the three results we desire.

Theorem 2.1. *For a generic sidelength vector $l \in \mathbb{R}^3_+$, the function A has only nondegenerate critical points on $M_2(P(l))$.*

Proof of Theorem 2.1. The proof is achieved by merely applying the parametric transversality paradigm to the gradient mapping ∇A of function A. In our notation ∇A depends on parameters b and c and we want to show that for almost all (b, c) the mapping $\nabla A_{(b,c)}$ is a submersion over the origin. To this end we consider the Jacobi matrix of ∇A with respect to all of its variables (x, y, b, c), which obviously has the form

$$\begin{pmatrix} A_{xx} & A_{xy} & A_{xb} & A_{xc} \\ A_{yx} & A_{yy} & A_{yb} & A_{yc} \end{pmatrix}. \quad (2.4)$$

We call it the *extended Hessian matrix* (EHM) of A. In order to apply the parametric transversality paradigm to the gradient map ∇A, we need to find out at which points (b, c) the rank of the extended hessian matrix is equal to 2. Consider first the minor obtained by deleting the first two columns:

$$\begin{pmatrix} A_{xx} & A_{xy} & c\cos[y - x] - a\cos[x] & -c\cos[y - x] \\ A_{yx} & A_{yy} & b\cos[y - x] & a\cos[y] - b\cos[y - x] \end{pmatrix}. \quad (2.5)$$

It is easy to see that, at the critical point, the EHM takes the form

$$\begin{pmatrix} A_{xx} & A_{xy} & 0 & -c\cos[y - x] \\ A_{yx} & A_{yy} & b\cos[y - x] & 0 \end{pmatrix}. \quad (2.6)$$

So the rank is maximal if $\cos[y - x] \neq 0$. In the case of equality, it follows that

$$\cos[y - x] = \cos[x] = \cos[y] = 0,$$

which gives a contradiction. Thus we can apply PTP to conclude that for generic (a, b, c) the area function is Morse, as was claimed. □

Remark 2.1. It is obvious that this reasoning is of general nature and virtually enables one to prove similar result for PnP with arbitrary n. However, we were not able to overcome the technical difficulties arising in the general case. Another application of PTP in our context is given in Theorem 2.4 below. In connection with Theorem 2.4, we notice that in our notation the Hessian (determinant of the hessian matrix) of A has the form

$$H(x, y) = a\sin[x]\sin[y] + b\sin[x]\sin[y - x] + c\sin[y - x]\sin[y].$$

Meanwhile, we also notice that two other approaches to proving the Morse property were suggested in [24, 29].

Since we have established that the oriented area is generically a Morse function, there arises a natural problem of finding its Morse indices. For a triple pendulum this can be easily done by "case-by-case" analysis using the above formulas for the Hessian. For simplicity of formulation, suppose that $l_1 > l_2 + l_3$. For each diacyclic configuration the circumscribed circle has a natural (counterclockwise) orientation.

Each link $v_i v_{i+1}$ of this configuration will be called *positive* if its direction coincides with the chosen orientation, and *negative* in the opposite case. Let $p(V)$ be the number of positive links of diacyclic configuration V, and let $n(V) = 3 - p(V)$ be the number of negative links. Notice also that since v_0v_3 coincides with the diameter, there is a (kind of) "combinatorial dichotomy" determined by the position of v_2: either it belongs to the arc v_1v_3, in which case we call the configuration *convex*, or it belongs to the arc v_3v_0, in which case we call it *non-convex*.

Theorem 2.2. *Let P be a P3P with a generic sidelengths vector $l \in \mathbb{R}^3_+$ satisfying $l_1 > l_2 + l_3$, and let V be its diacyclic configuration. If the first link is positive, then the Morse index of A at V is equal to $p(V) - 1$, and if the first link is negative, then the Morse index of A at V is equal to $n(V) - 1$.*

It follows that convex configurations are maxima or minima of A while non-convex ones are the saddle points. We omit a straightforward "calculational" proof of this result since in the last section it will be proved in a more conceptual way.

Remark 2.2. Thus, for P3P, the Morse index of A is determined by the "combinatorics" of diacyclic configuration; however, examples show that this is not the case for PnP with $n \geq 4$. Also, the formulas for the Hessian of A become much more complicated and not helpful for direct calculation of the Morse index, which was possible for $n = 3$. Thus the general problem appeared much more delicate. It has been solved only very recently in [28] using a numerical invariant of the configuration found in [29]. We refer to [28] for the formulation and proof of the general result.

In line with the paradigms suggested in [30], [23], we will now show that the critical points of A can be calculated using an analog of the generalized Heron polynomial [30].

Theorem 2.3. *For a generic sidelength vector $l \in \mathbb{R}^3_+$, there exists a polynomial P_l with real coefficients such that critical values of A on $M_2(P(l))$ are roots of P_l. The coefficients of P_l can be polynomially expressed via the lengths of the sides of $P(l)$.*

Proof of Theorem 2.3. This follows from the results of [30] and [11]. To show this we introduce a "double" of P defined as the hexagon linkage Φ with the sidelength vector $(l_1, l_2, l_3, l_1, l_2, l_3)$. Then each diacyclic configuration V of P gives a cyclic configuration W of Φ by adding the result of reflection of V at the midpoint of the connecting side (which coincides with the diameter of the circumscribed circle). By additivity of the signed area function we get $A(W) = 2A(V)$. On the other hand, from the results of [30] it follows that the signed area of each such W coincides with a certain root of the real polynomial introduced in [30]. Thus the first statement follows from an analogous result for inscribed hexagons established in [30]. The proof in [30] did not give an effective way of constructing P_l, but explicit algebraic formulas for the coefficients of P_l were given in [11]. These observations complete the proof. $\square$

Remark 2.3. This result is actually a particular case of general results on computation of critical values of algebraic functions on real algebraic sets (see., e.g., [2]). It also follows from the discussion in [23]. In fact, the same holds for critical values of area for generic PnP with arbitrary n. We presented it separately since, for a P3P, we can write down the polynomial P_l explicitly using the formulas for the generalized Heron polynomial for hexagons [11]. Starting with $n = 4$ we cannot do that since explicit formulas for coefficients of the generalized Heron polynomial are still unavailable in the case of an octagon.

Let us now find the maximal possible number of diacyclic configurations for a P3P. It is obvious that without loss of generality we can assume that $l_1 = 1, v_1 = (0, 0), v_2 = (1, 0)$. Let $v_3 = (x, y), v_4 = (u, v)$. Thus we are left with four variables and two parameters $l_2 = b, l_3 = c$.

Suppose V is a diacyclic configuration. Then the first condition of orthogonality immediately gives that $u = 1$. The second condition of orthogonality reads $x(1 - x) + y(v - y) = 0$. Adding the two conditions on the lengths of the links, we obtain a system of three equations with three variables:

$$x(1 - x) + y(v - y) = 0, \quad (x - 1)^2 + y^2 = b^2, \quad (x - 1)^2 + (v - y)^2 = c^2.$$

Using elementary algebraic manipulations to eliminate y, one easily finds that x should be a real root of the following cubic polynomial:

$$f(x) = 2(x-1)^3 + (b^2 + c^2 + 1)(x-1)^2 - b^2c^2.$$

The real roots of f can be counted and localized using several simple observations. First, notice that from the second equation it follows that a real root satisfying the above (3×3)-system should belong to the segment $[1-b, 1+b]$. Moreover, $f(1 \pm b) = b^2(b \pm 1)^2 > 0$, and it is also easy to see that f has a negative local minimum at $x = 1$.

This implies that f has exactly two (real) roots in the segment $[1-b, 1+b]$ and to each of them correspond two admissible values of y. Thus we conclude that there can be not more than four solutions of the above system, which yields the following result.

Proposition 2.1. *The number of diacyclic configurations of a P3P does not exceed four.*

We conclude this section with a result showing another application of PTP.

Theorem 2.4. *For a generic sidelengths vector $l \in \mathbb{R}^3_+$, the set of quasicyclic configurations $QC(P(l))$ of planar triple pendulum $P(l)$ is a closed smooth one-dimensional submanifold of $M_2(P(l))$.*

Proof of Theorem 2.4. Let us denote by g the determinant (1.1) written in the natural angular coordinates on $M_2(P) \cong T^2$. In our notation one has

$$g[x,y] = c\sin[x] - b\sin[y] - a\sin[y-x] + 2b\sin[x]cos[y-x].$$

So we to analyze the equation

$$c\sin[x] - b\sin[y] - a\sin[y-x] + 2b\sin[x]cos[y-x] = 0.$$

We can again fix the length a of the first side and consider g as depending on parameters b and c. To show that this g is submersion over the origin for almost all (b, c), we have to analyze the system

$$\frac{\partial g}{\partial b} = -\sin[y] + 2\sin[x]cos[y-x] = 0, \quad \frac{\partial g}{\partial c} = c\sin[x] = 0.$$

These equations are equivalent to $\sin[x] = \sin[y] = 0$. Substituting the solutions in the first two components of ∇g, we get a system

$$\frac{\partial g}{\partial x} = ccos[x] + acos[y-x] + 2bcos[x]cos[y-x]2b\sin[x]\sin[y-x] = 0,$$

$$\frac{\partial g}{\partial y} = -bcos[y] - acos[y-x] + 2b\sin[x]\sin[y-x] = 0,$$

which can only happen if

$$\pm c \pm b \pm 2b = 0, \quad \pm b \pm a = 0.$$

Hence the conditions of the parametric transversality theorem [1] are fulfilled for all (b, c) that do not satisfy the above linear relations. Thus we can again apply PTP to finish the proof. □

Remark 2.4. By definition, the set of quasicyclic configurations is the union of two sets: the set of cyclic configurations and the set of aligned configurations. It seems remarkable that the second set is also a critical set of another natural function on $M(P)$, which will be discussed in the next section.

Thus our approach yields sufficiently detailed information on the critical points of A for a triple pendulum. In the last section we give a few comments and outline some possible generalizations for PnPs with arbitrary n. Our next goal is to compare these results with the Morse theory of another natural function on $M(P)$ discussed in the next section.

3. Spread of Configuration as a Morse Function

Let V be a configuration of a planar n-pendulum $P = P_l$. The *spread* $s(V)$ of V is defined as the distance between v_0 and v_n. This gives a function $s_P : M(P) \to \mathbb{R}_+$. Let as assume that the first link of P is the longest one and its length l_1 is strictly bigger than the sum $\sum_{i\geq 2} l_i$ of the lengths of all other links. It is then obvious that s is a smooth function and $s(M(P))$ is contained in the segment

$$I(P) = \left[l_1 - \sum_{i\geq 2} l_i,\ l_1 + \sum_{i\geq 2} l_i \right].$$

Let us now assume that V is a normalized configuration of P. Then assigning to V the position of the last vertex v_n one obtains the so-called *tip-map* $t_P : M(P) \to \mathbb{R}^2$ from the moduli space into the plane. The tip-map and spread (under other names) have been discussed in many papers (see, e.g., [9, 19]). We will discuss some aspects that do not seem to have been considered earlier.

For $d \in I(P)$, let us denote by M_d the level surface $s^{-1}(d)$. The decomposition of $M(P)$ into disjoint union $\cup M_d$ will be called the *spread decomposition* of the moduli space $M(P)$. Notice that, for each $d \in I(P)$, M_d is naturally homeomorphic to the moduli space $M(L(l;d))$ of the $(n+1)$-gonal linkage $L(l;d)$. This observation was earlier used (see, e.g., [9, 19]) to obtain information on the topology of *polygon spaces* (i.e., the moduli spaces of polygonal linkages). Reversing this relation, we will use results on the topology of polygon spaces to answer some natural questions concerned with PnPs with $n \leq 5$.

In particular, since the complete lists of topological types of polygon spaces are known for $n \leq 6$ (see [16]), one may wish to consider the following problem. Given a PnP, which of these types appear as a homeomorphy type of a certain M_d? Concrete examples (see Proposition 3.1 below) show that all of these types need not be realized by the sets M_d for a given PnP, so the problem is nontrivial.

This problem can be completely solved for $n \leq 5$. We present here only two typical results for $n = 3$ because a complete description of results available for $n = 4, 5$ requires rather lengthy "case-by-case" formulations, which will be published elsewhere.

Recall that the complete list of homeomorphy types of planar moduli spaces of nondegenerate quadrilateral linkages is as follows: (1) circle; (2) disjoint union of two circles; (3) bouquet of two circles; (4) two circles with two common points; (5) three circles with pairwise intersections equal to one point.

Proposition 3.1. *For a P3P with generic sidelength vector (a, b, c) such that $a > b + c$, $b \neq c$, only the first three types in the above list are realized as homeomorphy types of sets M_d, $d \in I(P)$.*

Proposition 3.2. *For a P3P with (nongeneric) sidelength vector (a, a, a), all five types in the above list are realized as homeomorphy types of sets M_d, $d \in I(A)$.*

Both propositions are derived from the detailed description of homeomorphy types of planar moduli spaces of nondegenerate quadrilateral linkages in terms of relations between their sidelengths (see, e.g., [19]). Remembering that in this case the spread decomposition is a decomposition of a two-torus, we conclude that it may be visualized on the usual model of a torus as a square with identified opposite sides, which is left as a (pleasant) exercise for the interested reader.

In line with our strategy, let us now describe the critical points of s_P. We assume that side-lengths are independent over the field of rational numbers. In particular, for any n-tuple of signs $\alpha_i = \pm 1$, the sum $\sum \alpha_i l_i$ never vanishes. Such a PnP will be called generic. Recall that a configuration is called *aligned* if all of its vertices belong to the same straight line. For a normalized aligned configuration, a link will be called positive if it is parallel to the positive axis Ox (recall that by our convention the first link of a normalized configuration is always directed along Ox). Let $p(V)$ be the number of positive links. The results of [9, 19] yield the following information on the critical points of spread.

Theorem 3.1 (see [9, 19]). *For a planar n-pendulum P, the critical points of s_P are given by the aligned configurations of P. Generically there are 2^{n-1} distinct critical points, which are all nondegenerate. The Morse index of a nondegenerate normalized aligned configuration is equal to $p(V) - 1$.*

Remark 3.1. Notice the analogy with the case of A-critical configurations. In both cases critical configurations are quasi-cyclic, and the formulas for the Morse index are similar. An informal explanation that comes to one's mind is that a straight line is a "circle of infinite radius" but it would be interesting to find a more rigorous explanation of this analogy.

At the same time there is an essential difference. Since the moduli space of a PnP is homeomorphic to the $(n-1)$-torus and the sum of its Betti numbers is equal to 2^{n-1}, we conclude that, generically, the spread is an exact Morse function on the moduli space. This exhibits an essential difference with the case of oriented area A since there are examples showing that A is not an exact Morse function on the moduli space of PnP. For example, it was shown in [28] that this happens for the P4P with sidelength vector $l = (22, 17, 21.9, 19)$ for which A has twelve critical points while the sum of the Betti numbers in this case is 8.

We conclude this section by two observations concerned with tip-map, having in mind comparing them with analogous results on the cross-ratio map considered in the next section. It should be mentioned that the singular points of a tip-map are well studied (see, e.g., [14]). We note only that the results of [14] can be interpreted in terms of stable mappings [13], which leads to the following conclusion.

Proposition 3.3. *For a generic PnP, the tip-map is a stable map of the moduli space $M(P)$ into the plane. For a generic P3P with $l_1 > l_2 + l_3$, the image of a tip-map is a closed annulus centered at $(l_1, 0)$ with inner radius $l_2 - l_1$ and outer radius $l_1 + l_2$.*

4. Cross-Ratio Map and Darboux Transformation of a Planar Triple Pendulum

For a planar triple pendulum, the results of the previous section can be complemented by two results based on some recent results on the quadrilateral linkages from [7, 24]. To comply with the standard notation and definition of cross-ratio, let us renumber vertices as v_1, v_2, v_3, v_4 and (re)denote $l_1 = a, l_2 = b, l_3 = c$.

Let us first define an analog of the cross-ratio map (CRM) from [24] for a generic P3P with nonequal side-lengths (a, b, c). For every $V \in M(P), d \in I(P) \setminus \{a, b, c\}$, we can add the "connecting side" to V and obtain a configuration of a nondegenerate quadrilateral $Q = Q(a, b, c, d)$ with smooth $M_2(Q)$. Thus we can use the definition from [24] and put

$$Cr(V) = Cr((v_1, v_2, v_3, v_4)) = \frac{v_3 - v_1}{v_3 - v_2} : \frac{v_4 - v_1}{v_4 - v_2}.$$

It is easy to verify that Cr is actually defined on the whole of $M(P)$. Now the spread decomposition of $M(P)$ enables us to identify $Cr(M(P))$ using the results on cross-ratios of quadrilaterals established in [24]. The final result depends on relations between the lengths a, b, c. For brevity, we present here the result in the case where it is easiest to formulate. Other cases can be solved using a completely similar way.

Theorem 4.1. *For a P3P with length-code (a, b, c) such that $a > b + c$, the cross-ratio mapping is stable, and its image is contained in the closed annulus centered at point $(1, 0)$ with inner radius $\dfrac{ac}{b(a+b+c)}$ and outer radius $\dfrac{ac}{b(a-b-c)}$.*

This follows from the results of [24] on the cross-ratio image for quadrilateral linkages. Indeed, the image of cross-ratio mapping for P is the union of cross-ratio images of quadrilaterals $Q(a, b, c, d)$ with $d \in [a - b - c, a + b + c]$. According to [24], the cross-ratio image of $Q(a, b, c, d)$ is contained in a circle of radius $\frac{ac}{bd}$ centered at $(1, 0)$, which obviously implies the description of its image given in the theorem. The stability of cross-ratio mapping can be established by a routine argument of singularity theory.

Remark 4.1. As can be seen by considering concrete examples, the image of a cross-ratio map need not coincide with the annulus indicated in the proof of Theorem 4.1. In the same way, one can see that the shape of this image can vary considerably. Determining the exact shape of this image in terms of sidelengths remains an open problem. Notice also a certain analogy with Proposition 3.3. Analogous results may be proved for the mapping Φ into the plane with the components given by the spread and oriented area (i.e., $\Phi = (s, A)$), but we were not able to find a geometrically meaningful description of its singular points.

We proceed by discussing an analog of the Darboux transformation (DT) of a quadrilateral linkage [6, 7], which can be defined for a generic P3P in the following way. Assume that the sidelengths l_i are pairwise nonequal and consider a configuration of P3P such that $s(V)$ does not coincide with any of l_i. Since the side-lengths l_i of the resulting quadrilateral are pairwise nonequal, one can construct reflections of V in the diagonals v_1v_3 and v_2v_4, respectively. This obviously defines two *diagonal involutions* i_1 and i_2 on the whole of $M(P)$. These involutions do not commute, so in line with the usual practice it is advisable to consider their composition $= i_1 \circ i_2$. Following a suggestion from [7], we call τ the *Darboux transformation* (DT) of P3P.

It is obvious that i_1, i_2, and τ have no fixed points on $M_2(Q(l))$. However, τ may have periodic points, and their investigation led to interesting results [6, 7]. Since τ does not change the length of the "connecting side" v_1v_4, we conclude that τ acts on each $M_d, d \in I(P)$. According to the Poncelet porism for quadrilateral linkages [6], on each of the sets M_d one has the following dichotomy.

Proposition 4.1. *The Darboux transformation τ is either periodic with the same period $T(d)$ for each configuration $V \in M_d$ or the orbit $\{\tau^k(V), k = 1, 2, \ldots\}$ of each $V \in M_d$ is everywhere dense in M_d.*

Thus, one can distinguish between *periodic* and *nonperiodic* invariant subsets M_d. This setting suggests several natural problems. For example, which periods of DT arise for all admissible values of spread d? Are there different values of spread d with the same period of DT on M_d? Or, more precisely, what is the number of k-periodic values of d for all admissible periods k? It turns out that answers to these questions can be obtained using the results of [7].

Recall that the Darboux transformation of quadrilateral linkage is discussed in detail in Chapter 12 of the recent book of J. Duistermaat [7]. After establishing the connection between the Darboux transformation and the so-called *QRT mapping* of the associated biquadratic curve, the author investigates the dynamical properties of the Darboux transformation in a one-dimensional family of linkages that arises by varying the length of one side while keeping the remaining three lengths unchanged. In particular, J. Duistermaat calculates the number of k-periodic fibers of the so-called QRT automorphism of the corresponding family of biquadratic curves. These considerations are preceded by the following comment on page 514 of [7]: "Although it is debatable whether it is natural to vary only c while keeping a, b, and d constant, I cannot resist giving a short discussion of the elliptic surface that is defined by this pencil."

We find it remarkable that considerations of [7] have a natural interpretation in our context. Indeed, variation of one side-length while keeping the remaining lengths unchanged exactly corresponds to studying the action of the Darboux transformation on invariant sets $M_d \subset M(P)$. Let us now outline how this observation can be used to answer the two questions above.

To this end, notice that from the considerations in Chapter 12 of [7] it follows that the action of the Darboux transformation of our P3P on $M_d \cong Q(a, b, c, d)$ coincides with the action of the QRT automorphism of the resulting family of biquadratic curves. Hence k-periodic fibers of this family can be interpreted as values of d for which the Darboux transformation is k-periodic on M_d. So the following result, which gives answers to the above questions, follows directly from the table on p. 516 of [7].

Theorem 4.2. *For any natural n, the number of values of d for which the DT of a generic P3P is k-periodic on M_d is $3n^2+1$ when $k=3n$; $3n^2+2n$ when $k=3n+1$; $3n^2+4n+1$ when $k=3n+2$.*

Details of the proof and further applications of the results of [7] to our setting will be published elsewhere.

Remark 4.2. Taking into account Theorem 4.1, one may wish to consider the action of the Darboux transformation on the image of the cross-ratio mapping. There is some evidence that, for a nonperiodic fiber M_d, the orbit should be uniformly distributed, but we do not have a clue for the proof.

5. Concluding Remarks

We wish to add several comments on how one could complement and extend the above results on the critical configurations of a triple pendulum.

One obvious perspective of further research is to verify if straightforward analogs of our results for P3Ps hold for generic PnPs with arbitrary $n \geq 4$. This leads to a number of natural and pleasantly looking problems. In particular, one could wish to find the maximal number of diacyclic configurations of a planar PnP. We point out that certain estimates can be derived from the results of [30] and [11], but we could not prove that they are exact for arbitrary n, and so the problem remains largely open.

For completeness let us add a few words about the case of the double pendulum. It is obvious that the moduli space of a P2P is homeomorphic to the circle, and all configurations are quasicyclic. Moreover, we can use all the above formulas for the Hessian of A with $c=0$. This immediately gives that A has one maximum ab (for $\beta=\pi/2$) and one minimum $-ab$ (for $\beta=-\pi/2$). Both these critical points are Morse and there are no other critical points.

For a generic planar triple pendulum we can explain the results of Sec. 2 on the number of diacyclic configurations.

Proposition 5.1. *Generically, A has exactly four critical points on $M(P)$: two extrema and two saddles. Extrema are given by the convex diacyclic configurations. Each of these points is nondegenerate.*

Proof. The partial derivatives give the conditions for the critical point:

$$\frac{\partial A}{\partial x} = ab\cos[x] - bc\cos[y-x] = 0, \quad \frac{\partial A}{\partial y} = ac\cos[y] + bc\cos[y-x] = 0.$$

The orthogonality conditions are simply $\vec{a}\perp(\vec{b}+\vec{c})$ and $(\vec{a}+\vec{b})\perp\vec{c}$.

The next step is to show that there are exactly four critical points. This can be done as follows. One uses elementary geometry to obtain a cubic equation for the length d of the connecting edge OC:

$$d^3 = (a^2+b^2+c^2)d \pm 2abc. \tag{5.7}$$

We must solve these equations in d, taking into account that $d \geq a$, $d \geq b$, and $d \geq c$. Elementary calculations show that both the + equation and the equation have one solution satisfying these conditions. From $\cos\beta = \pm c/d$ and $\cos(\gamma-\beta) = \pm a/d$, it follows that there are exactly two solutions in each case. They occur in the pairs (β,γ) and $(-\beta,-\gamma)$, which yields the result. □

This reasoning applies to all cases, except when $a=b=c$. Then there are three critical points (one of which is a "monkey saddle" (see [1]).

Notice now that Eq. (5.7) actually determines the values of the circumradius of diacyclic configurations. So it can be considered as an analog of the generalized Heron polynomial for the circumradius. It is well known that the circumradius and oriented area of a cyclic configuration satisfy an algebraic relation [30]. It is an interesting problem to find the generalized Heron polynomials [32] of the minimal possible degree for the circumradius and oriented area. Using the above considerations, it is not difficult to show that, for a P3P, the minimal degree in both cases is equal to 3.

For a P3P, one can obtain the following two results concerned with quasicyclic configurations. They can be proved by elementary geometric considerations, and therefore we omit their proofs.

Proposition 5.2. *Generically, the set of quasicyclic configurations $QC(P)$ has two connected components. The extrema of A always belong to the same component, and the two saddles belong to another one.*

Recall that by Proposition 2.4 the subset of quasicyclic configurations is generically a smooth one-dimensional submanifold of $M(P)$, and so one can consider the critical points of the restriction $s|QC(P)$ of the spread function to the submanifold of quasicyclic configurations. Using a parametric transversality argument, it is easy to show that generically these critical points are nondegenerate minima and maxima. In particular, this happens in the case where $a > b > c$ and $a > b + c$. For brevity, we formulate the result only in this case.

Proposition 5.3. *Let P be a P3P with $a > b > c$ and $a > b + c$. Then the critical points of the spread restricted to $QC(P)$ coincide with the aligned configurations of P. There are two maxima and two minima. The maxima are attained at the configurations where the first two links or all three links are positive.*

Similar considerations are possible for the critical points of $A|QC(P)$. There is good geometric evidence that those critical points are generically nondegenerate and coincide with the diacyclic configurations of P, but we did not check this in detail. These results naturally suggest an open problem of obtaining their analogs in the case of generic PnP with arbitrary n.

We also wish to add a few words about the "double" of a planar n-pendulum P as defined at the end of Sec. 1. This is a $2n$-gon linkage Φ with the sidelength vector $(l_1, \cdots, l_n, l_1, \cdots, l_n)$ and its moduli space has dimension $2n - 3$. It follows from [19] that this moduli space is smooth except at the aligned configurations and its singularities are isolated and all of Morse type.

The proof of Theorem 2.3 tells us that each diacyclic configuration V of P gives a cyclic configuration W of Φ. This links the critical points of the oriented area function for a pendulum to critical points of the oriented area function for the corresponding "double" $2n$-gon, since the cyclic configurations of the latter are just critical points of A. Note that all these points belong to the smooth part of the moduli space of Φ.

There are more critical points of A on this moduli space. There are even non-isolated singularities. They occur as follows: Start with a planar pendulum in cyclic configuration (but not necessarily diacyclic). Consider its "pseudo-double" by reflecting the pendulum into the perpendicular bisector of v_0v_n. We get a configuration of $2n$-gon that starts with our pendulum and returns with the same pendulum in opposite order, and all vertices are still on the same circle.

This corresponds to a critical point of the oriented area function with value 0 in the smooth part of the moduli space of Φ. Since the radius of the cyclic configuration is a free parameter in the construction, we get a 1-parameter family of critical points. Thus A has nonisolated critical points on the moduli space of Φ.

These observations suggest that some interesting aspects are connected with the consideration of multiple penduli and polygonal linkages with a nongeneric vector of sidelengths, which is a promising research perspective.

Another promising perspective is connected with looking for analogs of our constructions and results in the case of spatial multiple penduli.

All this shows that the critical configurations discussed in this paper have a number of interesting aspects that are worthy of further investigation.

REFERENCES

1. V. Arnold, A. Varchenko, and S. Gusein-Zade, *Singularities of Differentiable Mappings* [in Russian], Nauka, Moscow (2005).

2. J. Bochnak, M. Coste, and M.-F. Roy, *Real Algebraic Geometry*, Springer-Verlag, Berlin–Heidelberg–New York (1998).
3. R. Connelly and E. Demaine, "Geometry and topology of polygonal linkages," in: *Handbook of Discrete and Computational Geometry*, CRC Press, Boca Raton (2004), pp. 197–218.
4. D. Cox and G. Scheya, *Ideals, Varieties, and Algorithms*, Springer-Verlag (1993).
5. H. Coxeter and S. Greitzer, *Geometry Revisited*, Toronto–New York (1967).
6. G. Darboux, "De l'emploi des fonctions elliptiques dans la théorie du quadrilatère plan," *Bull. Sci. Math. Astron.* **3**, 109–120 (1879).
7. J. Duistermaat, *Discrete Integrable Systems*, Springer-Verlag (2010).
8. E. Elerdashvili, M. Jibladze, and G. Khimshiashvili, "Cyclic configurations of pentagon linkages," *Bull. Georgian Natl. Acad. Sci.*, **2**, No. 4, 13–16 (2008).
9. M. Farber, *Invitation to Topological Robotics*, Eur. Math. Soc., Zürich (2008).
10. M. Farber, J.-C. Hausmann, and D. Schütz, "The Walker conjecture for chains in $\mathbb{R}^d$," *Math. Proc. Cambridge Philos. Soc.* **151**, No. 2, 283–292 (2011).
11. M. Fedorchuk and I. Pak, "Rigidity and polynomial invariants of convex polytopes," *Duke Math. J.*, **129**, No. 2, 371–404 (2005).
12. C. Gibson and P. Newstead, "On the geometry of the planar 4-bar mechanism," *Acta Appl. Math.*, **7**, 113–135 (1986).
13. M. Golubitsky and V. Guillemin, *Stable Mappings and Their Singularities*, Grad. Texts Math., **14**, Springer-Verlag, New York–Heidelberg (1973).
14. J.-C. Hausmann, "Contrôle des bras articulés et transformations de Möbius," *Enseign. Math.* **51**, Nos. 1–2, 87–115 (2005).
15. J.-C. Hausmann, "Geometric descriptions of polygon and chain spaces," *Contemp. Math.* **438**, 47–57 (2007).
16. J.-C. Hausmann and E. Rodriguez, "The space of clouds in Euclidean space," *Exp. Math.* **13**, No. 1, 31–47 (2004).
17. M. Hirsch, *Differential Topology*, Springer-Verlag (1976).
18. M. Kapovich and J. Millson, "Moduli spaces of linkages and arrangements," *Prog. Math.* **172**,730–770 (1999).
19. M. Kapovich and J. Millson, "On the moduli spaces of polygons in the Euclidean plane," *J. Differ. Geom.* **42**, 430–464 (1995).
20. A. Kempe, "A method of describing curves of the nth degree by linkwork," *Proc. London Math. Soc.* **7**, 213–216 (1876).
21. G. Khimshiashvili, "On configuration spaces of planar pentagons," *Zap. Nauch. Sem. POMI* **292**, 120–129 (2002).
22. G. Khimshiashvili, "Signature formulae and configuration spaces," *J. Math. Sci.* **61**, 89–98 (2009).
23. G. Khimshiashvili, "Extremal problems on configuration spaces," *Proc. A. Razmadze Math. Inst.*, **155**, 73–77 (2011).
24. G. Khimshiashvili, "Complex geometry of quadrilateral linkages," in: *Recent Developments in Generalized Analytic Functions and Their Applications*, Tbilisi State Univ. (2011), pp. 90–100.
25. G. Khimshiashvili and G. Panina, "Cyclic polygons are critical points of the area," *Zap. Nauch. Sem. POMI*, **360**, 31–40 (2008).
26. G. Khimshiashvili and G. Panina, "On the area of a polygonal linkage," *Dokl. Math.*, **85**, No. 1, 1–2 (2012).
27. G. Khimshiashvili and D. Siersma, "Cyclic configurations of planar multiple penduli," ICTP Preprint IC/2009/047 (2009).
28. G. Khimshiashvili, G. Panina, D. Siersma, and A. Zhukova, "Critical configurations of planar robot arms," *Centr. Eur. J. Math.*, **10**, No. 6 (2012).
29. G. Panina and A. Zhukova, "Morse index of a cyclic polygon," *Centr. Eur. J. Math.*, **9**, No. 2, 364–377 (2011).

30. D. Robbins, “Areas of polygons inscribed in a circle,” *Discr. Comput. Geom.*, **12**, 223–236 (1994).
31. W. Thurston, “Shapes of polyhedra and triangulations of the sphere,” *Geom. Topol. Monogr.*, **1**, 511–549 (1998).
32. V. Varfolomeev, “Inscribed polygons and Heron polynomials,” *Mat. Sb.*, **194**, 311–321 (2003).

G. Khimshiashvili
Ilia State University, Tbilisi, Georgia
E-mail: gogikhim@yahoo.com
D. Siersma
Department of Mathematics, Utrecht University, The Netherlands
E-mail: D.Siersma@uu.nl