Non reduced plane curve singularities with $b_1(F) = 0$ and Bobadilla’s question

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Abstract

If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to $x^r$.

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1. Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function germ. What can be said about functions whose Milnor fibre $F$ has the property $b_i(F) = 0$ for all $i \geq 1$? If $F$ is connected then $f$ is non-singular and equivalent to a linear function by A’Campo’s trace formula. The remaining question: What happens if $F$ is non-connected? is only relevant for non-reduced plane curve singularities.

This question is related to a recent paper [HM]. That paper contains a statement about the so-called Bobadilla conjectures [Bo] in case of plane curves. The invariant $\beta = 0$, used by Massey [Ma] should imply that the singular set of $f$ is a smooth line.

In this note we give a short topological proof of a stronger statement.

Proposition 1.1. If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to $x^r$.

Corollary 1.2. In the above case the singular set is a smooth line and the system of transversal singularities is trivial.

2. Non-reduced plane curves

Non-isolated plane curve singularities have been thoroughly studied by Rob Schrauwen in his dissertation [Sch1]. Main parts of it are published as [Sch2] and [Sch3]. The above Proposition 1.1 is an easy consequence of his work.
We can assume that $f = f_1^{m_1} \cdots f_r^{m_r}$ (partition in powers of reduced irreducible components).

**Lemma 2.1.** Let $d = \gcd(m_1, \ldots, m_r)$

(a.) $F$ has $d$ components, each diffeomorphic to the Milnor fibre $G$ of $g = g_1^{m_1} \cdots g_r^{m_r}$.

The Milnor monodromy of $f$ permutes these components,

(b.) if $d = 1$ then $F$ is connected.

**Proof.** (a.) Since $f = g^d$ the fibre $F$ consists of $d$ copies of $G$.
(b.) We recall here the reasoning from [Sch1]. Deform the reduced factors $f_i$ into $\hat{f}_i$ such that the product $\hat{f}_1 \cdots \hat{f}_r = 0$ contains the maximal number of double points (cf. Figure 1). This is called a network deformation by Schrauwen. The corresponding deformation $\hat{f}$ of $f$ near such a point has local equation are of the form $x^p y^q = 0$ (point of type $D[p, q]$).

![Figure 1: Deformation to maximal number of double points.](image)

Near every branch $\hat{f}_i = 0$ the Milnor fibre is a $m_i$-sheeted covering of the zero-locus, except in the $D[p, q]$-points. We construct the Milnor fibre $F$ of $f$ starting with $S = \sum m_i$ copies of the affine line $\mathbb{A}$. Cover the $i$th branch with $m_i$ copies of $\mathbb{A}$ and delete $(p + q)$ small discs around the $D[p, q]$-points. Glue in the holes $\gcd(p, q)$ small annuli (the Milnor fibres of $D[p, q]$). The resulting space is the Milnor fibre $F$ of $f$.

A hyperplane section of a generic at a generic point of $\hat{f}_i = 0$ defines a transversal Milnor fibre $F_1^{th}$. Start now the construction of $F$ from $F_1^{th}$, which consists of $m_1$ cyclic ordered points. As soon as $f_1 = 0$ intersects $f_k = 0$ it connects the sheets of $f_1 = 0$ modulo $m_k$. Since $\gcd(m_1, \ldots, m_r) = 1$ we connect all sheets.

**Proof of Proposition 1.1.** If $b_1(F) = 0$, then also $b_1(G) = 0$. The Milnor monodromy has trace($T_g$) = 1. According to A’Campo’s observation [AC] $g$ is regular: $g = x$. It follows that $f = x^r$.

**3. Relation to Bobadilla’s question**

We consider first in any dimension $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with a 1-dimensional singular set, see especially the 1991-paper [Si] for definitions, notations and statements.

We focus on the group $H_n(F, F^{th})$ which occurs in two exact sequences on p. 468 of [Si]:

$$0 \to H_{n-1}^i(F) \to H_{n-1}(F^{th}) \to H_n(F) \oplus H_{n-1}^{r}(F) \to 0$$
Here $F^\dagger$ is the disjoint union of the transversal Milnor fibres $F_i^\dagger$, one for each irreducible branch of the 1-dimensional singular set.\footnote{$F^\dagger$ was originally denoted by $F'$. In the second sequence a misprint $n$ in the third term has been changed to $n-1$.}

Note that $H_n(F)$, $H_n(F, F^\dagger)$ and $H_{n-1}(F^\dagger)$ are free groups. $H_{n-1}(F)$ can have torsion, we denote its free part by $H_{n-1}(F)'$ and its torsion part by $H_{n-1}(F)^t$. All homologies here are taken over $\mathbb{Z}$, but also other coefficients are allowed.

From both sequences it follows that the $\beta$-invariant, introduced in [Ma] has a 25 years history, since is nothing else than:

$$\dim H_n(F, F^\dagger) = b_n - b_{n-1} + \sum \mu_i^\dagger := \beta$$

From this definition is immediately clear that $\beta \geq 0$ and that $\beta$ is topological. The topological definition has as direct consequence:

**Proposition 3.1.** Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with a 1-dimensional singular set, then:

$$\beta = 0 \iff \chi(F) = 1 + (-1)^n \sum \mu_i^\dagger \iff H_n(F, \mathbb{Z}) = 0 \text{ and } H_{n-1}(F, \mathbb{Z}) = \mathbb{Z} \sum \mu_i$$

The original Bobadilla conjecture C [Bo] was in [Ma] generalized to the reducible case as follows: Does $\beta = 0$ imply that the singular set is smooth? As consequence of our main Proposition 1.1 we have:

**Corollary 3.2.** In the curve case $\beta = 0$ implies that the singular set is smooth; and that the function is equivalent to $x^r$.

**Remark 3.3.** In [HM] the first part of this corollary was obtained with the help of Lê numbers.

**Remark 3.4.** From the definition $\beta = H_n(F, F^\dagger)$ follow direct and short proofs of several statements from [Ma].

An other consequence from [Si] is the composition of surjections:

$$H_{n-1}(F^\dagger) = \oplus \mathbb{Z}^{\mu_i} \twoheadrightarrow H_{n-1}(\partial_2 F) = \oplus \frac{\mathbb{Z}^{\mu_i}}{A_i - I} \twoheadrightarrow H_{n-1}(F)$$

From this follows:

**Proposition 3.5.** If $\dim H_{n-1}(F) = \sum \mu_i$ (upper bound) then
a. $H_{n-1}(\partial_2 F)$ and $H_{n-1}(F)$ are free and isomorphic to $\mathbb{Z}^{\sum \mu_i}$.

b. All transversal monodromies $A_i$ are the identity.

The second part of [Ma] contains an elegant statement about $\beta = 1$ via the A’Campo trace formula. Also the reduction of the generalized Bobadilla conjecture to the (irreducible) Bobadilla conjecture. As final remark: The great work (the irreducible case) has still has to be done! Together with the Lê-conjecture this seems to be an important question in the theory of hypersurfaces 1-dimensional singular sets.

References


