Non reduced plane curve singularities with $b_1(F) = 0$ and Bobadilla's question

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Abstract

If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to x^r.

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1. Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function germ. What can be said about functions whose Milnor fibre F has the property $b_i(F) = 0$ for all $i \geq 1$? If F is connected then f is non-singular and equivalent to a linear function by A'Campo's trace formula. The remaining question: *What happens if F is non-connected ?* is only relevant for non-reduced plane curve singularities.

This question is related to a recent paper [HM]. That paper contains a statement about the so-called Bobadilla conjectures [Bo] in case of plane curves. The invariant $\beta = 0$, used by Massey [Ma] should imply that the singular set of f is a smooth line.

In this note we give a short topological proof of a stronger statement.

Proposition 1.1. *If the first Betti number of the Milnor fibre of a plane curve singularity is zero, then the defining function is equivalent to x^r.*

Corollary 1.2. *In the above case the singular set is a smooth line and the system of transversal singularities is trivial.*

2. Non-reduced plane curves

Non-isolated plane curve singularities have been thoroughly studied by Rob Schrauwen in his dissertation [Sch1]. Main parts of it are published as [Sch2] and [Sch3]. The above Proposition 1.1 is an easy consequence of his work.

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We can assume that $f = f_1^{m_1}.\cdots.f_r^{m_r}$ (partition in powers of reduced irreducible components).

Lemma 2.1. *Let $d = \gcd(m_1, \cdots, m_r)$*

(a.) F has d components, each diffeomorphic to the Milnor fibre G of $g = g_1^{\frac{m_1}{d}}.\cdots.g_r^{\frac{m_r}{d}}$. The Milnor monodromy of f permutes these components,

(b.) if $d = 1$ then F is connected.

Proof. (a.) Since $f = g^d$ the fibre F consists of d copies of G.
(b.) We recall here the reasoning from [Sch1]. Deform the reduced factors f_i into $\hat{f}_i$ such that the product $\hat{f}_1.\cdots.\hat{f}_r = 0$ contains the maximal number of double points (cf. Figure 1). This is called a network deformation by Schrauwen. The corresponding deformation $\hat{f}$ of f near such a point has local equation are of the form $x^p y^q = 0$ (point of type $D[p, q]$).

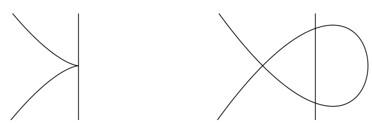

Figure 1: Deformation to maximal number of double points.

Near every branch $\hat{f}_i = 0$ the Milnor fibre is a m_i-sheeted covering of the zero-locus, except in the $D[p, q]$-points. We construct the Milnor fibre F of f starting with $S = \sum m_i$ copies of the affine line $\mathbb{A}$. Cover the ith branch with m_i copies of $\mathbb{A}$ and delete $(p + q)$ small discs around the $D[p, q]$-points. Glue in the holes $\gcd(p, q)$ small annuli (the Milnor fibres of $D[p, q]$). The resulting space is the Milnor fibre F of f.

A hyperplane section of a generic at a generic point of $\hat{f}_i = 0$ defines a transversal Milnor fibre $F_1^{\pitchfork}$. Start now the construction of F from $F_1^{\pitchfork}$, which consists of m_1 cyclic ordered points. As soon as $f_1 = 0$ intersects $f_k = 0$ it connects the sheets of $f_1 = 0$ modulo m_k. Since $\gcd(m_1, \cdots, m_r) = 1$ we connect all sheets.

Proof of Proposition 1.1. If $b_1(F) = 0$, then also $b_1(G) = 0$. The Milnor monodromy has $\mathrm{trace}(T_g) = 1$. According to A'Campo's observation [AC] g is regular: $g = x$. It follows that $f = x^r$.

3. Relation to Bobadilla's question

We consider first in any dimension $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with a 1-dimensional singular set, see especially the 1991-paper [Si] for definitions, notations and statements.

We focus on the group $H_n(F, F^{\pitchfork})$ which occurs in two exact sequences on p. 468 of [Si]:

$$0 \to H^{\mathfrak{f}}_{n-1}(F) \to H_{n-1}(F^{\pitchfork}) \to H_n(F) \oplus H^{\mathfrak{t}}_{n-1}(F) \to 0$$

$$0 \to H_n(F) \to H_n(F, F^{\pitchfork}) \to H_{n-1}(F^{\pitchfork}) \to H_{n-1}(F) \to 0$$

Here $F^{\pitchfork}$ is the disjoint union of the transversal Milnor fibres $F_i^{\pitchfork}$, one for each irreducible branch of the 1-dimensional singular set.[1]

Note that $H_n(F)$, $H_n(F, F^{\pitchfork})$ and $H_{n-1}(F^{\pitchfork})$ are free groups. $H_{n-1}(F)$ can have torsion, we denote its free part by $H_{n-1}(F)^{\mathrm{f}}$ and its torsion part by $H_{n-1}(F)^{\mathrm{t}}$. All homologies here are taken over $\mathbb{Z}$, but also other coeffecents are allowed.

From both sequences it follows that the β-invariant, introduced in [Ma] has a 25 years history, since is nothing else than:

$$\dim H_n(F, F^{\pitchfork}) = b_n - b_{n-1} + \sum \mu_i^{\pitchfork} := \beta$$

From this definition is immediately clear that $\beta \geq 0$ and that β is topological. The topological defintion has as direct consequence:

Proposition 3.1. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ with a 1-dimensional singular set, then:*

$$\beta = 0 \quad \Leftrightarrow \quad \chi(F) = 1 + (-1)^n \sum \mu_i^{\pitchfork} \quad \Leftrightarrow \quad H_n(F, \mathbb{Z}) = 0 \text{ and } H_{n-1}(F, \mathbb{Z}) = \mathbb{Z}^{\sum \mu_i}$$

The original Bobadilla conjecture C [Bo] was in [Ma] generalized to the reducible case as follows: *Does $\beta = 0$ imply that the singular set is smooth?* As consequence of our main Proposition 1.1 we have:

Corollary 3.2. *In the curve case $\beta = 0$ implies that the singular set is smooth; and that the function is equivalent to x^r.*

Remark 3.3. *In [HM] the first part of this corollary was obtained with the help of Lê numbers.*

Remark 3.4. *From the definition $\beta = H_n(F, F^{\pitchfork})$ follow direct and short proofs of several statements from [Ma].*

An other consequence from [Si] is the composition of surjections:

$$H_{n-1}(F^{\pitchfork}) = \oplus \mathbb{Z}^{\mu_i} \twoheadrightarrow H_{n-1}(\partial_2 F) = \oplus \frac{\mathbb{Z}^{\mu_i}}{A_i - I} \twoheadrightarrow H_{n-1}(F)$$

From this follows:

Proposition 3.5. *If $\dim H_{n-1}(F) = \sum \mu_i$ (upper bound) then*

[1] $F^{\pitchfork}$ was originally denoted by F'. In the second sequence a misprint n in the third term has been changed to $n-1$.

a. $H_{n-1}(\partial_2 F)$ *and* $H_{n-1}(F)$ *are free and isomorphic to* $\mathbb{Z}^{\sum \mu_i}$.

b. All transversal monodromies A_i *are the identity.*

The second part of [Ma] contains an elegant statement about $\beta = 1$ via the A'Campo trace formula. Also the reduction of the generalized Bobadilla conjecture to the (irreducible) Bobadilla conjecture. As final remark: The great work (the irreducible case) has still has to be done! Together with the Lê-conjecture this seems to be an important question in the theory of hypersurfaces 1-dimensional singular sets.

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