reprint from

Chinese Quaterly Journal of Mathematics

Volume 10, No. 4, December 1995
Special Issue of Proceedings of the 1994 Workshop on Topology and Geometry

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Vol. 10 No. 4 CHINESE QUARTERLY JOURNAL OF MATHEMATICS Dec. 1995

# Is the Polar Relative Monodromy of Finite Order? An Example

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**Abstract** We construct an example of an isolated singularity $f:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ for which one of the polar relative monodromies is not of finite order, by using a nonisolated singularity with transversal type the A'Campo example.

## § 1. Introduction

Let $f:(\mathbf{C}^n,0)\to(\mathbf{C},0);n\geqslant 2$, be a complex analytic function germ with isolated singularity. We recall the Monodromy Theorem. The function $f$ induces a $\mathbf{C}^\infty$-fibration:

$$f_|:B_\varepsilon\cap f^{-1}(D_\eta\setminus\{0\})\to D_\eta\setminus\{0\}$$

where $B_\varepsilon$ is a ball centred at $0\in\mathbf{C}^n$ and $D_\eta$ a disc centred at $0\in\mathbf{C}$. The radii $\varepsilon$ and $\eta$ are small enough, with $0<\eta\ll\varepsilon$. By Milnor's result [Mi, Theorem 6. 5], the fibre $F$ of this fibration (the Milnor fibre) is homotopy equivalent to a wedge of spheres of dimension $n-1$. The number of spheres is the Milnor number of $f$. A characteristic homomrphism of this Milnor fibration is called geometric monodromy. This induces an algebraic monodromy $h$: $H_{n-1}(F,\mathbf{C})\to H_{n-1}(F,\mathbf{C})$. The Monodromy Theorem states that $h$ is quasi-unipotent of index $\leqslant n-1$. This means that there exists $k\in\mathrm{N}$ such that $(h^k-I)^n=0$, where $I$ is the identity, which further means that the eigenvalues of $h$ are roots of unity and the Jordan blocks have size at most $n$. This was proved e. g. in [La], [G], [B]. See [vDS] for a supplementary statement. There is a geometric proof of the quasi-unipotency by Lê D. T. [Lê-2], completed by Looijenga in his book [Lo] with the proof of the index statement. The index $n-1$ is a sharp estimation, by well-known examples of Malgrange [Ma]. For $n=2$ and irreducibe f, Lê proves that the monodromy is of finite order (i. e. the index is 0). Examples of A'Campo [AC] show that this is not any more the case for a plane curve with several branches. Then Durfee [Du] gave a method to decide easily when a reducible curve

Received March 21, 1995.

singularity has monodromy of infinite order.

Let now $l:(\mathbf{C}^n,0)\to(\mathbf{C},0)$ be a linear form which is general with respect to f. Denote by $F_0$ the Milnor fibre of the isolated singularity $f_{|l=0}$. The monodromy acts on the exact sequence:

$$0\to H_{n-1}(F)\to H_{n-1}(F,F_0)\to H_{n-1}(F_0)\to 0.$$

The action in the middle $h_{rel}:H_{n-1}(F,F_0)\to H_{n-1}(F,F_0)$ is called relative monodromy and it was first considered by Thom. Lê discovered the carrousel method [Lê-2],[Lê-3] and gave the most effective use to the relative monodromy. He introduced the polar filtration on $F$. Roughly, this is a filtration:

$$F_0\subset F_1\subset\cdots\subset F_r=F,$$

such that a certain geometric monodromy acts on each term and induces what we may call polar relative monodromies:

$$h_{rel_i}:H_{n-1}(F_i,F_{i-1})\to H_{n-1}(F_i,F_{i-1}).$$

They are also quasi-unipotent, as the carrousel method shows [Lê-2]. In case $n=2$, the same method moreover shows that $h_{rel_i}$ is of finite order, for any $i\in\{1,\cdots,r\}$, see also [SZ] for another proof. There is a question wheather this is also true when $n>2$. A comment at the end of [SZ] refers to [Lê-3] for an afirmative answer. Inspection shows that a remark of this kind is not contained in [Lê-3].

In this note we give an example of an isolated singularity $f:(\mathbf{C}^3,0)\to(\mathbf{C},0)$ where one of the relative monodromies $h_{rel_i}$ is not of finite order. This gives a negative answer to the above question. Our example is $f=g+z^6$, where

$$g=x^2y^2z+x^5+y^5+pxy^4+qx^4y,\quad p,q\neq 0.$$

In fact $g$ is a non-isolated singularity, where the transversal type is just the A'Campo example. This plays in important role here. The term $z^6$ mades the singularity isolated.

We start giving the back-ground, then show how our example works.

## § 2. Comparing Monodromies within a Iomdin Series

We take a function germ $g:(\mathbf{C}^n,0)\to(\mathbf{C},0)$ with 1-dimensional singular locus $\sum:=\operatorname{Sing} g$. Let $\Omega_g$ be the set of linear forms $l:(\mathbf{C}^n,0)\to(\mathbf{C},0)$ such that $\sum\cap\{l=0\}=\{0\}$. We shall describe first how to compare the monodromy of $g_N:=g+l^N$, for $N\gg 0$, with the one of g. To do that, we use Lê's carrousel construction and Siersma's method for comparing monodromies within a so called Iomdin series [Si]($g_N$, for $N\in\mathbf{N}$ is such a one). In some sense, g is the "limit" of the series $g_N$, when $N\to\infty$.

First, some preliminary lemmas which may not by found under this form in the literature (proofs are also slightly different from the original ones). Let $\operatorname{Crt}\Phi$ denote the critical locus of the application $\Phi:=(l,g):(\mathbf{C}^n,0)\to(\mathbf{C}^2,0)$ and let $\Gamma(l,g):=\operatorname{closure}(\operatorname{Crt}\Phi\setminus\{g=0\})$ be the polar locus of $g$ with respect to $l$.

**Lemma 2.1** ([Lê-1],[Pe-1, Prop. 8.5],[Pe-2, Prop. 3.1],[Sch, p. 75]) If $l\in\Omega_g$ then $\dim \mathrm{Crt}\,\Phi\leqslant 1$ and $\mathrm{Crt}\,\Phi=\sum\cup\Gamma(l,g)\cup S$, where $S\subset\{g=l=0\}$.

**Proof** We choose coordinates on $\mathbf{C}^n$ such that $l=x_n$. Then $\mathrm{Crt}\Phi=V(\partial_1 g,\cdots,\partial_{n-1}g)$. By a standard application of Sard's Theorem, there is an open dense $\Lambda\subset C$ such that, for any $\lambda\in\Lambda$, the germ $g_\lambda:=g+\lambda x_n$ is an isolated singularity. This implies that $\dim V(\partial_1 g_\lambda,\cdots,\partial_{n-1}g_\lambda)\leq 1$. But $\partial_i g_\lambda=\partial_i g$, for $1\leq i\leq n-1$.

The rest of the proof is strightforward.

**Lemma 2.2** ([Lê-1],[Pe-2, Prop. 3.3]) If $l\in\Omega_g$ then $g_N:=g+l^N$ is an isolated singularity, for all $N\in\mathbf{N}$ except a finite number of values.

**Proof** We may take again $l=x_n$. Denote by $\partial g_N$ the Jacobian ideal of $g_N$. Consider the germ of $g_N$ at a point $p\in\mathbf{C}^n$. If $p\notin\mathrm{Crt}\Phi$ then $g_N$ is nonsiongular at $p$. Moreover,

$$V(\partial g_N)\cap\sum=V(l)\cap\sum=\{0\}.$$

But $V(\partial g/\partial x_n+Nx_n^{N-1})$ can contain a certain component of closure$(\mathrm{Crt}\Phi\setminus\sum)$ for at most one value of $N$. Therefore $V(\partial g_N)\cap$closure$(\mathrm{Crt}\Phi\setminus\sum)=\{0\}$, for all $N$ except a finite number of values.

One usually works with the oen dense subset $\Omega_g'\subset\Omega_g$ of admissible linear functions, as introduced by Lê in [Lê-1]: $l\in\Omega_g'$ if and only if $g_{|l=0}$ has isolated singularity. The inclusion is strict, as the following example due to Pellikaan shows (cf. [Sch, p. 75])

**Example 2.3** $g(x,y,z)=x(y^2+z^2)+y^2z^2$, and $l=0$.

Then $\sum=V(y,z)$ and $\sum\cap\{l=0\}=\{0\}$. But $g_{|l=0}=y^2z^2$ has nonisolated singularity and $S=V(x,y)\cup V(x,z)$. The polar curve is $\Gamma(l,g)=\{x+y^2=x+z^2=0\}$.

We shall work in the following under the hypothesis $l\in\Omega_g$. The next results of this section are usually proved for admissible $l$, but one can check that they remain valid (without any modification) in the more general case $l\in\Omega_g$.

**2.4** We recall and adapt to our situation Lê's carrousel construction, following [Lê-2]. The curve germ (with reduced structure) $\Delta(l,g):=\Phi(\Gamma(l,g))$ is the Cerf diagram (of $g$, with respect to $l$). Let $(u,\lambda)$ be local coordinates at $0\in\mathbf{C}^2$.

For small enough ball $B\subset\mathbf{C}^{+n}$ and small enough discs $D,D'\subset\mathbf{C}$, where $D'$ is small enough with respect to $D$, the following restriction of $\Phi$:

$$\Phi_|:B\cap\Phi^{-1}(D\times D')\to D\times D'$$

is a topolgical fibration over $D\times D'\setminus(\Delta(l,g)\cup\{\lambda=0\})$. Moreover, $g$ induces a topological fibration

$$g|:B\cap\Phi^{-1}(D\times D')\cap g^{-1}(D'\setminus\{0\})\to D'\setminus\{0\},$$

respectively

$$g':B\cap\Phi^{-1}(\{0\}\times D')\cap g^{-1}(D'\setminus\{0\})\to D'\setminus\{0\},$$

which is fibre homeomorphic to the Milnor fibration of $g$, respectively to the Milnor fibration of $g|\{l=0\}$. The disc $D'$ has been chosen small enough such that $\Delta(l,g)\cap\partial\bar{D}\times D'=\varnothing$.

Let $S' := \partial \overline{D'}$. One builds an integrable smooth vector field on $D \times S'$, tangent to $\triangle(l,g) \cap (D \times S')$ and lifting the unit vector field of $S'$ by the projection $D \times S' \to S'$. This vector field on $D \times S'$ has a lift by $\Phi$ which is tangent to the polar curve $\Gamma(l,g) \cap \Phi^{-1}(D \times S')$ and tangent to $\partial \overline{B} \cap \Phi^{-1}(D \times S')$. This lift can be integrated to get a characteristic homeomorphism of the fibration induced by $g$ over $S'$, hence a geometric monodromy **h** of the Milnor fibre $F_g$ of $g$. We call it the (geometric) carrousel monodromy.

We fix some $\eta \in S'$ and denote $\mathbf{D} = \mathbf{D}(l,g) := D \times \{\eta\}$. Let

$$l_| : B \cap \Phi^{-1}(\mathbf{D}) \to \mathbf{D}$$

be the restriction of $\Phi$ and notice that $F_g$ is homeomorphic to $l_|^{-1}(\mathbf{D})$.

The integration of the vector field on $D \times C'$ produces a homeomorphism $\mathscr{C} : \mathbf{D} \to \mathbf{D}$ which we call carrousel of the disc $\mathbf{D}$: the trajectory inside $D \times S'$ of some point $a \in \mathbf{D}$ is such that after one turn around the circle $S'$ we get some other point $a' := \mathscr{C}(a) \in \mathbf{D}$. By construction, the vector field rdstricted to $\{0\} \times S'$ is the unit vector field of $S'$, hence the centre $(0,\eta)$ of the carrousel disc $\mathbf{D}$ is indeed fixed; the circle $\partial \mathbf{D}$ is also pointwise fixed.

The distinguished points $\triangle(l,g) \cap \mathbf{D}$ of the disc have a complex motion around $(0,\eta)$, depending on the Puiseux parametrizations of the branches of $\triangle$. Let $\triangle_i$ be such a branch and consider a Puiseux parametrization of it, in coordinates $(u,\lambda)$: $u = \sum_{j \geq m_i} c_{i,j} t^j, \lambda = t^{n_i}$, where

$$m_i := \mathrm{mult}_0 \triangle_i$$

$$n_i = \mathrm{mult}_0(\triangle_i, \{\lambda = 0\}).$$

Let $\rho_i = m_i / n_i$ denote the Puiseux ratio of $\triangle_i$.

Lê D. T. defines the polar filtration of the disc $\mathbf{D}$ as follows [Lê-3]. Assume that the Puiseux ratios are decreasingly ordered: $\rho_1 \geq \rho_2 \geq \cdots$. Then there is a corresponding sequence of open discs $D_1 \subseteq D_2 \subseteq \cdots \subset \mathbf{D}$ centred at $(0,\eta)$, where $D_i = D_{i+1}$ if and only if $\rho_i = \rho_{i+1}$, such that $\triangle_{i+1} \cap \mathbf{D} \subset D_{i+1} \setminus \overline{D_i}$, for $i > 1$ and $\triangle_1 \cap \mathbf{D} \subset D_1$.

In esch annulus $A_i := D_i \setminus \overline{D_{i-1}}$, the carrousel is not that easy to describe (we send to [Lê-2], [Lê-3], [Ti] for details), but at the "first approximation", each point is rotated by $2\pi\rho_i$. There must be a continuous transition between successive annuli: within a thin enough annulus containing the circle $\overline{A_i} \cap \overline{A_{i+1}}$ each point will have a carrousel movement which is exactly a rotation by $2\pi\rho(r)$, where $r$ is the radius to that point and $\rho(r)$ is a continuous decreasing real function with values onto the interval $[\rho_{i+1}, \rho_i]$. Denoting by $F$, resp. $F_N$, the Milnor fibre of $g$, resp. $g_N$, we have the following result.

**Proposition 2.5 ([Lê-1] and [Si])** Let $l \in \Omega_g$ and let $\{\rho_i, i \in K\}$ be the set of polar ratios of $\triangle(l,g)$, where $\triangle(l,g) = \bigcup_{i \in K} \triangle_i$ is the decomposition into irreducible curves. If $N > 1/\rho_i, \forall\, i \in K$, then the polar filtration of $(l, g_N)$ and $(l,g)$ are the same, except that $(l, g_N)$ has one disc more at the end of the chain.

Consequently, the Milnor fibre $F$ is naturally embedded into the Milnor fibre $F_N$ and this embedding commutes with the carrousel monodromies.

**Proof** . The idea of the technique is due to Lê. One notices that $\triangle(l,g_N)$ has all the polar quotients of $\triangle(l,g)$ and one more only, namely $1/N$. This is because the singular locus of g becomes a component of the polar curve $\Gamma(l,g_N)$, hence the image of it by the map $(l,g_N)$ is the supplementary component of $\triangle(l,g_N)$. Actually, using Lemma 2.1, it is easy to check that $\sum(l,g_N)=\Gamma(l,g)\cup\sum$.

We look to the image of $\Phi$ in coordinates $(u,\lambda)$. Then $F=\Phi^{-1}(\mathbf{D})=\Phi^{-1}\{\lambda=\eta\}$ and $F_N=\Phi^{-1}(\lambda+u^N=\eta)$. Using the 1-parameter deformation $\lambda+\varepsilon u^N=\eta$, where $\varepsilon\in[0,1]$, one constructs a nonsingular vector field, tangent to $\triangle$ and such that, by integrating it, to define an embedding of the carrousel disc $\mathbf{D}$ of $(l,g)$ into $\{\lambda+u^N=\eta\}$. The image of this embedding does ont intersect $\Phi(\text{Sing } g)$. Therefore one can identify the carrousel disc $\mathbf{D}(l,g)$ with the last disc before $\mathbf{D}(l,g_N)$ in the polar filtration of the carrousel disc $\mathbf{D}(l,g_N)$. The vector field on $C^2$ can be lifted, then integrated to give an embedding $F\subset F_N$, which is actually the lift of the embedding $\mathbf{D}(l,g)\subset\mathbf{D}(l,g_N)$.

Let now $\Phi_N:=(l,g_N):(C^n,0)\to(C^2,0)$. We saw that $F\subset F_N$ and that

$$F_N\backslash F=\Phi_N^{-1}(\mathbf{D}(l,g_N)\backslash\mathbf{D}(l,g)),$$

where $\mathbf{D}(l,g_N)$ is identified with the corresponding disc in the polar filtration of $\mathbf{D}(l,g_N)$. Denote by $\mathbf{A}$ the annulus $\mathbf{D}(l,g_N)\backslash\mathbf{D}(l,g)$ and by $\triangle_\Sigma$ the component of $\triangle(l,g_N)$ which comes from the singular locus of $g$; this has the following parametrization: $u=t,\lambda=t^N$. Then $\mathbf{A}\cap\triangle(l,g_N)=\mathbf{A}\cap\triangle_\Sigma$ is a set of $N$ points equally distributed on a circle included in $\mathbf{A}$. By definition, the carrousel movement of any point in $\mathbf{A}$ is a rotation by $2\pi\frac{1}{N}$.

By the construction in the proof of Proposition 2.5, the geometric monodromy induces a monodromy action on the exact sequence:

$$0\to H_{n-1}(F)\to H_{n-1}(F_N)\to H_{n-1}(F_N,F)\to H_{n-2}(F)\to 0.$$

We denote by $h_{rel,N}$ the action on $H_{n-1}(F_N,F)$. If $F_i$ denotes $\Phi_N^{-1}(D_i)$ then we have the following polar filtration of $F_N$:

$$F_1\subset F_2\subset\cdots\subset F\subset F_N$$

and then $h_{rel,N}$ is one of the polar relative monodromies (in fact the last one).

One can describe the monodromy $h_{rel,N}$ as follows. There is a direct sum decomposition:

$$H_{n-1}(F_N,F)=\oplus_{j=1}^r\oplus_{k=1}^{Nd_j}\tilde{H}_{n-2}(F_{j,k}),$$

where $\sum=\sum_1\cup\cdots\cup\sum_r$ is the decomposition into irreducibles, $d_j$ is the multiplicity of $\sum_j$ and $F_{j,k}$ is the Milnor fibre of the transversal singularity at some point of $\sum_j\backslash\{0\}$. This Milnor fibre does not depend on the point, up to homeomorphisms; we shall denote it by $F_j$.

Actually $\tilde{H}_{n-2}(F_j,\mathbf{C})$ is the stalk of the local system defined on $\sum_j\backslash\{0\}$. The monodromy group on this local system is generated by the following two independent monodromies:

(a) the "horizontal" monodromy $T_j:\tilde{H}_{n-2}(F_j)\to\tilde{H}_{n-2}(F_j)$, which is the monodromy of the restriction of g to a transversal slice at some point $a\in\sum_j\backslash\{0\}$.

(b) the "vertical" monodromy $A_j:\tilde{H}_{n-2}(F_j)\to\tilde{H}_{n-2}(F_j)$, which is the characteristic mapping of the local system along a simple loop around 0 in $\sum_j\backslash\{0\}$.

The first author proved the following result:

**Proposition 2.6 ([Si])** The relative monodromy $h_{rel,N}$ acts on each vector space $V_j:=\oplus_{k=1}^{Nd}\tilde{H}_{n-2}(F_{j,k})$ and the matrix of this action has the following block decomposition:

$$\begin{pmatrix} 0 & \cdot & \cdot & \cdot & 0 & A_jT_j^{d_jN} \\ 1 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & 0 & \cdot & \cdot & \cdot \\ \cdot & 0 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ 0 & \cdot & \cdot & 0 & 1 & 0 \end{pmatrix}$$

## § 3. The Example

Let $g:\mathbf{C}^3\to\mathbf{C}, g=x^2y^2z+x^5+y^5+pxy^4+qx^4y,\ p,q\neq0$. This is homogeneous of degree $d=5$. The singular locus is the line $\sum=\{x=y=0\}$, hence irreducible $(r=1)$, and $\sum^{red}$ has multiplicity 1. The transversal singularity has normal form $cx^2y^2+x^5+y^5$, (type $Y^1_{1,1}$ from Arnold's list, with Milnor number 11). This is equivalent to A'Campo's example mentioned before. Therefore the horizontal monodromy $T$ has Jordan blocks of size 2 (i.e. $T$ is of index 1). Let us take $l=z$ and $N=6$; our example is then $f:=g+z^6$. Notice that $z\in\Omega_g$ and $f$ is an isolated singularity.

Since our $g$ is homogeneous, we may apply the following result.

**Lemma 3.1 ([St])** If $g$ is a homogeneous singularity of degree $d$, with 1-dimensional singular locus $\sum$, then $A_j=T_j^{-d}$.

**Proof (following an idea of A. Dimca)** We use the notations from Section 2. $\sum$ consists of lines because of the homogeneity of $g$. Take a branch $\sum_j$ and suppose (after a linear change of coordinates) that this is the $x_1$-axis. Consider the map: $p:=(x_1,g):\mathbf{C}^n\to\mathbf{C}^2$ and let $\mathbf{T}$ be a thin enough tubular neighbourhood of $\Delta_\Sigma\cap D\times S'$ in $D\times S'$. We identify $\partial\mathbf{T}$ with the torus $S^1\times S^1$. Then $P_|:Y:=p^{-1}(S^1\times S^1)\to S^1\times S^1$ is a locally trivial fibre boundle with fibre $F_j$. Note that $\pi_1(S^1\times S^1)=\mathbf{Z}\times\mathbf{Z}$ and that $A_j$ and $T_j$ are the canonical generators of the monodromy representation $\pi_1(S^1\times S^1)\to\mathrm{Aut}(\tilde{H}_{n-2}(f_j))$. Then consider the family of automorphisms $h_s:Y\to Y, h_s(x)=e^{is}x$, where $s\in[0,2\pi]$. Since $p\circ h_s$, for $s\in[0,2\pi]$, is the element $(1,d)$ in $\mathbf{Z}\times\mathbf{Z}$, it follows that $h_{2\pi}$ represents $A_jT_j^d$. On the other hand $h_{2\pi}$ is the identity. Hence $A_jT_j^d$ is the identity.

In our case we get $AT^6=T$. Then by Proposition 2.6, the matrix of the polar relative monodromy $h_{rel,6}$ has a simple block decomposition and one can easily check that:

$$(h_{rel,6})^6 = \begin{bmatrix} T & 0 & 0 & 0 & 0 & 0 \\ 0 & T & 0 & 0 & 0 & 0 \\ 0 & 0 & T & 0 & 0 & 0 \\ 0 & 0 & 0 & T & 0 & 0 \\ 0 & 0 & 0 & 0 & T & 0 \\ 0 & 0 & 0 & 0 & 0 & T \end{bmatrix}.$$

This shows that the index of $h_{rel,6}$ is the same as the one of $T$, thus the monodromy is not of finite order.

The only thing to do more is to show that $h_{rel,6}$ is a polar relative monodromy for $f$. The polar relative monodromies of $f$ are defined by the polar filtration associated to the polar curve $\Gamma(l,f)$, where $l \in \Omega_f$. Now $l=z$ is general for $g$, but not anymore for $f$! A general linear form for $f$ would be:

$$z_t := z + t(\alpha x + \beta y),$$

where $t$ is a real parameter in a neighbourhood of 0 and $\alpha,\beta$ are general complex constants. Then $\Gamma(z_t,f)$ depends continuously on $t$. We have $\Gamma(z_0,f) = \Gamma(z,g) \cup \sum$, where $\Gamma \cap \sum = \{0\}$. Then we get a decomposition $\Gamma(z_t,f) = \Gamma'(z_t,f) \cup \Gamma''(z_t,f)$, $\Gamma' \cap \Gamma'' = \varnothing$, where $\Gamma'(z_t,f)$ denotes the branches that specialize to branches of $\Gamma(z,g)$ and $\Gamma''(z_t,f)$, those branches that specialize to $\sum$.

There is just one polar ratio for all the branches that specialize to $\Gamma(z,g)$; it is $1/5$, one can see this from the equations and from the fact that $\Gamma(z,g)$ has this single polar ratio.

We have to find the polar ratios of $\Gamma''(z_t,f)$. For that, we employ a slightly more sophisticated argument, as follows. We have:

$$\Gamma(z_t,f) = \{\frac{\partial f}{\partial x} = \alpha t \frac{\partial f}{\partial z}, \frac{\partial f}{\partial y} = \beta t \frac{\partial f}{\partial z}\}.$$

Bolw-up by substituting $z:=z, x:=xz, y:=yz$. We get the strict transform given by the following system:

$$\begin{cases} 2xy^2 + 5x^4 + py^4 + 4qx^3y = \alpha t x^2y^2 + 6\alpha t z & (1) \\ \beta(2xy^2 + 5x^4 + py^4 + 4qx^3y) = \alpha(2x^2y + 5y^4 + qx^4 + 4pxy^3). & (2) \end{cases}$$

This is the graph of a certain function $z(x,y)$ over the plane curve (2). By straightforward computations, each branch of the plane curve (2) can be parametrized in such a way (let $s$ be the parameter) that in all cases we get: ord $f$/ord $z_t = 6$, which means that all those branches have a polar ratio $1/6$.

This proves that the polar filtration of $f$ defined by $\Gamma(z,f)$ is the same as the one defined by $\Gamma(z,f)$ and therefore the polar relative monodromies are the same.

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