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PERIODICITIES IN ARNOLD'S LISTS OF SINGULARITIES

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Abstract

V. I. Arnold discovered experimentally periodicities in the classification of singularities. These periodicities are explained for functions from C^2 to C, using the blowing up construction.

Moreover the singularities of multiplicity 5 are classified.

In his paper 'Local forms of functions' Arnold [2] gives a list of normal forms of functions in the neighborhood of critical points (the classification of all singularities with number of modules $m=0,1,2$ or with multiplicity $\mu \leqq 16$ included). In his introduction he mentions a *periodicity* in the decomposition of singularities into μ-equivalence classes. According to Arnold the phenomenon of periodicity is only partially explained and for quasi-homogeneous singularities only. The explanation is based upon some root technique for the quasihomogeneous Lie algebra, related to work of Enriques and Demazure [6].

The aim of this paper is to explain the periodicity for all isolated singularities of corank $\leqq 2$ and some singularities of corank 3 (including all singularities in Arnold's lists), using the theory of resolutions. We also give a list of singularities of corank 2 with multiplicity equal to 5.

This paper is an elaborated version of the lecture I gave at the Institut des Hautes Études Scientifiques (Bûres-sur-Yvette, France) in May 1975 about Arnold's paper: 'Local forms of functions'.

I thank the I.H.E.S. for their hospitality.

Added in proof:

Arnold communicated to me that:

1° Some of his students have also classified the singularities of corank 2 with multiplicity 5 (unpublished).

2° His remark about periodicities also concerns the actual computations that occur.

AMS(MOS) subject classification scheme (1970): 32 C 40, 58 C 25

Contents

1. Introduction

For results and definitions, mentioned in this introduction see Arnold [2] and Siersma [10].

(1.1) The group $\mathscr{D}$ of germs (or jets) of biholomorphic mappings $(\boldsymbol{C}^n, O) \to (\boldsymbol{C}^n, O)$ acts on $\mathscr{E}$, the set of germs at O (or jets) of holomorphic functions $\boldsymbol{C}^n \to \boldsymbol{C}$ by right-multiplication. A *singularity class* is a subset of $\mathscr{E}$, invariant under this action. Each orbit is a singularity class. Two germs (or jets) belonging to the same orbit are called *right-equivalent.*

(1.2) Let $m = \{f \in \mathscr{E} \mid f(O) = 0\}$ and let $\Delta(f) = (\partial_1 f, \cdots, \partial_n f)$. The codimension of $f \in m^2$ is defined by $\operatorname{codim}(f) = \dim_{\boldsymbol{C}}(m^2/m\Delta(f))$ and the Milnor number of f is defined by $\mu(f) = \dim_{\boldsymbol{C}}(\mathscr{E}/\Delta(f))$. These are related by: $\mu(f) = 1 + \dim(m/\Delta(f)) = 1 + \operatorname{codim}(f)$.

(1.3) A germ $f \in m$ is called *k-determined* (or k-sufficient) if for any $g \in m : j^k(f) = j^k(g) \Rightarrow f$ is right-equivalent with g. It is well-known that the following are equivalent:
1° $\mu(f) < \infty$.
2° O is an isolated critical point of f.
3° there exists $k \in \boldsymbol{N}$ such that f is k-determined.

(1.4) Two germs f and g are called:
(1) *Right-left-equivalent* if there is a $\phi \in \mathscr{D}_n$ and a $\psi \in \mathscr{D}_1$ such that $\psi g = f\phi$.
(2) *Contact-equivalent* if there is a $\phi \in \mathscr{D}_n$ and for every x near O a $\psi_x \in \mathscr{D}_1$ (analytically depending on x) such that $\psi_x g = f\phi$.

(3) *Topological-equivalent* if there exist homeomorphisms $\phi : \boldsymbol{C}^n \to \boldsymbol{C}^n$ and $\psi : \boldsymbol{C} \to \boldsymbol{C}$ such that $\psi g = f\phi$.
(4) *μ-equivalent* if there exists a family f_t with $\mu(f_t) = \mu$ (constant) and $f_0 = f$ and $f_1 = g$.
If $n \neq 3$: μ-equivalence$\Leftrightarrow$top. equivalence (cf. Lê Dũng Tráng and Ramanujam [8] and Teissier [12]).
A μ-equivalence class will also be called *μ-class.*

Remark. A μ-class can consist of several orbits; examples are:

$$x^4 + tx^2y^2 + y^4$$
$$x^5 + y^5 + ux^2y^3 + vx^3y^2 + wx^3y^3$$
$$x^3 + y^3 + z^3 + uxyz.$$

(1.5) The *modality* $m(f)$ of $f \in m$ is the smallest number k such that some neighborhood of (a sufficient k-jet of) f at O is covered by a finite number of no more than m-parametrized families of orbits of $\mathscr{D}$ in m.

Another characterization of modality is as follows.

Let $(\phi_1, \cdots, \phi_{\mu-1})$ be a basis of $m/\Delta(f)$ and let $F_u(x) = f(x) + \Sigma u_i\phi_i$ (versal deformation).

Define $S = \{\mu \in \boldsymbol{C}^{\mu-1} \mid F_u$ is μ-equivalent with $f\}$.

Gabrielov [7] proved that the modality of f is equal to the dimension of S. So $m(f) = \mu(f) - c(f) - 1$ where $c(f) = \operatorname{codim}_f \{g \in m \mid \mu(g) = \mu(f)\}$.

(1.6) Splitting Lemma. *Let $f \in m^2$ then: $f(x_1, \cdots, x_n) \sim g(x_1, \cdots, x_\rho) + Q(x_{\rho+1}, \cdots, x_n)$ with $g \in m^3$ and Q a non-degenerate quadratic form. (Here $\sim$ means Right-equivalent.)*

The number ρ is called the *corank* of f.

Codimension, sufficiency, and modality of f are equal to those of g. Moreover the classification of f follows from the classification of g by adding $Q(x_{\rho+1}, \cdots, x_n)$.

(1.7) Arnold [2] classified in 105 theorems:
1° all singularities with Milnor-number $\mu(f) \leqq 16$.
2° all singularities with modality $m(f) \leqq 2$.

The classification of singularities follows:
1° increasing corank of f.
2° increasing multiplicity ν of f; (f has multiplicity ν if $f \in m^\nu \setminus m^{\nu+1}$).
3° different factorizations of $j^\nu(f)$ (decreasing the number of factors).
In this way one finds (see also Siersma [10], [11]).

Corank 0,1. type $A_n : f = x^{n+1} (n \geqq 0)$.

CORANK 2:

(a) multiplicity $3 \to j^2(f)=0$, $j^3(f)\neq 0$. Factorizations of $j^3(f)$:

(a1) three linear factors

x^2y+y^3

(a2) two linear factors

x^2y

} $f=x^2y+y^k$ type D_k $(k\geqq 4)$

(a3) one linear factor

x^3

$f=x^3+y^4(E_6)$
$f=x^3+xy^3(E_7)$
$f=x^3+y^5(E_8)$
$f=x^3+\alpha xy^4+\beta y^6(J_{10})$, etc.

(b) multiplicity 4:5 different factorizations, etc.

If μ increases the classification becomes more complicated. To work more systematically and to reduce some of the computations we propose, in the case of corank 2, the use of the blowing up construction.

2. The blowing up construction

We consider $f: \boldsymbol{C}^2 \to \boldsymbol{C}$, a holomorphic mapping.

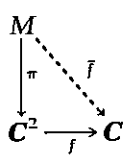
(2.1) Replace $O \in \boldsymbol{C}^2$ by the set of all its tangent-directions (isomorphic to $\boldsymbol{P}^1(\boldsymbol{C})$). We get a manifold M that can be covered by two charts (x_1, y_1) and (x_2, y_2), together with a projection $\pi: M \to \boldsymbol{C}^2$.

The charts and the projection are related by the formulae:

$$\pi(x_1, y_1)=(x_1y_1, y_1)=(x, y)$$
$$\pi(x_2, y_2)=(x_2, x_2y_2)=(x, y).$$

$C=\pi^{-1}(O)=\boldsymbol{P}^1(\boldsymbol{C})$ is given by $y_1=0 \vee x_2=0$ and is called the *exceptional divisor.*

$f: \boldsymbol{C}^2 \to \boldsymbol{C}$ extends in a natural way to a map $\bar{f}: M \to \boldsymbol{C}$:



If f has multiplicity k we have:

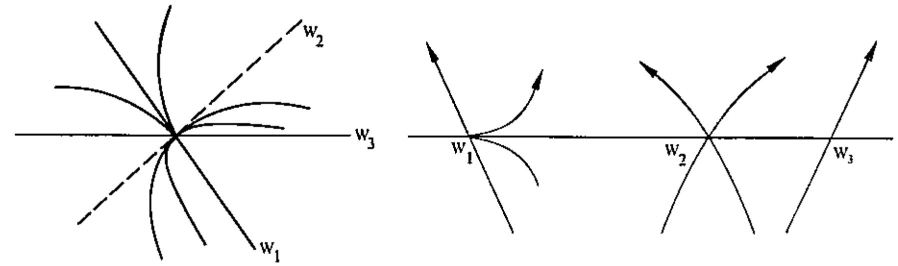
$$\bar{f}(x_1, y_1)=f(x_1y_1, y_1)=y_1{}^k g_1(x_1, y_1)$$

and

$$\bar{f}(x_2, y_2)=f(x_2, x_2y_2)=x_2{}^k g_2(x_2, y_2)$$

for certain g_1 and g_2 defined in a neighborhood of C.

(2.2) The set $\bar{f}^{-1}(O)$ consists of C and some other branches, intersecting C in points $w_1, \cdots, w_s$, corresponding to the tangent directions of $f^{-1}(O)$ in O. Those branches are given by $g_1(x_1, y_1)=0$ or $g_2(x_2, y_2)=0$.



In the neighborhoods of the points $w_1, \cdots, w_s$ we can choose local coordinates (ξ, η) such that $g_i(\xi, \eta)=\xi^n+\eta q(\xi, \eta) (i=1, 2)$ where C is given by $\eta=0$.

If an intersection point is covered by 2 charts then $g_2(x_2, y_2)=x_1{}^k g_1(x_1, y_1)$, so g_1 and g_2 are μ-equivalent near that intersection point.

(2.3) In the following paragraphs we shall obtain the following *results*:

1. The classification of μ-classes of f can be done by local investigations around every intersection point.
2. Different intersection points are treated independently.
3. In each intersection point there is a periodicity in the classification of f.

First we compare the Milnor number of f with the Milnor numbers at the intersection points. The following proposition is due to Pham [9].

(2.4) PROPOSITION. *Let f have an isolated critical point at $O \in \boldsymbol{C}^2$. Let $f^{-1}(O)$ have s different tangent directions $w_1, \cdots, w_s$ in $O \in \boldsymbol{C}^2$. Let M be constructed from $\boldsymbol{C}^2$ by blowing up O and let $\bar{f}: M \to \boldsymbol{C}$ be the natural extension of f. Denote by f_i $(i=1, \cdots, s)$ the restriction of g_1 or g_2 (defined above) to neighborhoods of the points $w_1, \cdots, w_s \in M$.*

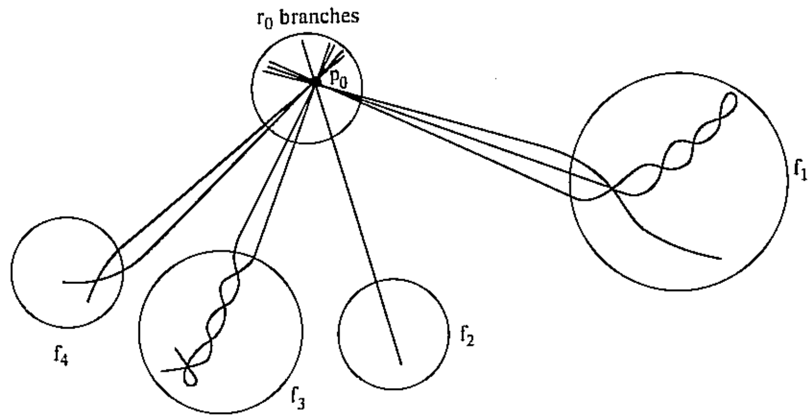
Then: $\mu(f)=r_0(r_0-1)+\sum_{i=1}^{s}\mu(f_i)-s+1$ where r_0 is the multiplicity of f.

PROOF. We consider the real morsification $\tilde{f}$ of f, as constructed by A'Campo [1]. Let $\tilde{C}$ be the set of zeros of $\tilde{f}$, intersected by a small real 2-disc D_ε. The curve $\tilde{C}$ has as its only singularities multiple normal crossings of branches. Let $p_1, \cdots, p_l$ be the points of crossing of $\tilde{C}$, and r_k the number of branches of $\tilde{C}$ around the points p_k.

A'Campo proved: $\mu=\sum_{k=1}^{l} r_k(r_k-1)-q+1$ where q is the number of branches of f.

The morsifications of f_i $(i=1, \cdots, s)$ are constructed from $\tilde{f}$ by restricting

$\bar{f}$, as shown in the following figure:



In this way the set $\{p_k \mid k=0,\cdots,l\}$ is subdivided. We write now $p_{i,j}$ and $r_{i,j}$ in the obvious way. Let p_0 and r_0 correspond to the exceptional divisor C. Let q_i the number of branches of f_i.

Then:

$$\mu(f)=\sum_{k=0}^{l} r_k(r_k-1)-q+1=r_0(r_0-1)+\sum_{i=1}^{s}\sum_j r_{i,j}(r_{i,j}-1)-q+1$$

$$=r_0(r_0-1)+\sum_{i=1}^{s}\left(\sum_j r_{i,j}(r_{i,j}-1)-q_i+1\right)+\sum_{i=1}^{s}(q_i-1)-q+1$$

$$=r_0(r_0-1)+\sum_{i=1}^{s}\mu(f_i)-s+1.$$

3. Periodicity of μ-classes in x^n+m^{n+1}

(3.1) Let $f(x,y)=x^n+p(x,y)$ with multiplicity $\nu(p)\geqq n+1$. We blow up $O\in \mathbf{C}^2$.

Then:

$$\bar{f}(x,y)=f(x_1y_1,y_1)=x_1^ny_1^n+p(x_1y_1,y_1)=y_1^n[x_1^n+y_1q_1(x_1,y_1)]$$

$$\bar{f}(x,y)=f(x_2,x_2y_2)=x_2^n+p(x_2,x_2y_2)=x_2^n[1+x_2q_2(x_2,y_2)]$$

where $q_1(x_1,y_1)=y_1^{-n-1}p(x_1y_1,y_1)$ and $q_2(x_2,y_2)=x_2^{-n-1}p(x_2,x_2y_2)$.
We consider $x_1^n+y_1q_1(x_1,y_1)=0$ and $1+x_2q_2(x_2,y_2)=0$.
The solutions have only one intersection point with the exceptional divisor, namely $(x_1,y_1)=(0,0)$. We next consider the map-germ:
$f_1(x_1,y_1)=y_1^{-n}f(x_1y_1,y_1)=x_1^n+y_1q_1(x_1,y_1)$ in a neighborhood of $(x_1,y_1)=(0,0)$.

(3.2) In order to classify f up to R-equivalence it now seems enough to classify f_1 up to R-equivalence. However we have to take into account the

following:

1° Diffeomorphisms of $\mathbf{C}^2$ lift only to diffeomorphisms of M with a special form.
2° The polynomial $q_1(x_1,y_1)$ is not arbitrary since $q_1(x_1,y_1)=y_1^{-n-1}p(x_1y_1,y_1)$.
Since $\mu(f)=n(n-1)+\mu(f_1)$, we see that $\mu(f)$ depends only on $\mu(f_1)$.

Moreover the μ-class of f follows from the configuration of its resolution. If we apply diffeomorphisms to f_1 such that this resolution is still the same, then we stay in the same μ-class.

The resolution of f depends only on the different possibilities for tangencies of $f_1^{-1}(O)$ to the exceptional divisor C. This can give a subdivision of every μ-class of f_1, each subclass giving a μ-class for f.

(3.3) We now return to $f_1(x_1,y_1)=x_1^n+y_1q_1(x_1,y_1)$.
1° Let the multiplicity $\nu(y_1q_1)\leqq n$.
In this case a detailed study is necessary to find the possibilities for f_1 and the corresponding classes for f. In some sense the singularity f_1 is less complicated than f and is already treated in an earlier part of the classification.

As an example consider $n=3: x_1^3+y_1q_1(x_1,y_1)$.

This singularity must be of type A or D if $\nu(y_1q_1)\leqq 3$.
A detailed study (cf. §5) gives just the following possibilities for f_1:

$$A_0\ A_1\ A_2\quad\times\quad D_4\ D_5\ D_6\ D_7\ D_8\qquad D_k$$

and the following corresponding classes for f:

$$E_6\ E_7\ E_8\quad\times\quad J_{2.0}\ J_{2.1}\ J_{2.2}\ J_{2.3}\ J_{2.4}\qquad J_{2,k-4}$$

(Here and in the following we use the same notations as in Arnold [2]).
2° Let the multiplicity $\nu(y_1q_1)\geqq n+1$.

We now have more or less the same situation as before with $f(x,y)=x^n+p(x,y)$. So we blow up a second time and get: $f_1^1(x_1^1,y_1^1)=(x_1^1)^3+y_1^1q_1^1(x_1^1,y_1^1)$. Now we can omit the detailed study, mentioned above, because of the following lemma:

(3.4) Periodicity Lemma. *There is a 1-1-correspondence between μ-classes of f, defined by f_1 and μ-classes of f defined by f_1^1.*

Proof. Let $f(x,y)=x^n+p(x,y)$ with $\nu(p)\geqq 4$.
1° Arrange the normal form:

$$f(x,y)=x^n+x^{n-2}A_{n-2}(y)+\cdots+xy^nA_1(y)+y^{n+1}A_0(y)$$

$$=x^n+x^{n-2}y^3\sum_{k=0}^{\infty}a_{n-2,k}y^k+\cdots+xy^n\sum_{k=0}^{\infty}a_{1,k}y^k+y^{n+1}\sum_{k=0}^{\infty}a_{0,k}y^k.$$

2° After blowing up:

$$f_1(x_1, y_1)=x_1^n+x_1^{n-2}y_1\sum_{k=0}^{\infty} a_{n-2,k}y_1^k+\cdots+x_1y_1\sum_{k=0}^{\infty} a_{1,k}y_1^k+$$
$$+y_1\sum_{k=0}^{\infty} a_{0,k}y_1^k.$$

So

$$\nu(y_1q_1)\geqq n+1 \Leftrightarrow \begin{cases} a_{n-2,0}=a_{n-2,1}=0 \\ a_{n-3,0}=a_{n-3,1}=a_{n-3,2}=0 \\ a_{0,0}=\cdots\cdots\cdots=a_{0,n-1}=0 \end{cases}$$

If $\nu(y_1q_1)\geqq n+1$ then:

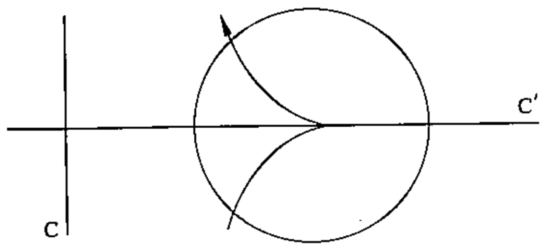
$f_1(x_1, y_1)=$

$$x_1^n+x_1^{n-2}y_1^3\sum_{k=0}^{\infty} a_{n-2,k+2}y_1^k+\cdots+x_1y_1^n\sum_{k=0}^{\infty} a_{1,n-1+k}y_1^k+y_1^{n+1}\sum_{k=0}^{\infty} a_{0,n+k}y_1^k.$$

The normal forms of f and f_1 differ only by a shift of indices

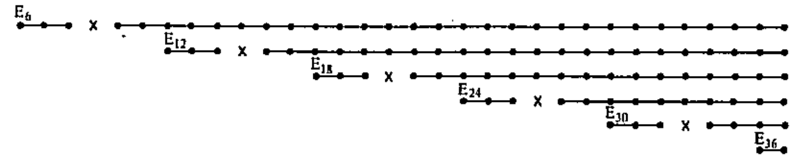
$$\begin{cases} a_{n-2,k+2}\to a_{n-2,k} \\ \quad\cdot \\ \quad\cdot \\ \quad\cdot \\ a_{1,n-1+k}\to a_{1,k} \\ a_{0,n+k}\to a_{0,k} \end{cases}$$

So if we blow up our f a second time we have to consider for $f_1^1(x_1^1, y_1^1)$ the same set of germs as before with $f_1(x_1, y_1)$. However the induced action of diffeomorphisms has become even more complicated. But also in this case the μ-class of f depends only on the possibilities for tangencies of $(f_1^1)^{-1}(O)$ to the exceptional divisors C and C':



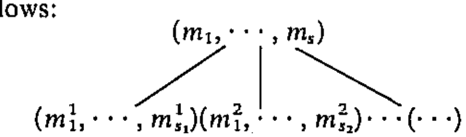
Since $(f_1^1)^{-1}(O)$ and C intersect C' in different points, every μ-class of f_1^1 is subdivided in the same classes as before, when we blow up once.

(3.5) From this lemma follows the periodicity in the classification of μ-classes in $x^n+\mathfrak{m}^{n+1}$. In the case $n=3$ we get the following pattern:



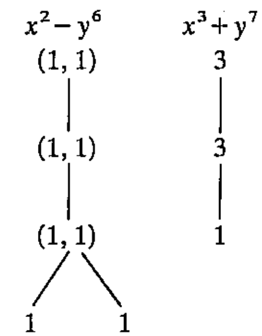
(3.6) Following Zariski's definition of equisingularity, all information about the topological type is contained in the 'resolution-tree', which can be constructed as follows:

Write down the multiplicities $(m_1,\cdots,m_s)$ of the irreducible components of $f^{-1}(O)$, blow up once and do the same for the multiplicities of each f_i. Write this as follows:



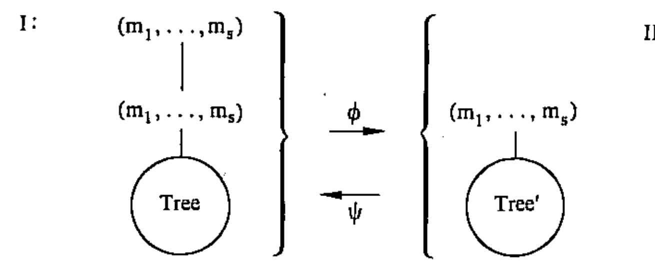
Repeat this until you get everywhere one branch of multiplicity one.

Examples



This construction and the following alternative proof of the periodicity-lemma were communicated to me by P. Slodowy.

Proof. There is a bijective map between trees of class I and trees of class II:



ϕ is defined by blowing up and ψ by transversal blowing down. The multiplicities of the new branches are the same as before.

Transversal blowing down is always possible; use e.g. the normal form:

$$f(x, y) = x^n + x^{n-2}A_{n-2}(y) + \cdots + xy^nA_1(y) + y^{n+1}A_0(y)$$

(3.7) It is possible to start the classification of those resolution-trees and to get in that way the classification of μ-classes. We prefer to use map-germs and to get the classification as a chain of semi-algebraic sets in some jet-space, since it is then possible to compare the modality of f and f_1 (cf. (5.6)).

4. Independence

(4.1) Let f have multiplicity k and let f have s different tangent directions. The k-jet of f can be given by a homogeneous polynomial $j^k(f)$ of degree k, which factors into linear forms as follows:

$$j^k(f) = (\alpha_1 x + \beta_1 y)^{k_1} \cdot \cdots \cdot (\alpha_s x + \beta_s y)^{k_s}.$$

From Hensel's lemma it follows that we can factor f (at least over $\mathbf{C}[[x_1, x_2]]$): $f = f_1 \cdot \cdots \cdot f_s$ where each f_i is given by $f_i(x, y) = (\alpha_i x + \beta_i y)^{k_i} + p(x, y)$, where $\nu(p) > k_i$.

So we can change coordinates such that $f_i(\xi, \eta) = \xi^{k_i} + p(\xi, \eta)$ with $\nu(p) > k_i$.

Definitions. Let Π^k be the set of μ-classes with multiplicity k. Let $k_1 \geqq k_2 \geqq \cdots \geqq k_s \geqq 1$ with $\sum_{i=1}^s k_i = k$. We define $\Pi^k[k_1, \cdots, k_s]$ to be the subset of Π^k, consisting of those $f \in \Pi^k$ such that f has s different tangent-directions $w_1, \cdots, w_s$ such that the corresponding $f_1, \cdots, f_s$ have multiplicities $k_1, \cdots, k_s$.

(4.2) Independence-Lemma. *The map* $\Psi: \Pi^k[k_1, \cdots, k_s] \to \Pi^{k_1}[k_1] \times \cdots \times \Pi^{k_s}[k_s]/\sim$ *defined by* $\Psi(f) = (f_1, \cdots, f_s)$ *is an isomorphism. Here*:

$$(f_1, \cdots, f_i, \cdots, f_j, \cdots, f_s) \sim (f_1, \cdots, f_j, \cdots, f_i, \cdots, f_s) \Leftrightarrow k_i = k_j.$$

Proof. The surjectivity of Ψ is clear. The injectivity follows from the fact that a μ-class is determined by the homeomorphism class of its resolution (cf. Zariski [13]).

Example. (a) $\Pi^4[2, 2] = \Pi^2[2] \times \Pi^2[2]/\sim$: double A-series.
(b) $\Pi^3[2, 1] = \Pi^2[2] \times \Pi^1[1] \cong \Pi^2[2]$: A-series.

(4.3) If we combine the periodicity lemma with the independence lemma we see that singularity classes are repeated in several ways. Moreover in the detailed study of the 'period' in $\Pi^n[n]$ we meet a 1-1-correspondence with earlier treated cases.

This happens e.g. if $f_1 = x_1^n + y_1 q_1(x_1, y_1)$ and $\nu(y_1 q_1) = n$.

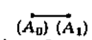
We first apply a coordinate transformation of the type $\begin{cases} x_1 = x_1 + \alpha y_1 \\ y_1 = y_1 \end{cases}$ such that the coefficient of $x_1^{n-1} y_1$ is equal to zero. If the n-jet of f_1 was given by $(x_1 + \beta y_1)^n$ then, after this transformation, $\nu(y_1 q_1) > n$ and the corresponding μ-classes can be studied in the usual way by the periodicity-lemma. In the remaining cases $\nu(y_1 q_1) = n$ and f_1 has two or more tangent directions. The possible μ-classes are already treated in an earlier part of the classification, since in each tangent direction the corresponding singularity has multiplicity less than n. Moreover it is not difficult to show that all these possibilities can occur.

Since $f_1 = x_1^n + y_1 q_1(x_1, y_1)$ and $\nu(y_1 q_1) \geqq n$, all tangent directions are transversal to the exceptional divisor. Consequently there is no subdivision, but a 1-1-correspondence between μ-classes of f_1 and μ-classes of f.

So the full pattern of singularities of multiplicity n and with two or more tangent directions is repeated with a jump in μ of $n(n-1)$.

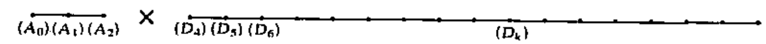
This pattern becomes a part of the 'period' of $\Pi^n[n]$ and shall be repeated by the periodicity lemma.

(4.4) Examples.

$n = 2: \Pi^2[2]$: Period: 

(A_1) has multiplicity 2 and 2 tangent directions.

$n = 3: \Pi^3[3]$: Period:



The D-series has multiplicity 3 and $\geqq 2$ tangent directions.

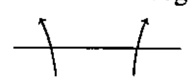
$n = 4: \Pi^4[4]$ see §5.

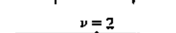
5. Classification of singularities of corank $\leqq 2$ and multiplicity $\nu \leqq 4$

(5.1) As an application of the independence lemma and the periodicity lemma we treat the classification of the singularities, mentioned in the title of this paragraph.

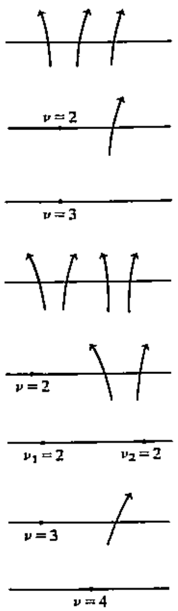
We shall only indicate results, since the classification itself is well-known (cf. Arnold [2], [3], [4], Siersma [11], for $\nu = 3$ see also Briancon [5]). We have to consider the following cases:

$\nu = 1$: The function is regular at O, and can be given by $f(x,y) = x$.

$\nu = 2$: (a)  $\Pi^2[1, 1]$

(b) 

$\Pi^2[2]$



$\nu=3$: (a) $\Pi^3[1, 1, 1]$

(b) $\Pi^3[2, 1]$

(c) $\Pi^3[3]$

$\nu=4$: (a) $\Pi^4[1, 1, 1, 1]$

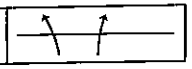
(b) $\Pi^4[2, 2]$

(c) $\Pi^4[2, 2]$

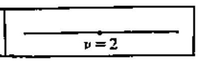
(d) $\Pi^4[3, 1]$

(e) $\Pi^4[4]$

In the above pictures the transform of $f^{-1}(0)$ is given after one blowing up.

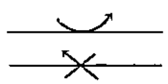
(5.2)	$\Pi^2[1, 1]$	

$\Pi^2[1, 1]=\Pi^1[1]\times\Pi^1[1]$ only one μ-class, type A_1, given by $f(xy)=xy$ or equivalently by $f(x, y)=x^2+y^2$.

(5.3)	$\Pi^2[2]$	

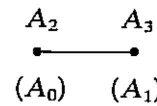
Let $f(x, y)=x^2+p(x, y)$ with multiplicity $\nu(p)\geqq 3$.
We blow up once and get: $f_1(x_1, y_1)=x_1^2+y_1q_1(x_1, y_1)$.
We study the case, that multiplicity $\nu(y_1q_1)\leqq 2$, so:
$f_1(x_1, y_1)=x_1^2+ay_1+bx_1y_1+cy_1^2+\cdots$ and
$f(x, y)=x^2+ay^3+bxy^2+cy^4+\cdots$.

The singularities of f_1 have been classified before, so they must be of type A_0 or A_1. We get the following detailed study:

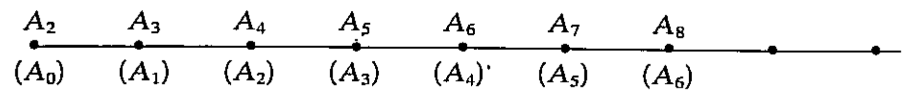
μ		generic $f_1(x_1, y_1)$				generic $f(x, y)$
2		$x_1^2+y_1$		(A_0)	A_2	x^2+y^3
3	$a_1=0$	$x_1^2+y_1^2$		(A_1)	A_3	x^2+y^4

If $b^2-4c=0$, then $x_1^2+bx_1y_1+(b^2/4)y_1^2+\cdots\sim x_1^2+\cdots$ (diffeo respects exc. divisor).

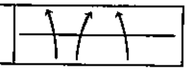
So the detailed study gives the 'period':



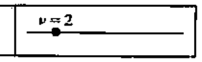
The periodicity lemma gives the classes $A_4, A_5, \cdots, A_{2k}, A_{2k+1}, \cdots$.
So in the $\Pi^2[2]$ we get the following singularity classes:



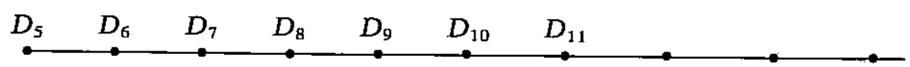
A_k is given by $f(x, y)=x^2+y^{k+1}$.

(5.4)	$\Pi^3[1, 1, 1]$	

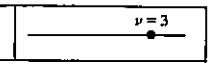
$\Pi^3[1, 1, 1]=\Pi^1[1]\times\Pi^1[1]\times\Pi^1[1]$: only one μ-class, D_4, given by $f(x, y)=x^3+y^3$, or equivalently by $f(x, y)=x^2y+y^3$.

(5.5)	$\Pi^3[2, 1]$	

$\Pi^3[2, 1]\cong\Pi^2[2]\times\Pi^1[1]\cong\Pi^2[2]=A$-series $D_{k+5}\leftrightarrow(A_k, A_0)\leftrightarrow(A_k)$.
We get the types:



A class of type $D_k (k\geqq 4)$ can be given by: $f(x, y)=y(x^2+y^{k-1})$.

(5.6)	$\Pi^3[3]$	

Let $f(x, y)=x^3+p(x, y)$ with multiplicity $\nu(p)\geqq 4$.
Blowing up once we get: $f_1(x_1, y_1)=x_1^3+y_1q_1(x_1, y_1)$.
We study the case, that multiplicity $\nu(y_1q_1)\leqq 3$, so:

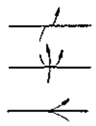
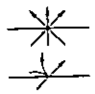
$$f_1(x_1, y_1)=x_1^3+a_1y_1+b_1x_1y_1+b_2y_1^2+c_1x_1^2y_1+c_2x_1y_1^2+c_3y_1^3+\cdots$$

and:

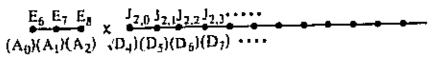
$$f(x, y)=x^3+a_1y^4+b_1xy^3+b_2y^5+c_1x^2y^2+c_2xy^4+c_3y^6+\cdots.$$

The singularities of f_1 have been classified earlier, so they must be of type

A or D. We get the following detailed study:

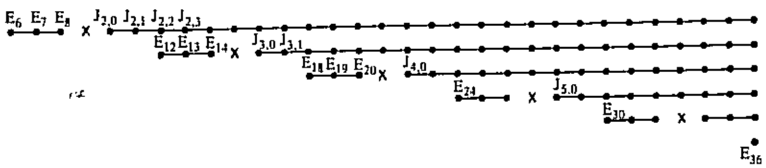
μ		generic $f_1(x_1, y_1)$			generic $f(x, y)$	
6 7 8	 $a_1=0$ $b_1=0$	$x_1^3+y_1$ $x_1^3+x_1y_1$ $x_1^3+y_1^3$		(A_0) (A_1) (A_2)	E_6 E_7 E_8	x^3+y^4 x^3+xy^3 x^3+y^5
×	$b_2=0$	The A_k-singularities with $k \geqq 3$ don't occur				
10 $10+i$		$x_1^3+y_1^2$ $x_1^3+x_1^2y_1+y_1^{3+i}$		(D_4) (D_{i+4})	$J_{2,0}$ $J_{2,i}$	x^3+y^6 $x^3+x^2y^2+y^{6+i}$

So the detailed study gives the 'period':



The periodicity lemma gives classes E_{12}, E_{13}, E_{14}, $J_{3,0}, \cdots, J_{3,i}, \cdots, \cdots,$ $E_{6k}, E_{6k+1}, E_{6k+2}, J_{k+1,0}, \cdots, J_{k+1,i}, \cdots$.

Thus in $\Pi^3[3]$ we get the following pattern of singularity classes:



Remark about modality. First we recall the formula $m(f) = \mu(f) - c(f) - 1$. Consider again the case $\Pi^3[3]$. It is known that $\mu(E_6) = 6$; $c(E_6) = 5$ and $\mu(E_7) = 7$; $c(E_7) = 6$.

The semi-algebraic set A_2 can be constructed from the semi-algebraic set A_1 by one defining equation (of course one has also to change the defining inequalities). After blowing up we see that also E_8 can be constructed from E_7 by one defining equation.

Next we try to do the same for $A_2 \to A_3$. Since the (A_3)-type doesn't occur we see, after blowing up, that the corresponding defining equation for (A_3) must imply the equation for (D_4).

So $c(J_{2,0}) = c(E_8) + 1$ but $\mu(J_{2,0}) = \mu(E_8) + 2$.

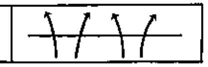
So $m(J_{2,0}) = m(E_8) + 1$.

In the same way one shows that $m(J_{2,i}) = 1$, $m(E_{12}) = m(E_{13}) = m(E_{14}) = 1$. But $m(J_{3,0}) = 2$, etc.

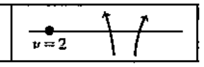
In general: $m(J_{k+1,i}) = k$

$$m(E_{6k}) = m(E_{6k+1}) = m(E_{6k+2}) = k-1.$$

This way of computation of the modality works not only in this example but can be used in general.

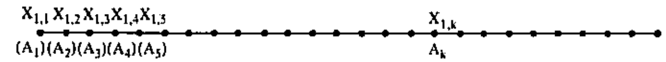
(5.7) $\Pi^4[1, 1, 1, 1]$ 

We get $\Pi^4[1, 1, 1, 1] = \Pi^1[1] \times \Pi^1[1] \times \Pi^1[1]\Pi^1[1]$: one μ-class $X_9 = X_{1,0}$ given by $f(x, y) = x^4 + y^4$.

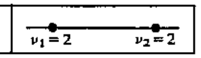
(5.8) $\Pi^4[2, 1, 1]$ 

We get $\Pi^4[2, 1, 1] = \Pi^2[2] \times \Pi^1[1] \times \Pi^1[1] \cong \Pi^2[2] \cong A$-series where $X_{i+9} = X_{1,i} \leftrightarrow (A_i, A_0, A_0) \leftrightarrow (A_i)$.

We get the types:



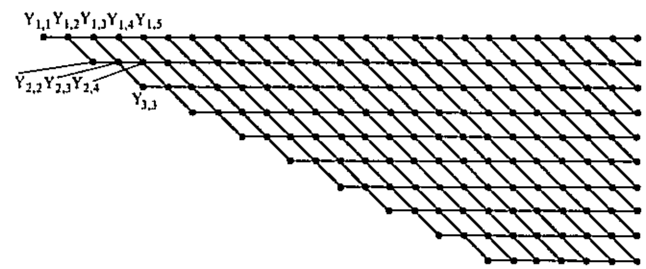
A class of type $X_{1,i}$ can be given by $f(x, y) = x^4 + x^2y^2 + y^{4+i}$.

(5.9) $\Pi^4[2, 2]$ 

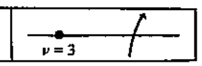
We get $\Pi^4[2, 2] = \Pi^2[2] \times \Pi^2[2]/\sim$.

The type $Y_{p,q}$ corresponds to (A_{p-1}, A_{q-1}).

We have a double-A-series:

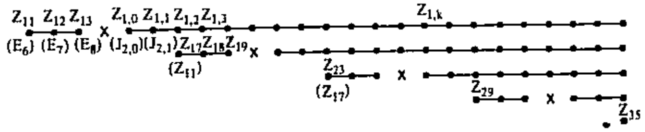


The class $Y_{p,q}$ can be given by $f(x, y) = x^{p+4} + x^2y^2 + y^{q+4}$.

(5.10) $\Pi^4[3, 1]$ 

We get $\Pi^4[3, 1] = \Pi^3[3] \times \Pi^1[1] \cong \Pi^3[3]$.

The type Z_{i+5} corresponds to E_i and $Z_{k-1,i}$ corresponds to $J_{k,i}$.



(5.11) $\Pi^4[4]$ $\nu=4$

Let $f(x, y)=x^4+p(x, y)$ with multiplicity $\nu(p)\geqq 5$.
Blowing up once we get: $f_1(x_1, y_1)=x_1^4+y_1q_1(x_1, y_1)$.
We study the case, that multiplicity $\nu(y_1q_1)\leqq 4$, so:

$$f_1(x_1, y_1)=x_1^4+a_1y_1+b_1x_1y_1+b_2y_1^2+c_1x_1^2y_1+c_2x_1y_1^2+ +c_3y_1^3+d_1x_1^3y_1+d_2x_1^2y_1^2+d_3x_1y_1^3+d_4y_1^4+\cdots$$

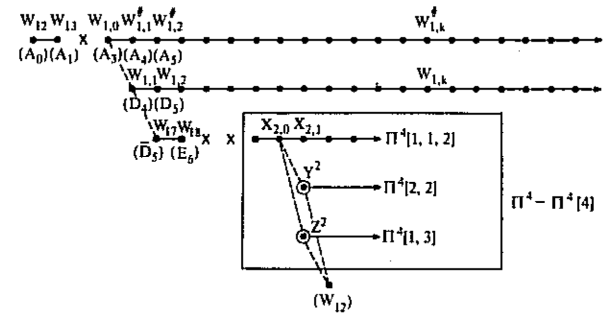
and:

$$f(x, y)=x^4+a_1y^5+b_1xy^4+b_2y^6+c_1x^2y^3+c_2xy^5+c_3y^7+ +d_1x^3y^2+d_2x^2y^4+d_3xy^6+d_4y^8+\cdots.$$

The singularities of f_1 have been classified before; so must be of type A, D, E, J, X, Y or Z.
We get the following detailed study:

μ		generic $f_1(x_1, y_1)$				generic $f(x, y)$
12		$x_1^4+y_1$		(A_0)	W_{12}	x^4+y^5
13	$a_1=0$	$x_1^4+x_1y_1$		(A_1)	W_{13}	x^4+xy^4
	$b_1=0$	The A_2-singularity cannot occur, so we get next:				
15		$x_1^4+y_1^2$		(A_3)	$W_{1,0}$	x^4+y^6
We next get two possibilities: $c_1=0$ and $b_2=0$						
16	$4b_2=c_1^2$	$(y_1+x_1^2)^2+x_1y_1^2$		(A_4)	$W^\#_{1,1}$	$(x^2+y^3)^2+xy^5$
17	$c_2=0$	$(y_1+x_1^2)^2+x_1^2y_1^2$		(A_5)	$W^\#_{1,2}$	$(x^2+y^3)^2+x^2y^4$
$2q+14$		$(y_1+x_1^2)+x_1y_1^{q+1}$		(A_{2q+2})	$W^\#_{1,2q-1}$	$(x^2+y^3)^2+xy^{q+4}$
$2q+15$		$(y_1+x_1^2)+x_1^2y_1^{q+1}$		(A_{2q+3})	$W^\#_{1,2q}$	$(x^2+y^3)^2+x^2y^{q+3}$
16	$b_2=0$	$x_1^4+x_1^2y_1+y_1^3$		(D_4)	$W_{1,1}$	$x^4+x^2y^3+y^7$
17		$x_1^4+x_1^2y_1+y_1^4$		(D_5)	$W_{1,2}$	$x^4+x^2y^3+y^8$
$k+15$		$x_1^4+x_1^2y_1+y_1^{2+k}$		(D_{k+3})	$W_{1,k}$	$x^4+x^2y^3+y^{6+k}$
17		$x_1^4+x_1y_1^2$		(D_5)	W_{17}	x^4+xy^5
18		$x_1^4+y_1^3$		(E_6)	W_{18}	x^4+y^7

The classes E_7 and E_8 don't occur, because of x_1^4.
Next $\nu(y_1q_1)=4$ and because of (4.3) we find all singularities of $\Pi^4-\Pi^4[4]$, so the classes X_1, Y and Z.
Hence the 'period' is:



The periodicity lemma gives the classes W_k, X_k, Y_k, Z_k, etc.

6. Corank 3

(6.1) In the case of corank 3 we need 3 coordinates. The blowing up process now gives $P^2(C)$ as exceptional divisor. The intersections of the branches with the exceptional divisor are now not necessarily isolated but form a 1-dimensional algebraic variety X. So the induced functions $f_\sigma(\sigma\in X)$ can have non-isolated singularities.

If the 3-jet of f has no multiple factor, then f_σ has only singularities in the multiple points of X; in the other points f_σ is of type A_0.

In the cases we consider most f_σ has a singularity-type that has been studied before.

We list the topological classes of f_σ that can occur. Next we assume that different intersection points can be treated independently and compare the results with Arnold's list. The lists are identical and this means that we have found, in this way, all the μ-classes for f.

We cannot prove this directly since (μ constant $\Leftrightarrow$ resolution constant) is not true if $n \geqq 3$.

(6.2) We consider $f(x, y, z)$ with the multiplicity of f equal to 3. The corresponding variety $X \subset P^2(\mathbf{C})$ is given by $j^3(f) = 0$.
We have the following possibilities:
(a) $f = xyz + x^3 + y^3 + z^3 + p(x, y, z)$
X is given by $xyz + x^3 + y^3 + z^3 = 0$ and is an elliptic curve without multiple points.
The type of this singularity is called P_8.
(b) $f = xyz + x^3 + y^3 + p(x, y, z)$
X is given by $xyz + x^3 + y^3 = 0$ and has one double point σ, where $f_\sigma = x_\sigma y_\sigma + x_\sigma^3 + y_\sigma^3 + z_\sigma q_\sigma$.
We get a singularity of type $\Pi^2[2]$, which all occur.
f_σ of type $A_k \leftrightarrow f$ of type P_{k+2} $(k \geqq 1)$.
(c) $f = x^2 z + y^3 + p(x, y, z)$
X is given by $x^2 z + y^3 = 0$ and has one cusp point $\sigma = (0{:}0{:}1)$ where $f_\sigma = x_\sigma^2 + y_\sigma^3 + z_\sigma q_\sigma$.
This singularity is of type E or J and all singularities of $\Pi^3[3]$ occur.
f_σ of type $E_k \leftrightarrow f$ of type Q_{k+4}
f_σ of type $J_{k,i} \leftrightarrow f$ of type $Q_{k,i}$.
(d) $f = xyz + x^3 + p(x, y, z)$
X is given by $xyz + x^3 = 0$ and has two double points σ and τ, where:

$$f_\sigma = x_\sigma z_\sigma + x_\sigma^3 + y_\sigma q_\sigma$$
$$f_\tau = x_\tau y_\tau + x_\tau^3 + z_\tau q_\tau$$

Both are singularities of type A, and all possible combinations can be found.

$$\left\{\begin{matrix} f_\sigma \text{ of type } A_{p-4} \\ f_\tau \text{ of type } A_{q-4} \end{matrix}\right\} \leftrightarrow f \text{ of type } R_{p,q} = T_{3,p,q}.$$

This singularity class is isomorphic to $\Pi^2[2] \times \Pi^2[2]/\sim \,\cong \Pi^4[2, 2]$.
(e) $f = x^2 z + yz^2 + p(x, y, z)$
X is given by $x^2 z + yz^2 = 0$ and has one multiple point σ, where

$$f_\sigma = x_\sigma^2 z_\sigma + z_\sigma^2 + y_\sigma q_\sigma \sim z_\sigma^2 + x_\sigma^4 + y_\sigma q_\sigma.$$

We get the singularities of $\Pi^4[4]$, which all occur.

$$f_\sigma \text{ of type } W_p \leftrightarrow f \text{ of type } S_{p-1}$$
$$f_\sigma \text{ of type } W_{k,i} \leftrightarrow f \text{ of type } S_{k,i}$$
$$f_\sigma \text{ of type } W_{k,i}^{\#} \leftrightarrow f \text{ of type } S_{k,i}^{\#}$$

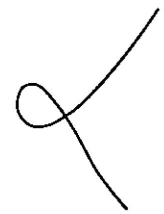
etc., in general:

$$f_\sigma \text{ of type } W_k \leftrightarrow f \text{ of type } S_k$$
$$f_\sigma \text{ of type } X_k \leftrightarrow f \text{ of type } SP_k$$
$$f_\sigma \text{ of type } Y_k \leftrightarrow f \text{ of type } SR_k$$
$$f_\sigma \text{ of type } Z_k \leftrightarrow f \text{ of type } SQ_k.$$

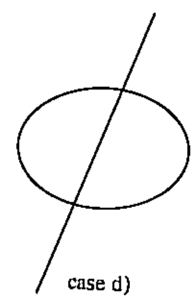
(f) $f = xyz + p(x, y, z)$
X is given by $xyz = 0$ and has three multiple points σ, τ and μ, where

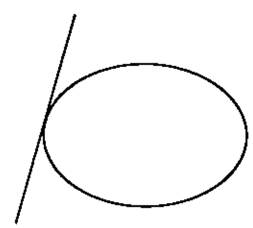
$$f_\sigma = y_\sigma z_\sigma + x_\sigma q_\sigma$$
$$f_\tau = x_\tau z_\tau + y_\tau q_\tau$$
$$f_\mu = x_\mu y_\mu + z_\mu q_\mu$$

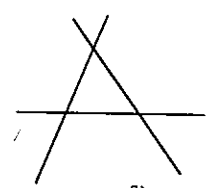

case a)

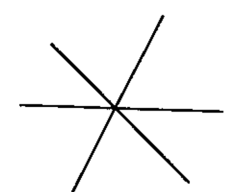

case b)


case c)


case d)


case e)


case f)


case g)

We get three singularities of type A and all combinations can be obtained:

$$\left.\begin{matrix} f_\sigma \text{ of type } A_{p-4} \\ f_\tau \text{ of type } A_{q-4} \\ f_\mu \text{ of type } A_{r-4} \end{matrix}\right\} \leftrightarrow f \text{ of type } T_{p,q,r}.$$

(g) $f = x^3 + xz^2 + p(x, y, z)$
X is given by $x^3 + xz^2 = 0$ and has one multiple point, where

$$f_\sigma = x_\sigma^3 + x_\sigma z_\sigma^2 + y_\sigma q_\sigma.$$

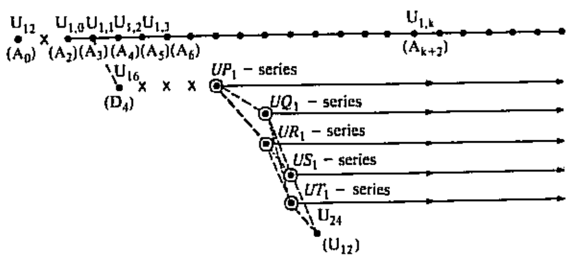
The following possibilities can occur:

μ	generic $f_\sigma(x_\sigma, y_\sigma, z_\sigma)$			generic $f(x, y, z)$
12	$x_\sigma^3 + x_\sigma z_\sigma^2 + y_\sigma$	(A_0)	U_{12}	$x^3 + xz^2 + y^4$
×	The A_1-*singularity cannot occur*			
$2q+14$ $2q+15$ 16	$x_\sigma^3 + x_\sigma z_\sigma^2 + x_\sigma y_\sigma + z_\sigma y_\sigma^{q+1}$ $x_\sigma^3 + x_\sigma z_\sigma^2 + x_\sigma y_\sigma + z_\sigma^2 y_\sigma^{q+1}$ $x_\sigma^3 + x_\sigma z_\sigma^2 + y_\sigma^2$	(A_{2q+2}) (A_{2q+3}) (D_4)	$U_{1,2q}$ $U_{1,2q+1}$ U_{16}	$x^3 + xz^2 + xy^3 + zy^{3+q}$ $x^3 + xz^2 + xy^3 + z^2y^{2+q}$ $x^3 + xz^2 + y^5$
	The higher order corank 2 singularities don't occur.			

The next case is: $x_\sigma^3 + x_\sigma z_\sigma^2 + y_\sigma q_\sigma$ with multiplicity $\nu(y_\sigma q_\sigma) \leqq 3$.
The 3-jet is

$$x_\sigma^3 + x_\sigma z_\sigma^2 + y_\sigma(\alpha x_\sigma^2 + \beta y_\sigma^2 + \gamma z_\sigma^2 + \delta x_\sigma y_\sigma + \varepsilon y_\sigma z_\sigma + \eta z_\sigma x_\sigma).$$

We have no restrictions on the type of the 3-jet and find all singularities of type P, R, T, Q and S.
We call the corresponding types of f: UP_1, UR_1, UT_1, UQ_1 and US_1.
So the 'period' of $x^3 + xz^2$ is given by:



7. Classification of corank 2 and multiplicity $\nu = 5$

$$\Pi^5 = \Pi^5[1,1,1,1,1] \cup \Pi^5[2,1,1,1] \cup \Pi^5[3,1,1] \cup \Pi^5[2,2,1] \cup \Pi^5[4,1] \cup \Pi^5[3,2] \cup \Pi^5[5].$$

(7.1) (a) $\Pi^5[1,1,1,1,1] \cong \Pi^1[1]$: one μ-class.
(b) $\Pi^5[2,1,1,1] \cong \Pi^2[2]$: A-series.
(c) $\Pi^5[3,1,1] \cong \Pi^3[3]$: E/J-series.
(d) $\Pi^5[2,2,1] \cong \Pi^2[2] \times \Pi^2[2]/\sim$: double A-series.
(e) $\Pi^5[4,1] \cong \Pi^4[4]$.
(f) $\Pi^5[3,2] \cong \Pi^3[3] \times \Pi^2[2]$: $E/J \times A$ series.
(g) $\Pi^5[5]$ shall be described next:

(7.2) $\boxed{\Pi^5[5]}$ $\nu = 5$

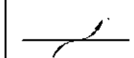
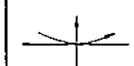
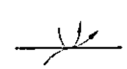
Let $f(x, y) = x^5 + p(x, y)$ with $\nu(p) \geqq 5$. Blowing up once we get: $f_1(x_1, y_1) = x_1^5 + y_1 q_1(x_1, y_1)$. We first study the case $\nu(y_1 q_1) \leqq 5$ so:

$$\begin{aligned} f_1(x_1, y_1) = x_1^5 &+ a_1 y_1 + b_1 x_1 y_1 + b_2 y_1^2 + c_1 x_1^2 y_1 + c_2 x_1 y_1^2 \\ &+ c_3 y_1^3 + d_1 x_1^3 y_1 + d_2 x_1^2 y_1^2 + d_3 x_1 y_1^3 + d_4 y_1^4 \\ &+ e_1 x_1^4 y_1 + e_2 x_1^3 y_1^2 + e_3 x_1^2 y_1^3 + e_4 x_1 y_1^4 + e_5 y_1^5 + \cdots \end{aligned}$$

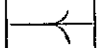
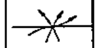
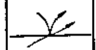
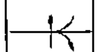
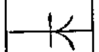
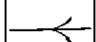
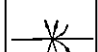
and

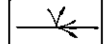
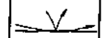
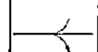
$$\begin{aligned} f(x, y) = x^5 &+ a_1 y^6 + b_1 x y^5 + b_2 y^7 + c_1 x^2 y^4 + c_2 x y^6 + c_3 y^8 \\ &+ d_1 x^3 y^3 + d_2 x^2 y^5 + d_3 x y^7 + d_4 y^9 + e_1 x^4 y^2 \\ &+ e_2 x^3 y^4 + e_3 x^2 y^6 + e_4 x y^8 + e_5 y^{10} + \cdots \end{aligned}$$

The singularities of f_1 must be classified before.
We get the following detailed study:

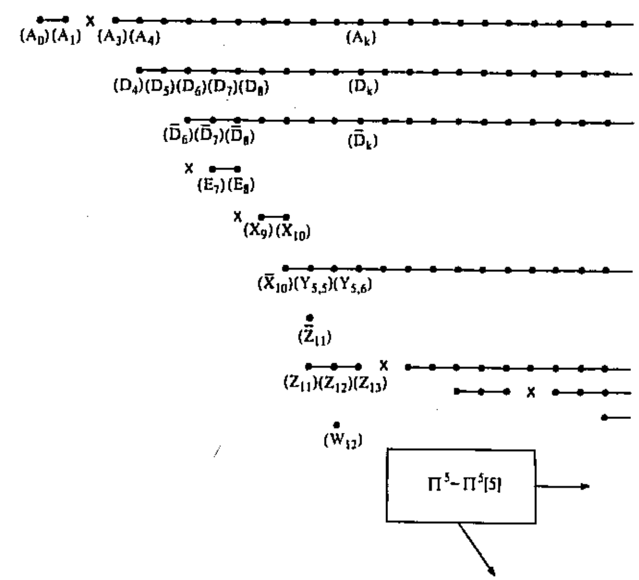
μ		generic $f_1(x_1, y_1)$				generic $f(x, y)$
20		$x_1^5 + y_1$		(A_0)		$x^5 + y^6$
21	$a_1 = 0$	$x_1^5 + x_1 y_1$		(A_1)		$x^5 + xy^5$
×	$b_1 = 0$	The A_2-singularity cannot occur, so we have:				
23		$x_1^5 + y_1^2 + x_1^2 y_1$		(A_3)		$x^5 + y^7 + x^2 y^4$

We next get two possibilities: $c_1=0$ and $b_2=0$

24	$c_1=0$	$x_1^5+y_1^2$		(A_4)	x^5+y^7
–		Other singularities of type A_k don't occur, because of x_1^5.			
24	$b_2=0$	$x_1^5+x_1^2y_1+y_1^3$		(D_4)	$x^5+x^2y^4+y^8$
25		$x_1^5+x_1^2y_1+y_1^4$		(D_5)	$x^5+x^2y^4+y^9$
$20+k$		$x_1^5+x_1^2y_1+y_1^{k-1}$		(D_k)	$x^5+x^2y^4+y^{k+4}$
26		$x_1^5+x_1y_1^2$		$(\bar{D}_6)$	x^5+xy^6
27		$x_1(y_1-x_1^2)^2+y_1x_1^4$		$(\bar{D}_7)$	$x(y^3-x^2)^2+x^4y^2$
$20+2n$ $21+2r$		$x_1(y_1-x_1^2)^2+y_1^{n-1}x_1 (n\geqq 4)$ $x_1(y_1-x_1^2)^2+y_1^n (n\geqq 4)$		$(\bar{D}_{2n})$ $(\bar{D}_{2n+1})$	$x(y^3-x^2)^2+xy^{n+3}$ $x(y^3-x^2)^2+y^{n+5}$
x		The E_6 singularity cannot occur			
27		$x_1^5+x_1^3y_1+y_1^3$		(E_7)	$x^5+x^3y^3+y^8$
28		$x_1^5+y_1^3$		(E_8)	x^5+y^8
–		Other singularities of type J or E cannot occur			
29		$x_1^5+x_1^3y_1+y_1^4$		(X_9)	$x^5+x^3y^3+y^9$
30		$x_1^5+x_1^2y_1^2+y_1^4$		(X_{10})	$x^5+x^2y^5+y^9$
30		$y_1(x_1-y_1)^2(x_1-2y_1)+x_1^5$		$(\bar{X}_{10})$	$x^5+(x-y^2)^2 \times(x-2y^2)y^3$

–		Other singularities of type X cannot occur.			
31		$x_1^5+x_1^2y_1^2+y_1^5$		$(Y_{5,5})$	$x^5+x^2y^5+y^{10}$
$26+k$		$x_1^5+x_1^2y_1^2+y_1^k$		$(Y_{5,k})$	$x^5+x^2y^5+y^{5+k}$
–		Other singularities of type Y and other tangentdirections don't occur.			
31		$x_1^5+x_1y_1^3$		$(\bar{Z}_{11})$	x^5+xy^7
31		$x_1^5+x_1^3y_1+y_1^5$		(Z_{11})	$x^5+x^3y^3+y^{10}$
32		$x_1^5+x_1^3y_1+x_1y_1^4$		(Z_{12})	$x^5+x^3y^3+xy^8$
33		$x_1^5+x_1^3y_1+y_1^6$		(Z_{13})	$x^5+x^3y^3+y^{11}$
		Also all other singularities of type Z occur.			
32		$x_1^5+y_1^4$		(W_{12})	x^5+y^9
–		Other singularities of type W don't occur.			

Next $\nu(y_1q_1)=5$ and because of (4.3) we find all singularities of $\Pi^5-\Pi^5[5]$. So the 'period' is:



8. Morsifications and intersection forms

The general reference for this paragraph is A'Campo [1].

(8.1) A *morsification* h of f is a nearby germ h, with only non-degenerate critical points.

A'Campo constructed for every μ-class of $f:\mathbf{C}^2 \to \mathbf{C}$ a morsification with real critical points, using the blowing up construction.

The intersection $h^{-1}(O) \cap \mathbf{R}^2$ consists of a curve C with only double points and:

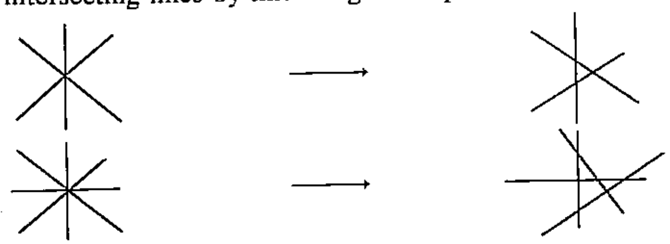
$$\text{(number of double points)} + \text{(number of regions)} = \mu.$$

The vanishing cycles and their intersections are determined by the curve C.

EXAMPLE. (E_8)

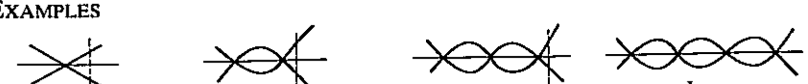
We simplify this picture to

In the following pictures (and those of table III) one has to replace the multiple intersecting lines by lines in general position, as follows:



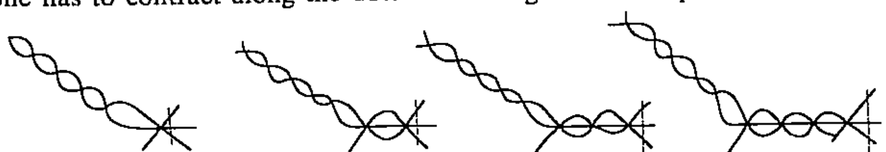
(8.2) The periodicity of the classification induces a periodicity of morsifications and intersection forms.

EXAMPLES

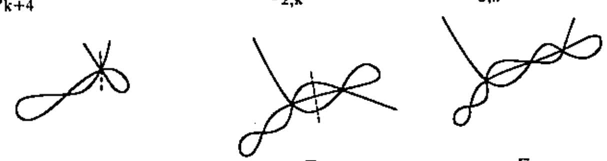


D_4 $J_{2,0}$ $J_{3,0}$ $J_{4,0}$

(one has to contract along the dotted line to get the next picture).



D_{k+4} $J_{2,k}$ $J_{3,k}$ $J_{4,k}$



E_8 E_{14} E_{20} etc.

TABLE I. Classification of corank ≤ 2 and multiplicity $\nu \leq 4$.

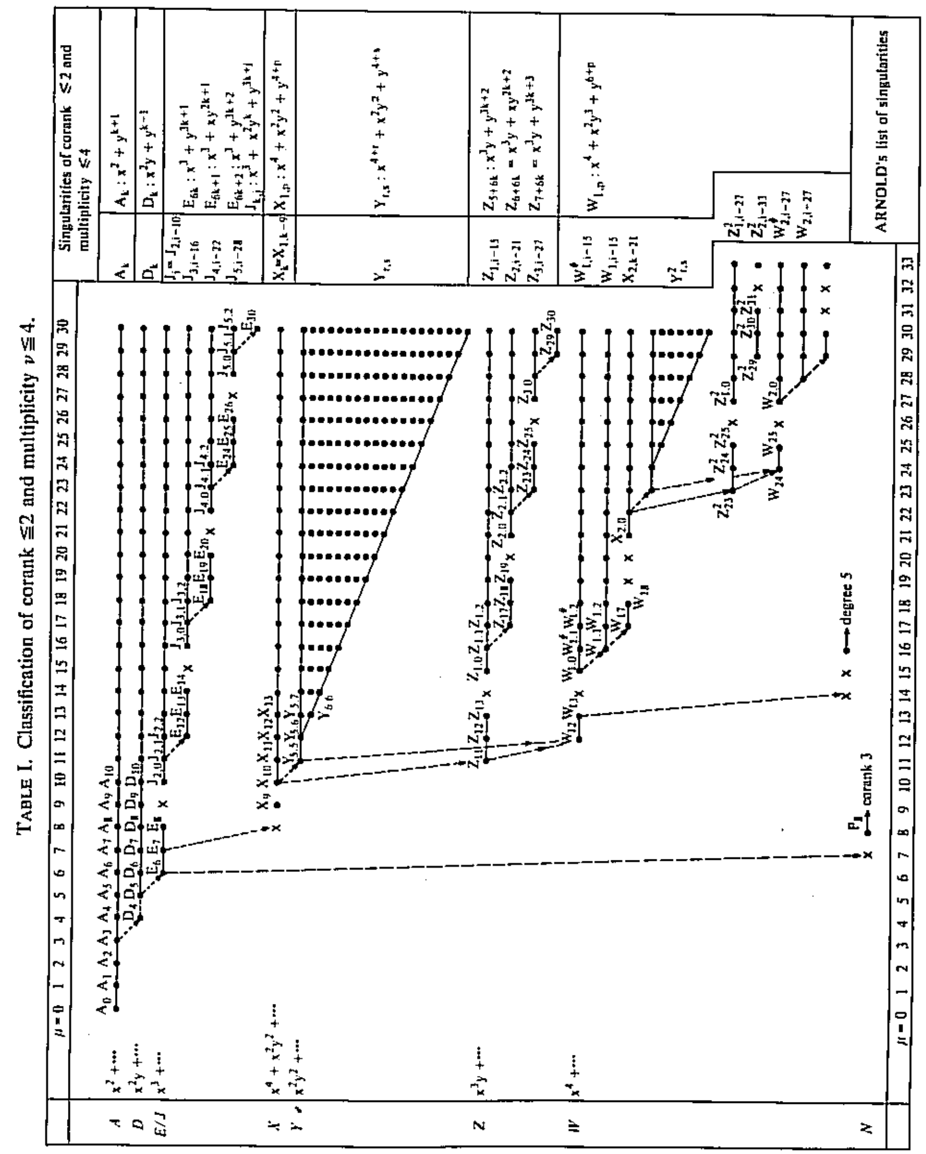


TABLE II. Singularities with $\rho = 3$ and reduced 3-jets.

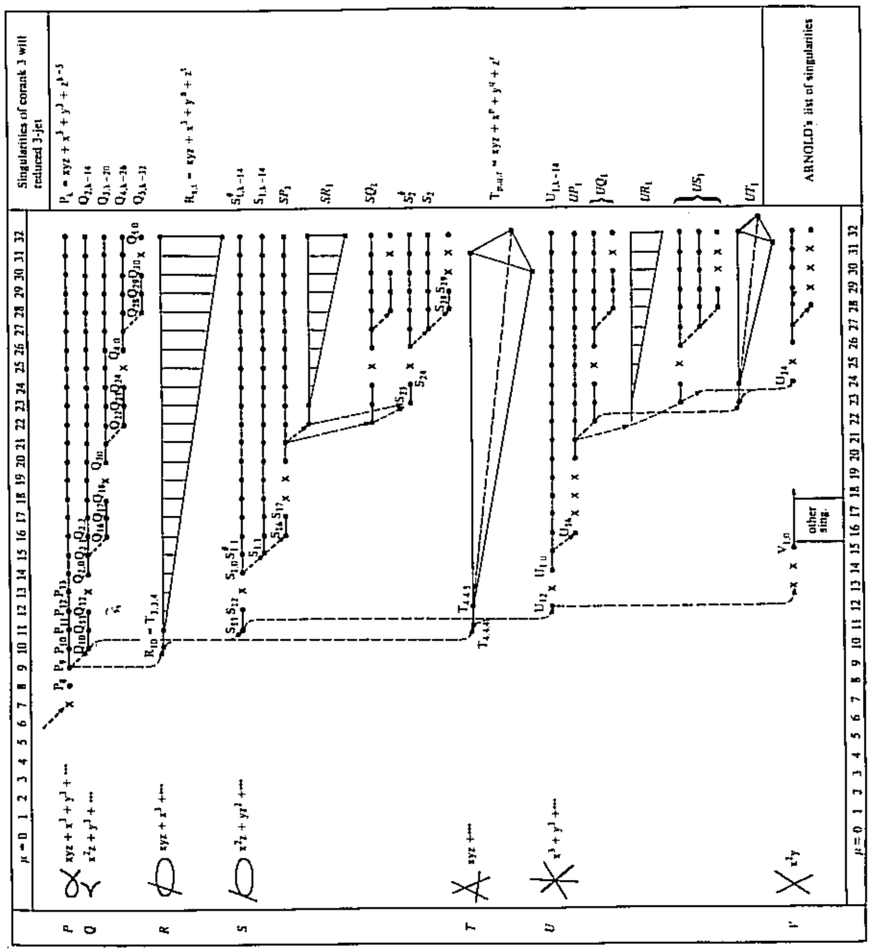
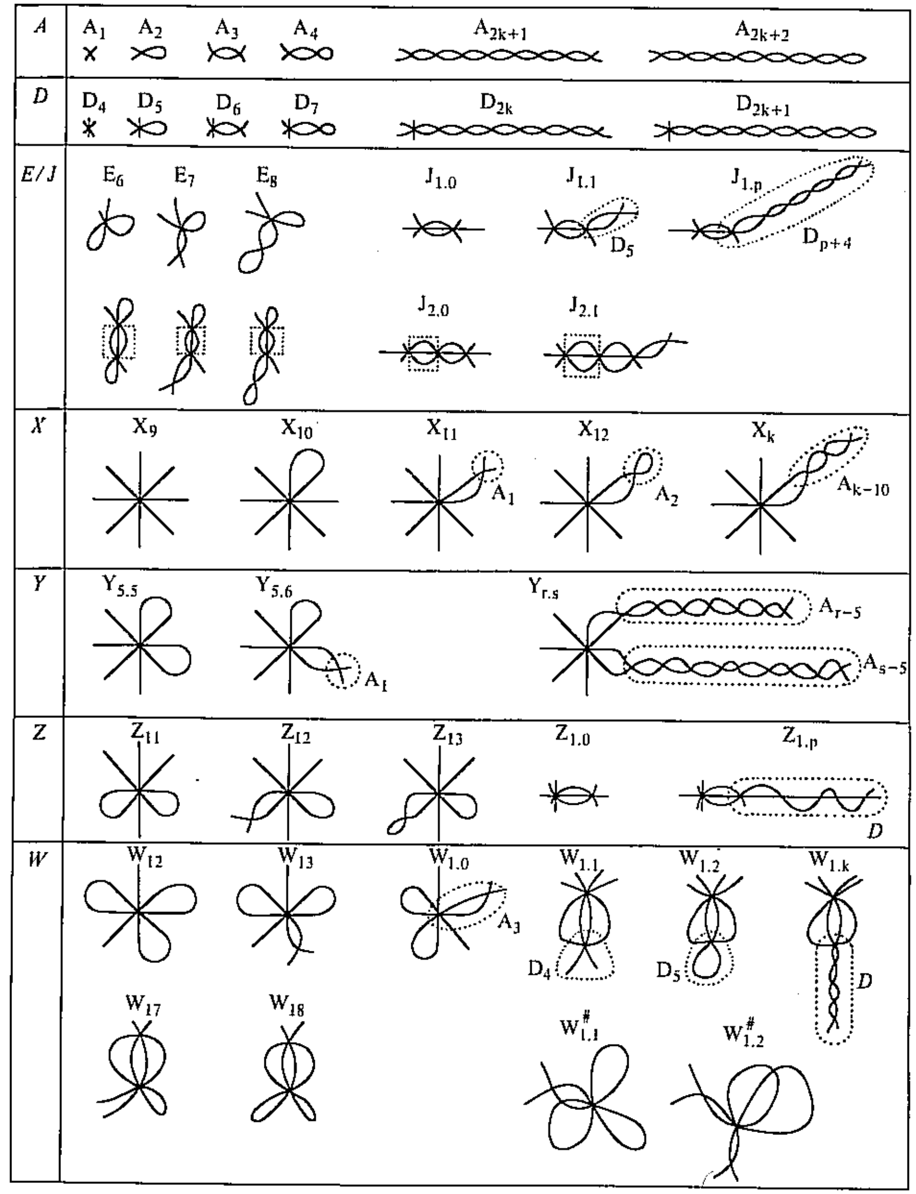


TABLE III. List of morsifications.



REFERENCES

[1] A'Campo, N., Le groupe de monodromie du déploiment des singularité's isolées de courbes planes I. Math. Ann. 213, p. 1–32 (1975).

[2] Arnold, V. I., Local normal forms of functions. Inventiones math. 35, 87–109 (1976). The Russian version is also contained in Uspekhi Mat. Nauk, tome XXX, 5 (1975) 3–65.

[3] Arnold, V. I., Critical points of smooth functions. The Vancouver International Congress of Mathematicians 1974. The Russian version is also contained in Uspekhi Mat. Nauk, tome XXX, 5 (1975) 3–65.

[4] Arnold, V. I., Normal forms of functions in the neighborhoods of degenerate critical points. Russ. Math. Surveys, *29:* 2, 11–49 (1974).

[5] Briançon, J., Quelques calculs inutiles sur les germes de courbes planes d'ordre 3. Université de Nice, dec. 1973.

[6] Demazure, M., Sousgroupes algébrique maximum du groupe de Cremona. Annales Sc. de l'E.N.S., 4–e serie, t. 3, fasc. *4*, 507–588 (1970).

[7] Gabrielov, A. M., Bifurcations, Dynkin diagrams and modality of isolated singularities. Funktional'nyi Analiz i Ego Prilozheniya, Vol. 8, No. 2, 7–12 (1974).

[8] Lê Dũng Tráng, Ramanujam, C. P., The invariance of Milnor's number implies the invariance of the topological type. Centre de Mathématique de l'École Polytechnique, Paris 1973.

[9] Pham, F., Singularités planes d'ordre 3. Astérisque 7–8, 363–391 (1973).

[10] Siersma, D., The singularities of C^∞-functions of right-codimension smaller or equal than eight. Indag. Math. 35, No 1, p. 31–37, (1973).

[11] Siersma, D., Classification and deformation of singularities. Thesis. University of Amsterdam, (1974).

[12] Teissier, B., Cycles evanescent, sections plane et conditions de Whitney. Astérisque 7–8, 285–362 (1973).

[13] Zariski, O., Studies in equisingularity I. Equivalent singularities of plane algebroid curves. Amer. J. Math. *87* (1965) 507–536.

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