CURVATURE AND GAUSS-BONNET DEFECT OF GLOBAL AFFINE HYPERSURFACES

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Abstract. The total curvature of complex hypersurfaces in $\mathbb{C}^{n+1}$ and its variation in families appear to depend not only on singularities but also on the behaviour in the neighbourhood of infinity. We find the asymptotic loss of total curvature towards infinity and we express the total curvature and the Gauss-Bonnet defect in terms of singularities and tangencies at infinity.

1. Introduction

Let $Y \subset \mathbb{C}^{n+1}$ be a global algebraic hypersurface, the zero locus of a polynomial in $n+1$ complex variables. By curvature, denoted by K, we mean the Lipschitz-Killing curvature of a real codimension two analytic space Y, with respect to the metric induced by the flat Euclidean metric of $\mathbb{C}^{n+1}$. Let $\mathrm{d}v$ denote the associated volume form. The integral of the curvature $\int_Y K\mathrm{d}v$ will be called "total curvature" of Y.

We study here the influence of the position of Y at infinity upon the total curvature of Y. Computing the total curvature of the projectivised $\bar{Y}$ wouldn't help, since the metrics on $\mathbb{P}^{n+1}$ and $\mathbb{C}^{n+1}$ are different. We shall therefore exploit two ways of computing the total curvature of Y: (1). by comparing it with the Euler characteristic $\chi(Y)$, and (2). by comparing it to the total curvature of a general hypersurface, after embedding Y into a family.

The first approach goes back to extrinsic proofs of the Gauss-Bonnet theorem. The failure of this celebrated theorem in case of *open surfaces* is a theme which has been under constant attention ever since Cohn-Vossen's pioneering work [Co] in 1935. Since our space Y is not compact, and possibly singular, we consider the *Gauss-Bonnet defect*:

$$GB(Y) := \omega_n^{-1} \int_Y K\mathrm{d}v - \chi(Y),$$

where ω_n is a universal constant, see §2.1.

The second approach is based on the work of Langevin [La1, La3] and Griffiths [Gr] in the late 70's on the influence of an isolated singularity upon the total curvature of the local Milnor fibre in case of analytic hypersurface germs. Langevin found the "loss of total

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curvature" of the Milnor fibre at an isolated hypersurface singularity[1] and expressed it in terms of certain Milnor-Teissier numbers μ^*, see §4.

In our global case, we start from the interplay between the total curvature and the *affine class* of Y, defined as the number of tangent hyperplanes to $Y_{\rm reg}$ in a general global affine pencil of hyperplanes in $\mathbb{C}^{n+1}$. For such pencils one defines global polar loci which are affine curves. We point out here that any global pencil of affine hypersurfaces, even if general, has as a "limit" the hyperplane at infinity H^∞, which may be not in general position with respect to the projectivised hypersurface $\bar{Y}$.

We show in Theorem 4.1 that there is a second possible way of losing curvature when specialising in some family of affine hypersurfaces: towards infinity. The formula for the total curvature can be interpreted as a Plücker-type formula for the class of *affine hypersurfaces*, see §4.2. In order to get more grip on the meaning of the quantity of curvature absorbed at infinity, we release generality: we consider Y with isolated singularities and such that $\bar{Y} \cap H^\infty$ has singularities of dimension ≤ 1. This includes the most studied cases in the literature, see §5.1. We then express the total curvature, as well as the Gauss-Bonnet defect, in terms of invariants associated to singularities of $\bar{Y}$ and to the non-generic section $\bar{Y} \cap H^\infty$ of $\bar{Y}$ (Proposition 5.2). We discuss in §6 several examples of deformations of affine hypersurfaces Y with isolated and also non-isolated singularities.

2. Background on the total curvature

2.1. **Real submanifolds.** For a real orientable hypersurface of $\mathbb{R}^N$ one has a well-defined Gauss map. One defines the Gauss-Kronecker curvature $K(x)$ as the Jacobian of the Gauss map at x. For a submanifold V in $\mathbb{R}^N$ Fenchel [Fe] computes the curvature as follows. For a given point x on V one considers a unit normal vector $\mathbf{n}$, projects V orthogonally to the affine subspace W generated by the affine tangent space to V and this normal vector. The projection of V to W is a hypersurface, which has a well defined Gauss-Kronecker curvature $K(x, \mathbf{n})$. The Lipschitz-Killing curvature $K(x)$ of V in x is defined (see e.g.[ChL, p. 246-247]) as the integral of these curvatures over all normal directions, up to a universal constant u: $K(x) = u \int_{N_xV} K(x, \mathbf{n}) \mathrm{d}\mathbf{n}$.

The classical *Gauss-Bonnet theorem* says that if V is compact and of even dimension $2n$ then the total curvature is equal, modulo an universal constant, to the Euler characteristic:

$$\omega_n^{-1} \int_V K \mathrm{d}v = \chi(V),$$

where $\mathrm{d}v$ denotes the restriction of the canonical volume form and where $\omega_n = \frac{(2\pi)^n}{1 \cdot 3 \cdots (2n-1)}$ is half the volume of the sphere S^{2n}.

2.2. **Complex hypersurfaces.** Langevin [La1, La3] studied the integral of curvature of complex hypersurfaces $Y \subset \mathbb{C}^{n+1}$, using Milnor's approach [Mi] to the computation of the total curvature from the number of critical points of orthogonal projections on generic lines. We recall here some results and fix our notations.

[1]More about this topic can be found in Griffiths' paper [Gr].

The curvature $K(x)$ of a smooth complex hypersurface is the Lipschitz-Killing curvature of $Y_{\rm reg}$ as a codimension 2 submanifold of $\mathbb{R}^{2n+2}$, where $Y_{\rm reg}$ denotes the regular part of Y. A computation due to Milnor allows one to express the Lipschitz-Killing curvature of Y in terms of the complex Gauss map $\nu_{\mathbb{C}} : Y_{\rm reg} \to \mathbb{P}^n_{\mathbb{C}}$ which sends a point $x \in Y_{\rm reg}$ to the complex tangent space of $Y_{\rm reg}$ at x, cf [La1, pag. 11]:

$$(-1)^n K(x) = |K(x)| = \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} |\mathrm{Jac}\ \nu_{\mathbb{C}}|^2. \tag{2.1}$$

In the complex case the curvature K is well-known to have the constant sign $(-1)^n$. Using (2.1) one can prove an *exchange formula*, as follows.[2] Let H be a hyperplane in $\mathbb{P}^n$, defined by a linear form $l_H : \mathbb{C}^{n+1} \to \mathbb{C}$. For almost all $H \in \check{\mathbb{P}}^n$ the restriction of l_H to $Y_{\rm reg}$ has only complex Morse critical points. Let $\alpha_Y(l_H)$ be the number of those critical points (which is finite, since Y is algebraic). On the complement of the zero-set of its Jacobian, the complex Gauss map is a local diffeomorphism with locally constant degree $\alpha_Y(l_H)$. It is shown in §3.2 that this number does not depend on H running in some Zariski open set of $\check{\mathbb{P}}^n$, the dual projective space of all hyperplanes of $\mathbb{P}^n$. So we may denote it by α_Y. From the above discussion and from Langevin's [La3, Theorem A.III.3], one may draw the following result:

Lemma 2.1. (Langevin) *Let $Y \subset \mathbb{C}^{n+1}$ be any affine hypersurface. Then:*

$$\int_Y |K| \mathrm{d}v = \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots (2n-1)} \int_{\check{\mathbb{P}}^n} \alpha_Y(l_H)\ \mathrm{d}H = \omega_n \alpha_Y.$$

□

Here the integral $\int_Y |K| \mathrm{d}v$ is by definition the integral over $Y_{\rm reg}$. This makes sense since Y differs from $Y_{\rm reg}$ by a set of measure zero. The above formula shows in particular that, up to the constant ω_n, $\int_Y |K| \mathrm{d}v$ is a non-negative integer. The real version of the exchange principle can be used to give an extrinsic proof of the Gauss-Bonnet theorem for compact even dimensional manifolds.

In order to measure the failure of the Gauss-Bonnet theorem in case of singular or non-compact spaces, we use the *Gauss-Bonnet defect* of Y defined in the Introduction: $GB(Y) := \omega_n^{-1} \int_Y K \mathrm{d}v - \chi(Y)$. By the above, the Gauss-Bonnet defect of a complex affine hypersurface Y is an integer. It may be interpreted as the correction term due to the "boundary at infinity" of Y, at least in case Y has isolated singularities, as follows.

Let $B_R \subset \mathbb{C}^{n+1}$ be a ball centered at the origin and denote $Y_R := Y \cap B_R$ and $\partial Y_R := Y \cap \partial \bar{B}_R$. Since Y has isolated singularities and is affine, the intersection $Y \cap \partial \bar{B}_R$ is transversal and Y_R is diffeomorphic to Y, for large enough radius R. By applying the Gauss-Bonnet formula for the manifold with boundary Y_R, see Griffith [Gr, p. 479], we

[2] The *exchange principle* was originally used in the framework of total absolute curvature of knots and embedded real manifolds, by Milnor [Mi], Chern-Lashof [ChL], Kuiper [Ku].

get:

$$\omega_n^{-1}\int_Y K\mathrm{d}v - c\int_{\partial Y_R} k\mathrm{d}s = \chi(Y_R),$$

where k is the generalised 'geodesic curvature' of ∂Y_R and c is a universal constant (which will be not made precise here). It then follows:

$$GB(Y) = \lim_{R\to\infty} c\int_{\partial Y_R} k\mathrm{d}s.$$

This interpretation suggests that $GB(Y)$ should be related to singularities which occur at infinity. We shall find such a relation in (5.1).

2.3. **Plücker's class formula.** Let $V \subset \mathbb{P}^{n+1}$ be a projective hypersurface of degree d. The space of tangent hyperplanes to V_{reg} is a subset in the dual $\check{\mathbb{P}}^{n+1}$ and its closure $\check{V}$ is called the dual of V. The degree of $\check{V}$, denoted by $d^*(V)$, is the number of intersection points of $\check{V}$ with a generic projective line in $\check{\mathbb{P}}^{n+1}$. This is the same as the *class* of V, the number of tangent hyperplanes to V_{reg} in a generic pencil on $\mathbb{P}^{n+1}$. Plücker's class formula describe d^* in terms of d and of certain invariants of the singularities of V. The one proven by Plücker himself in 1834 considers curves with nodes and cusps. Teissier generalized it in 1975 to the case of projective hypersurfaces with isolated singularities, and Laumon [Lau] found the following equivalent formula, in terms of Milnor-Teissier numbers of isolated singularities (see §4.3.1):

$$d^*(V) = d(d-1)^n - \sum[\mu^{\langle n\rangle} + \mu^{\langle n-1\rangle}]. \tag{2.2}$$

Later Langevin [La2, La3] showed the connection with the complex Gauss map and provided the integral-geometric interpretation of (2.2). Further generalisations, for arbitrary *projective* varieties with isolated singularities, and then without conditions on singularities, were found notably by Kleiman, Pohl and respectively Thorup, see e.g. [Th].

Turning now back to the affine case: the positive integer $\alpha_Y(l_H)$ defined at §2.2 can be interpreted as the degree of the dual variety $\check{Y}$. We shall derive in §4.2 a formula for the affine class of Y.

3. Polar invariants, singularities and Euler characteristic

The use of polar methods is naturally suggested by the exchange principle. On the other hand, the affine polar invariants determine, via the Lefschetz slicing theory, a CW-complex structure of the space, and therefore its Euler characteristic.

3.1. **Polar curves in affine families, after** [Ti1]. Let $\{X_s\}_{s\in\delta}$ be a family of affine hypersurfaces $X_s \subset \mathbb{C}^{n+1}$, where δ is a small disk at the origin of $\mathbb{C}$. We assume that the family is polynomial, i.e. there is a polynomial $F : \mathbb{C}\times\mathbb{C}^{n+1} \to \mathbb{C}$ such that $X_s = \{x \in \mathbb{C}^{n+1} \mid F_s(x) = F(s,x) = 0\}$. Let us denote by $X = \cup_{s\in\delta} X_s$ the total space of the family, which is itself a hypersurface in $\delta\times\mathbb{C}^{n+1}$. Let $\sigma : X \to \delta \subset \mathbb{C}$ denote the projection of X to the first factor of $\mathbb{C}\times\mathbb{C}^{n+1}$.

Let our affine hypersurface $X \subset \mathbb{C} \times \mathbb{C}^{n+1}$ be stratified by its canonical (minimal) Whitney stratification $\mathcal{S}$, cf. [Te3]. This is a finite stratification, having $X \setminus \operatorname{Sing} X$ as a stratum. For instance, if X_s has no singularities and X_0 has at most isolated ones, then $\mathcal{S}$ has as lower dimensional strata only these singular points. We shall use the same notation l_H for the application $\mathbb{C} \times \mathbb{C}^{n+1} \to \mathbb{C}$, $(s, x) \mapsto l_H(x)$, as well as for its restriction to X. The *polar locus* of the map $(l_H, \sigma) : X \to \mathbb{C}^2$ with respect to $\mathcal{S}$ is the following analytic set:

$$\Gamma_{\mathcal{S}}(l_H, \sigma) := \text{closure}\{\operatorname{Sing}_{\mathcal{S}}(l_H, \sigma) \setminus (\operatorname{Sing}_{\mathcal{S}} l_H \cup \operatorname{Sing}_{\mathcal{S}} \sigma)\},$$

where $\operatorname{Sing}_{\mathcal{S}} \sigma := \bigcup_{\mathcal{S}_i \in \mathcal{S}} \operatorname{Sing} \sigma_{|\mathcal{S}_i}$ is the singular locus of σ with respect to $\mathcal{S}$. The singular loci $\operatorname{Sing}_{\mathcal{S}} l_H$ and $\operatorname{Sing}_{\mathcal{S}}(l_H, \sigma)$ are similarly defined.

Lemma 3.1. [Ti1] *There is a Zariski-open set $\Omega_\sigma \subset \check{\mathbb{P}}^n$ such that, for any $H \in \Omega_\sigma$, the polar locus $\Gamma_{\mathcal{S}}(l_H, \sigma)$ is a curve or it is empty.* □

Let Ω_σ be the Zariski-open set from Lemma 3.1. We denote by $\Omega_{\sigma,0}$ the Zariski-open set of hyperplanes $H \in \Omega_\sigma$ which are transversal to the canonical Whitney stratification of the projective hypersurface $\overline{X_0} \subset \mathbb{P}^{n+1}$. This supplementary condition insures that $\dim(\Gamma_{\mathcal{S}}(l_H, \sigma) \cap X_0) \leq 0$, $\forall H \in \Omega_{\sigma,0}$.

3.2. **The α^* sequence and the Euler characteristic.** Let $\{X_s\}_{s \in \delta}$ be any family as above. We have defined in [Ti2, §3] generic polar intersection multiplicities for such a family. We shall paraphrase that definition by considering only the regular part $(X_s)_{\text{reg}}$ of the hypersurfaces, as follows:

Definition 3.2. Let $H \in \Omega_{\sigma,0}$. The following global generic polar intersection multiplicity:

$$\alpha_{X_s}^{(n)} = \text{mult}(\Gamma_{\mathcal{S}}(l_H, \sigma), (X_s)_{\text{reg}}). \tag{3.1}$$

is well defined for any $s \in \delta$ and does not depend on the choice of $H \in \Omega_{\sigma,0}$.

The geometric interpretation of $\alpha_{X_s}^{(n)}$ is the number of Morse points of a generic linear function on $(X_s)_{\text{reg}}$. We shall next define the lower global polar intersection multiplicities $\alpha_{X_s}^{(i)}$ by following [Ti2, §3]. The idea is to consider successively general hyperplane slices of our family and apply Definition 3.2. This idea comes from Teissier's construction of polar multiplicities [Te1, Te2, Te3].

One takes a general hyperplane $\mathcal{H} \in \Omega_{\sigma,0}$ and denotes by $\alpha_{X_s}^{(n-1)}$ the global generic polar intersection multiplicity at $s \in \delta$ of the family of affine hypersurfaces $X' = X \cap \mathcal{H}$. One pursues in this way and defines step-by-step $\alpha_{X_s}^{(n-i)}$, for $1 \leq i \leq n-1$. We set $\alpha_{X_s}^{(0)} := \deg X_s$.

By a standard connectivity argument, the polar intersection multiplicities $\alpha_{X_s}^{(i)}$ do not depend on the choices of generic hyperplanes. They are also invariant up to linear changes of coordinates but not invariant up to nonlinear changes of coordinates (e.g. $\deg X_s$ is not invariant). The numbers $\alpha_{X_s}^{(i)}$ are constant on $\delta \setminus \{0\}$, provided that δ is small enough.

The geometric interpretation of the sequence of global generic polar multiplicities $\alpha^{(i)}$ also follows, as we have shown above. For instance, if we apply this construction to a single non-singular hypersurface $Y \subset \mathbb{C}^{n+1}$, by the Lefschetz slicing principle we get that Y has the structure of a CW-complex of dimension $\leq n$, with $\alpha_Y^{(i)}$ cells in dimension i. Consequently, one may expressed its Euler characteristic as follows:

$$\chi(Y) = \sum_{i=0}^{n} (-1)^i \alpha_Y^{(i)}. \tag{3.2}$$

In case of a singular Y, the formula needs correction; we explain here the case of isolated singularities, which we shall use in §5 (and send to §6.1 for non-isolated singularities and several examples). By the stratified Morse theory [GM] and Lefschetz slicing principle, the space Y is obtained from the generic slice $Y \cap \mathcal{H}$ by attaching cones over the complex links of each Morse stratified singularity of the generic pencil on Y. The singularities of the pencil on Y_{reg} contribute by $\alpha_Y^{(n)}$. In case Y has only isolated singularities, the contribution at each such point-stratum is precisely the Milnor number of the generic local hyperplane section (since our pencil is locally generic at those points), which, by a standard argument, is equal to the sectional Milnor-Teissier number $\mu^{\langle n-1 \rangle}$ (see after (4.9) for the notation). The slice $Y \cap \mathcal{H}$ and the lower dimensional ones are non-singular. We therefore get, in case Y has isolated singularities, the following formula:

$$\chi(Y) = \sum_{i=0}^{n} (-1)^i \alpha_Y^{(i)} + (-1)^n \sum_{q \in \mathrm{Sing}\, Y} \mu_q^{\langle n-1 \rangle}(Y). \tag{3.3}$$

4. Vanishing curvature and an affine Plücker formula

4.1. **The vanishing curvature.** We show here that in case of a family of affine hypersurfaces, part of the "loss of total curvature" may occur at infinity. We shall denote by $\complement B_R$ the complement in $\mathbb{C}^{n+1}$ of the ball B_R centered at the origin and of radius R. We shall use the shorter notation $\alpha_s^{(n)}$ for $\alpha_{X_s}^{(n)}$ in the rest of the paper.

Theorem 4.1. *Let $Y \subset \mathbb{C}^{n+1}$ be any hypersurface. Let $\{X_s\}_{s \in \delta}$ be a one-parameter deformation of $X_0 := Y$ such that X_s is non-singular for all $s \neq 0$. Then the following limit exists:*

$$\lim_{s \to 0} \omega_n^{-1} \int_{X_s} |K| \mathrm{d}v = \omega_n^{-1} \int_{X_0} |K| \mathrm{d}v + \mathrm{mult}(\Gamma_{\mathcal{S}}(\sigma, l_H), X_0) + \alpha_0^{(n)}(\infty), \tag{4.1}$$

where $\alpha_0^{(n)}(\infty)$ is a non-negative integer defined as:

$$\alpha_0^{(n)}(\infty) := \omega_n^{-1} \lim_{R \to \infty} \lim_{s \to 0} \int_{X_s \cap \complement B_R} |K| \mathrm{d}v. \tag{4.2}$$

Proof. We deduce from Lemma 2.1 the following general formula, by using Definition (3.2):

$$\omega_n^{-1} \int_{X_s} K \mathrm{d}v = (-1)^n \alpha_s^{(n)}. \tag{4.3}$$

It has been remarked in §3.2 that $\alpha_s^{(n)}$ is constant for $s \in \delta \setminus \{0\}$, if the disk δ is small enough. Therefore the limit $\lim_{s\to 0} \omega_n^{-1} \int_{X_s} |K| \mathrm{d}v$ is equal to $\alpha_s^{(n)}$.

Let us take $H \in \Omega_{\sigma,0}$ as in §3. From the definition (3.1) of $\alpha_s^{(n)}$ we get the following decomposition into a sum of intersection numbers:

$$\alpha_s^{(n)} = \alpha_0^{(n)} + \alpha_0^{(n)}(crt, H) + \alpha_0^{(n)}(\infty, H). \tag{4.4}$$

The first term is the intersection multiplicity $\mathrm{mult}(\Gamma_{\mathcal{S}}(l_H, \sigma), (X_0)_{\mathrm{reg}})$ and we know that it does not depend on the choice of H as above and that it is equal to $\omega_n^{-1} \int_{X_0} |K| \mathrm{d}v$, which is the first term in our claimed formula (4.1). The second term of the sum (4.4) counts the number of those intersection points of $\Gamma_{\mathcal{S}}(\sigma, l_H)$ with $(X_s)_{\mathrm{reg}}$ which tend to points $q \in \mathrm{Sing}\, X_0$. This multiplicity does not depend on the choice of generic H. It then follows that the third term from (4.4), namely $\alpha_0^{(n)}(\infty, H)$, is also independent on $H \in \Omega_{\sigma,0}$. It counts the asymptotic loss of intersection points of the polar curve $\Gamma_{\mathcal{S}}(l_H, \sigma)$ with $(X_s)_{\mathrm{reg}}$, as $s \to 0$. In other words, we have:

$$\alpha_0^{(n)}(\infty, H) = \lim_{R\to\infty} \lim_{s\to 0} \mathrm{mult}(\Gamma_{\mathcal{S}}(\sigma, l_H), (X_s)_{\mathrm{reg}} \cap \complement B_R). \tag{4.5}$$

Let us see that this is exactly the double limit defined by (4.2). By the exchange formula (Lemma 2.1) we have that:

$$\int_{X_s \cap \complement B_R} |K| \mathrm{d}v = u \int_{\check{\mathbb{P}}^n} \alpha_{X_s \cap \complement B_R}(l_H)\ \mathrm{d}H,$$

where u is a constant defined in Lemma 2.1. Since this integral is, by definition, bounded from above by $\omega_n \alpha_s^{(n)}$, we may apply Lebesgues's theorem of dominated convergence (also used by Langevin in his local proof [La1]). This allows us to interchange each of the limits with the integral, thus we get:

$$\lim_{R\to\infty} \lim_{s\to 0} \int_{X_s \cap \complement B_R} |K| \mathrm{d}v = u \lim_{R\to\infty} \lim_{s\to 0} \int_{\check{\mathbb{P}}^n} \alpha_{X_s \cap \complement B_R}(l_H) \mathrm{d}H = u \int_{\check{\mathbb{P}}^n} [\lim_{R\to\infty} \lim_{s\to 0} \alpha_{X_s \cap \complement B_R}(l_H)] \mathrm{d}H.$$

Since $\alpha_{X_s \cap \complement B_R}(l_H) = \mathrm{mult}(\Gamma_{\mathcal{S}}(\sigma, l_H), (X_s)_{\mathrm{reg}} \cap \complement B_R)$, by using now (4.5) we get our claimed equality. □

In case of a nonsingular X_0, the non-negative integer $\alpha_0^{(n)}(\infty)$ is precisely the "polar defect at infinity" which has been introduced in [Ti2] under the notation λ_0^n. We shall see in §5 how $\alpha_0^{(n)}(\infty)$ can be expressed in terms of singularities occurring at infinity, in certain situations.

4.2. **A general Plücker-type formula for the class of affine hypersurfaces.** Let $Y \subset \mathbb{C}^{n+1}$ be a hypersurface of degree d. The degree $\deg(\check{Y})$ of the affine dual $\check{Y}$ is equal to the number of tangent hyperplanes to Y_{reg} in a generic affine pencil of hyperplanes in $\mathbb{C}^{n+1}$. We shall call it the *affine class* of Y in analogy to the projective case (see §2.3), and we shall denote it by $d^{@}(Y)$. The affine pencils (see §3) differ from the projective pencils especially in a neighbourhood of infinity, since after projectivising, the hyperplane at infinity H^∞

becomes a member of the pencil and our hypersurface Y may be asymptotically tangent to H^∞.

We say that an affine hypersurface of degree d is *general* when its projective closure is non-singular and transverse to the hyperplane at infinity. Its Euler characteristic is equal to $1+(-1)^n(d-1)^{n+1}$. The polar intersection number $\alpha^{(n)}$ (Definition (3.1)) is then maximal; by Bézout theorem, it is equal to $d(d-1)^n$.

Next, let us remark that one may always deform $Y := X_0$ in a constant degree family such that X_s is general, for $s \neq 0$. For instance, for $X_0 := \{f = 0\}$, define $f_s = (1-s)f + s(g_d - 1)$, where $g_d = x_1^d + \cdots + x_{n+1}^d$. Then $X_s := \{f_s = 0\}$ has this property, for small enough $s \neq 0$.

Considering some deformation of $Y = X_0$ in a constant degree family of general hypersurfaces, we may derive from Theorem 4.1 the following formula for the affine class:

$$d^{@}(X_0) = d(d-1)^n - \operatorname{mult}(\Gamma_{\mathcal{S}}(\sigma, l_H), X_0) - \alpha_0^{(n)}(\infty). \tag{4.6}$$

This can be made more explicit in case of isolated singularities, with help of the forthcoming formulas (4.7) and (5.3).

4.3. **Case of isolated affine singularities.**

4.3.1. *Polar multiplicity.* In our global case, if $X_0 = Y$ has only isolated singularities, then one may identify the intersection multiplicity in the formula (4.1) as follows:

$$\operatorname{mult}(\Gamma_{\mathcal{S}}(\sigma, l_H), X_0) = \sum_{q \in \operatorname{Sing} X_0} [\mu_q^{\langle n-1\rangle}(X_0) + \mu_q^{\langle n\rangle}(X_0)]. \tag{4.7}$$

This comes from the equality for the generic local polar multiplicity:

$$\operatorname{mult}_q(\Gamma_{\mathcal{S}}(\sigma, l_H), X_0) = \mu_q^{\langle n\rangle}(X_0) + \mu_q^{\langle n-1\rangle}(X_0) \tag{4.8}$$

proved by Teissier [Te2, Te3] when X_0 is the germ of the zero locus of a holomorphic function $(\mathbb{C}^{n+1}, 0) \to \mathbb{C}$. It is actually well-known that the local equality (4.8) is valid for any smoothing of X_0. In our case the local smoothing is embedded in the global smoothing $\sigma : X \to \mathbb{C}$.

In the local case, for a *germ of a holomorphic function* with isolated singularities $g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$, Langevin's formula [La1, Théorème 1] shows that the loss of total curvature at an isolated singularity is measured by the sum of the first two Milnor numbers of the sequence μ^* defined by Teissier [Te2]:

$$\lim_{\varepsilon \to 0} \lim_{t \to 0} \int_{g^{-1}(t) \cap B_\varepsilon} |K| \mathrm{d}v = \omega_n(\mu^{\langle n\rangle} + \mu^{\langle n-1\rangle}). \tag{4.9}$$

Here $\mu^{\langle n\rangle}$ denotes the usual Milnor number and $\mu^{\langle n-1\rangle}$ is the Milnor number of a generic hyperplane section[3]. The sum $\mu^{\langle n\rangle} + \mu^{\langle n-1\rangle}$ is precisely the local *generic polar number* of g, i.e. the intersection number of the local polar curve $\Gamma(g, l_H)$ with $g^{-1}(0)$.

[3]Indices are shifted by -1 from the original Teissier notation.

4.3.2. *Gauss-Bonnet defect.* For affine Y with isolated singularities (still without any condition at infinity), the following formula follows from (3.3) and (4.3):

$$GB(Y) = (-1)^{n-1} \sum_{q \in \operatorname{Sing} Y} \mu_q^{\langle n-1 \rangle}(Y) - \sum_{i=0}^{n-1} (-1)^i \alpha_Y^{(i)}, \tag{4.10}$$

where the sum $\sum_{i=0}^{n-1}(-1)^i \alpha_Y^{(i)}$ is just $\chi(Y \cap \mathcal{H})$.

5. Total curvature and singularities at infinity

We focus in the remainder on explaining the loss at infinity of the total curvature and the Gauss-Bonnet defect in some distinguished classes of hypersurfaces.

5.1. Some natural classes of hypersurfaces.

Definition 5.1. (i) Y is a $\mathcal{F}$*-type* hypersurface if $\bar{Y}$ and $\bar{Y} \cap H^\infty$ have at most isolated singularities.

(ii) Y is a $\mathcal{B}_0$*-type* hypersurface if $\bar{Y}$ has at most isolated singularities.[4]

(iii) Y is a $\mathcal{B}_1$*-type* hypersurface if Y has at most isolated singularities and $\bar{Y} \cap H^\infty$ has at most isolated 1-dimensional singularities.

It is easy to see that $\mathcal{F}$-type $\subset \mathcal{B}_0$-type $\subset \mathcal{B}_1$-type.

In order to introduce the main result of this section, we need to consider generic hyperplanes, in the following sense. Let $\mathcal{W}$ be some Whitney stratification of $\bar{Y}$ such that $\bar{Y} \cap H^\infty$ is a union of strata. Let $\mathcal{H} \subset \mathbb{C}^{n+1}$ be a hyperplane such that $\bar{\mathcal{H}}$ is generic with respect to the strata of $\mathcal{W}$. There exists a Zariski-open subset of such hyperplanes, see §3 for a similar discussion.

Proposition 5.2. (a) *If Y is a $\mathcal{B}_1$-type hypersurface of degree d then:*

$$GB(Y) = (-1)^n (d-1)^n - 1 + (-1)^{n+1} \sum_{q \in \operatorname{Sing} Y} \mu_q^{\langle n-1 \rangle}(Y) + \tag{5.1}$$
$$(-1)^{n+1}[\mu_p(\bar{Y} \cap \bar{\mathcal{H}}) + \mu(\bar{Y} \cap \bar{\mathcal{H}} \cap H^\infty)].$$

(b) *If Y is a $\mathcal{B}_0$-type hypersurface of degree d then:*

$$\omega_n^{-1} \int_Y K \mathrm{d}v = (-1)^n d(d-1)^n + (-1)^{n+1} \sum_{q \in \operatorname{Sing} Y} [\mu_q^{\langle n \rangle}(Y) + \mu_q^{\langle n-1 \rangle}(Y)] + \tag{5.2}$$
$$(-1)^{n+1} \sum_{p \in (\operatorname{Sing} \bar{Y}) \cap H^\infty} \mu_p(\bar{Y}) + (-1)^{n+1} \mu(\bar{Y} \cap \bar{\mathcal{H}} \cap H^\infty) + \chi^{n,d} - \chi(\bar{Y} \cap H^\infty),$$

---

[4]The topology of $\mathcal{B}_0$-type polynomials has been studied in several papers, see e.g. Broughton's [Br] and [ST].

where $\chi^{n,d}$ denotes the Euler characteristic of the generic hypersurface of degree d in $\mathbb{P}^n$ and where $\mu(\bar{Y}\cap\bar{\mathcal{H}}\cap H^\infty)$ is a notation for $\sum_{p\in\bar{\mathcal{H}}\cap\mathrm{Sing}(\bar{Y}\cap H^\infty)}\mu_p(\bar{Y}\cap\bar{\mathcal{H}}\cap H^\infty)$.

Proof. We have by (4.10): $GB(Y) = (-1)^{n-1}\sum_{q\in\mathrm{Sing}\, Y}\mu_q^{\langle n-1\rangle}(Y) - \chi(Y\cap\mathcal{H})$. Now $Y\cap\mathcal{H}$ is $\mathcal{F}$-type and we may compute its Euler characteristic by taking a deformation of $Y = X_0$ in a constant degree family such that X_s is general for $s\neq 0$, as follows:

$$\chi(X_s\cap\mathcal{H})-\chi(X_0\cap\mathcal{H}) = [\chi(\bar{X}_s\cap\bar{\mathcal{H}})-\chi(\bar{X}_0\cap\bar{\mathcal{H}})]+[-\chi(\bar{X}_s\cap\bar{\mathcal{H}}\cap H^\infty)+\chi(\bar{X}_0\cap\bar{\mathcal{H}}\cap H^\infty)]$$
$$= (-1)^{n-1}\sum_{p\in\mathrm{Sing}(\bar{X}_0\cap\bar{\mathcal{H}})}\mu_p(\bar{X}_0\cap\bar{\mathcal{H}}) + (-1)^{n-1}\sum_{p\in\mathrm{Sing}(\bar{X}_0\cap\bar{\mathcal{H}}\cap H^\infty)}\mu_p(\bar{X}_0\cap\bar{\mathcal{H}}\cap H^\infty).$$

We then get (5.1) since $\chi(X_s\cap\mathcal{H}) = 1+(-1)^{n-1}(d-1)^n$. Let us prove (b) now. For the $\mathcal{B}_0$-type hypersurface $Y = X_0$, the singularities of $\bar{Y}$ are isolated but those of $\bar{Y}\cap H^\infty$ are of dimension at most 1. We therefore have:

$$\chi(X_0)-\chi(X_s) = \chi(\bar{X}_0)-\chi(\bar{X}_s)-\chi(\bar{X}_0\cap H^\infty)+\chi(\bar{X}_s\cap H^\infty) =$$
$$(-1)^{n+1}\sum_{q\in\mathrm{Sing}\, X_0}\mu_q^{\langle n\rangle}(X_0) + (-1)^{n+1}\sum_{p\in(\mathrm{Sing}\,\bar{X}_0)\cap H^\infty}\mu_p(\bar{X}_0) + \chi^{n,d} - \chi(\bar{X}_0\cap H^\infty).$$

We then get our result from the definition of GB, by using the equality (5.1). □

Comparing (5.2) to (4.1) and to (4.7) we get, for a deformation of $Y = X_0$ in a constant degree family such that X_s is general for $s\neq 0$:

$$(5.3)\quad \alpha_0^{(n)}(\infty) = \sum_{p\in(\mathrm{Sing}\,\bar{Y})\cap H^\infty}\mu_p(\bar{Y}) + \mu(\bar{Y}\cap\bar{\mathcal{H}}\cap H^\infty) + (-1)^{n+1}[\chi^{n,d}-\chi(\bar{Y}\cap H^\infty)].$$

Remark 5.3. As a particular case of (5.2), the following formula holds for an $\mathcal{F}$-type hypersurface:

$$(5.4)\quad \omega_n^{-1}\int_Y |K|\mathrm{d}v = d(d-1)^n - \sum_{q\in\mathrm{Sing}\, Y}[\mu_q^{\langle n\rangle}(Y)+\mu_q^{\langle n-1\rangle}(Y)]$$
$$-\sum_{p\in\mathrm{Sing}(\bar{Y}\cap H^\infty)}[\mu_p(\bar{Y})+\mu_p(\bar{Y}\cap H^\infty)].$$

The contribution from the affine singularities is contained in the first of the two sums: one recognizes the Milnor-Teissier numbers of formula (2.2). The second sum is due to the "singularities at infinity": the number $\mu_p(\bar{Y})+\mu_p(\bar{Y}\cap H^\infty)$ is exactly the local polar number $\lambda_p = \mathrm{mult}_p(\Gamma(\sigma, x_0), \bar{X}_0)$ of the polar curve of the family $\{X_s\}_s$ with respect to the local coordinate at infinity x_0, which is *not* a locally generic coordinate[5] (compare to §4.3.1). Local polar numbers, introduced by Teissier in [Te1], are well defined as soon as the polar locus is a curve.

[5]In this context, it was used in [Ti2, 3.7].

We shall give an example of a $\mathcal{F}$-type family specialising to a $\mathcal{B}_0$-type hypersurface, such that the Euler characteristic is constant but the total curvature jumps (Example 6.2).

5.2. **Concentration of the loss of total curvature at infinity.** In the case of $\mathcal{F}$-type hypersurfaces there is pointwise concentration of the loss of total curvature at infinity, see (5.4). This might be no longer the case for $\mathcal{B}$-type or more general classes of hypersurfaces: in formula (5.2) we have Euler characteristics and dependence on the slice $\bar{\mathcal{H}}$. The loss of total curvature at infinity is nevertheless concentrated at the singular locus of the set $\bar{Y} \cap H^\infty$.

5.3. **Affine curves and the correction term at infinity.** Let $C := \{f = 0\} \subset \mathbb{C}^2$ be a non-singular complex affine curve of degree d. We get from (5.1):

$$GB(C) = -d. \tag{5.5}$$

The well-known inequality due to Cohn-Vossen [Co] tells that $GB(M) \leq 0$ if M is a complete, finitely connected Riemann surface having absolutely integrable Gauss curvature.

Let now r be the number of asymptotic directions of C, i.e. the number of points in the set $\{f_d = 0\}$, where f_d denotes the degree d homogeneous part of f. Let us point out that since $\bar{C} \cap H^\infty$ consists of r points, the sum of Milnor numbers $\sum_{p \in \mathrm{Sing}(\bar{C} \cap H^\infty)} \mu_p(\bar{C} \cap H^\infty)$ is precisely $d - r$. By applying formula (5.4), since non-singular plane curves are of $\mathcal{F}$-type, we get:

$$\begin{aligned} \omega_n^{-1} \int_C |K| \mathrm{d}v &= d(d-1) - \textstyle\sum_{p \in \mathrm{Sing}(\bar{C} \cap H^\infty)} \mu_p(\bar{C}) - d + r = \\ &= d^2 - 2d + r - \textstyle\sum_{p \in \mathrm{Sing}(\bar{C} \cap H^\infty)} \mu_p(\bar{C}). \end{aligned} \tag{5.6}$$

Comparing this to the formula found by Risler [Ri, Proposition 4.2, (15)] for a non-singular complex affine curve C, one notices that the latter does not contain the sum $\sum_{p \in \mathrm{Sing}(\bar{C} \cap H^\infty)} \mu_p(\bar{C})$. Therefore Risler's formula would not be valid when the compactification $\bar{C}$ is singular. However, Risler uses his formula in *loc.cit.* only in the case $r = d$, which implies that the affine curve C is general at infinity. In this special case indeed formula (5.6) reduces as such.

5.4. **Semi-continuity and extrema of curvature integrals.** For any family $\{X_s\}_{s \in \delta}$ of affine hypersurfaces we have:

$$\omega_n^{-1} \int_{X_0} |K| \mathrm{d}v = \alpha_0^{(n)} \leq \alpha_s^{(n)} = (\omega_n)^{-1} \int_{X_s} |K| \mathrm{d}v. \tag{5.7}$$

The total curvature is therefore bounded as follows: $0 \leq \omega_n^{-1} \int_{X_0} |K| \mathrm{d}v \leq d(d-1)^n$.

For a general hypersurface X_0, the equality $\omega_n^{-1} \int_{X_0} |K| \mathrm{d}v = d(d-1)^n$ holds. We claim that the reciprocal is true. Indeed, if X_0 is not general then there exists a deformation $\{X_s\}_s$ such that: X_s is of $\mathcal{F}$-type for $s \neq 0$, $\bar{X}_0$ is non-singular and $\bar{X}_0 \cap H^\infty$ is non-singular except at one point, say p, where the singularity is of type A_1, i.e. $\mu_p(\bar{X}_0 \cap H^\infty) = 1$. According to (5.4) we then have: $\omega_n^{-1} \int_{X_s} |K| \mathrm{d}v = d(d-1)^n - 1$, which, together with the semi-continuity relation (5.7), gives a contradiction.

What happens now when the minimum occurs, i.e. the total curvature of X_0 is zero? For the case of non-singular X_0, the answer is the following: $(\omega_n)^{-1}\int_{X_0}|K|\mathrm{d}v = 0$ implies that the map $l_H : X_0 \to \mathbb{C}$, for $H \in \Omega_\sigma$, is a trivial fibration; in particular $b_n(X_0) = 0$. This is a consequence of the fact that $\alpha_0^{(n)} = 0$ implies that there are no n-cells in the CW model of X_0, see §3.2.

6. EXAMPLES

EXAMPLE 6.1. We show first how to compute the total absolute curvature directly from equation (4.3). Let $f : \mathbb{C}^3 \to \mathbb{C}$, $f(x, y, z) = x + x^2yz$. We consider the family $X_s = \{f = s\}$, see [Ti2, Example 3.8]. The generic polar intersection multiplicities and the defects at infinity in the neighbourhood of the value 0 are given in [Ti2]; from those results we may extract the folowing data: $\alpha_s^{(2)} = 5$, $\alpha_s^{(1)} = 8$, $\alpha_s^{(0)} = 4$ for $s \neq 0$, and $\alpha_0^{(2)} = 3$, $\alpha_0^{(1)} = 6$, $\alpha_0^{(0)} = 4$. We get: $\omega_2^{-1}\int_{X_s}|K|\mathrm{d}v = 5$ if $s \neq 0$ and $\omega_2^{-1}\int_{X_0}|K|\mathrm{d}v = 3$.

The variation of total curvature is 2 and is equal to the vanishing curvature at infinity $\alpha_0^2(\infty)$, as defined in Theorem 4.1. Therefore the curvature of X_s is not constant in the family, even if X_s is nonsingular and $\chi(X_s) = 1$ for all $s \in \mathbb{C}$ (see *loc.cit.*). It is also clear that the family is not topologically trivial, since the number of connected components of the fibers change at $s = 0$.

EXAMPLE 6.2. Consider the double parametre family $X_{s,t} = \{f_s = x^4 + sz^4 + z^2y + z = t\}$. This deforms the $\mathcal{B}_0$-type hypersurface $X_{0,t}$ into a $\mathcal{F}$-type one $X_{s,t}$ for $s \neq 0$, see [ST, Example 6.5]. We recall that, for all s, f_s has a generic fibre, which is homotopy equivalent to a bouquet of three 2-spheres. There are no affine critical points and $t = 0$ is the only atypical value of f_s.

In order to compute the total curvature we use formulas (5.4) and (5.2) for the $\mathcal{B}_0$-type ($s = 0$) and (5.4) for the $\mathcal{F}$-type ($s \neq 0$). The input for the formulas is in the table below. The computation of $\chi(X_{s,t})$ is via the curvature by using the Gauss-Bonnet defect. Let us recall a few facts from [ST]:

(1). $\bar{X}_{s,t}$ has isolated singularities at infinity in $p := ([0 : 1 : 0], 0)$ for all s and in $q := ([1 : 0 : 0], 0)$ for $s = 0$. The μ's are listed in the table.

(2). The singularities of $\bar{X}_{s,t} \cap H^\infty \subset \mathbb{P}^2$ change from a single smooth line $\{x^4 = 0\}$ into the isolated point p with $\tilde{E}_7$ singularity.

(3). The space $\bar{X}_{s,t} \cap \bar{\mathcal{H}} \cap H^\infty$ has a single singularity of type A_3 for $s = 0$ and is smooth if $s \neq 0$.

(4). The change on the level of $\chi(\bar{X}_{s,t} \cap H^\infty)$ is from 2 to 5, so $\Delta\chi^\infty = -3$. Note that $\chi^{2,4} = -4$. In the table we use as notation $\Delta\chi = \chi^{2,4} - \chi(\bar{X}_{s,t} \cap H^\infty)$.

(s,t)	$\mu_p(\bar{X}_{s,t}) + \mu_q(\bar{X}_{s,t})$	$\mu(\bar{X}_{s,t} \cap \bar{\mathcal{H}} \cap H^\infty)$	$(-1)^{n+1}\Delta\chi$	$\alpha_{s,t}^{(2)}$	$\chi(X_{s,t})$
$(0,0)$	$18+3$	3	$4+2$	$36-30=6$	$6-6=0$
$(0,t)$	$15+3$	3	$4+2$	$36-27=9$	$9-6=3$
$(s,0)$	$18+0$	–	9	$36-27=9$	$9-9=0$
(s,t)	$15+0$	–	9	$36-24=12$	$12-9=3$

NB. In notations like $(0,t)$ we mean here that $t \neq 0$.

Let us point out that in this example we have, for each fixed t, a χ-constant family $X_{s,t}$ of constant degree, but with non-constant total curvature. It turns out (by using a coordinate change in the variable y) that actually this family is topologically trivial.

6.1. **Examples with non-isolated affine singularities.** In case Y is singular, one may correct the formula (3.2) by defining the level n correction terms $\beta_Y^{(n)}$ as follows:

$$\chi(Y) = \chi(Y \cap \mathcal{H}) + (-1)^n[\alpha_Y^{(n)} + \beta_Y^{(n)}]. \tag{6.1}$$

Remark that we have found $\beta_Y^{(n)}$ more explicitely in case Y has only isolated singularities, see (3.3). For Y with non-isolated singularities, one needs lower level corrections $\beta_Y^{(i)}$, $i \leq n$, which one defines by using equalities analogous to (6.1) for successive slices. It follows that $\beta_Y^{(n-i)} = 0$ for $i > \dim \operatorname{Sing} Y$. We show in the following Examples 6.3 and 6.4 how the lower β's occur in case of Y has one-dimensional singularities.

Example 6.3. Consider the family given by a single polynomial $X_s = \{f = x^2+x^3y+z^4 = s\}$ and note that f has a non-isolated singularity. The critical set is the y-axis, with constant transversal type A_3, and the only atypical value turns out to be 0. A generic affine pencil produces a polar curve, which has 12 intersections points with X_s if $s \neq 0$. It has 6 intersections with $(X_0)_{\mathrm{reg}}$ and no intersection with $\operatorname{Sing} X_0$, therefore six points disappear at infinity. This gives the values of $\alpha^{(2)}$ in the table below. We have $\beta_{X_s}^{(2)} = 0$ for all s since X_s is non-singular for $s \neq 0$ and X_0 has a non-singular 1-dimensional singular locus with constant transversal type.

We consider next the restriction of f to a generic hyperplane section. We use the plane $\mathcal{H}$ defined by $y = px + qz + r$. This gives us the polynomial

$$g = x^2 + px^4 + qx^3z + z^4 + rx^2z = s$$

The direct computation of $\alpha^{(1)}$ turns out to be involved, so we choose the following way. For generic (p,q,r), the fibers of g are general at infinity, of degree 4. So $\chi(X_s \cap \mathcal{H}) = -8$ for $s \neq 0$. If $s = 0$ then $X_0 \cap \mathcal{H}$ has a A_3 singularity, which has as effect $\chi(X_0 \cap \mathcal{H}) = -5$. By slicing again g we get 4 points: this gives $\alpha^{(0)} + \beta^{(0)}$ in the table below.

Next the complex links: the fibre X_0 has a singular stratum which is linear and with transversal A_3 singularity. Its complex link contributes with $\beta^{(1)} = 1$. If $s \neq 0$ the fibre is smooth, so all betas are zero. Using the notations $\chi^2 = \chi(X_s)$, $\chi^1 = \chi(X_s \cap \mathcal{H})$, $\chi^0 = \chi(X_s \cap \mathcal{H} \cap \mathcal{H}')$, the table with all information looks as follows:

$\alpha^{(i)}$	$\beta^{(i)}$	$\alpha^{(i)}+\beta^{(i)}$	χ^i	i	$\alpha^{(i)}$	$\beta^{(i)}$	$\alpha^{(i)}+\beta^{(i)}$	χ^i
6	0	6	1	2	12	0	12	4
8	1	9	-5	1	12	0	12	-8
4	0	4	4	0	4	0	4	4
$s=0$					$s \neq 0$			

We get $\int_{X_s} |K| \mathrm{d}v = 12\omega_2$ if $s \neq 0$ and $\int_{X_0} |K| \mathrm{d}v = 6\omega_2$. The vanishing of curvature is only due to the concentration at infinity: although we have an affine non-isolated singularity, there is no loss of total curvature in the affine part. Note also: $\int_{X_s\cap\mathcal{H}} |K|\mathrm{d}v = 12\omega_1$ if $s \neq 0$ and $\int_{X_0\cap\mathcal{H}} |K|\mathrm{d}v = 8\omega_1$ and that on this level there is an affine loss of total curvature.

EXAMPLE 6.4. Consider $f = x^2y + x^3y^2 + z^5 = s$. This can be treated in the same way. The polynomial has a non-isolated smooth 1-dimensional critical set (y-axis), but with a non-trivial complex link on the level $i = 2$ (modelled on the Whitney umbrella) and an isolated singularity on level $i = 1$. There is also an affine contribution to the loss of total curvature. The corresponding table is:

$\alpha^{(i)}$	$\beta^{(i)}$	$\alpha^{(i)} + \beta^{(i)}$	χ^i	i	$\alpha^{(i)}$	$\beta^{(i)}$	$\alpha^{(i)} + \beta^{(i)}$	χ^i
10	2	12	1	2	32	0	32	17
15	1	16	-11	1	20	0	20	-15
5	0	5	5	0	5	0	5	5
$s = 0$					$s \neq 0$			

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REFERENCES

[Br] S.A. Broughton, *On the topology of polynomial hypersurfaces*, Proceedings A.M.S. Symp. in Pure. Math., vol. 40, I (1983), 165-178.

[ChL] S. S. Chern, R. K. Lashof, *On the total curvature of immersed manifolds, I, II*, American J. of Math 79 (1957), 306-318; and Michigan Math. J., 5 (1958), 5-12.

[Co] S. Cohn-Vossen, *Kürzeste Wege and Totalkrümmung auf Flächen*, Compositio Math. 2 (1935), 69-133.

[Fe] W. Fenchel, *On total curvatures of Riemannian manifolds. 1*, J. Lond. Math. Soc. 15 (1940), 15-22.

[GM] M. Goresky, R. MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 14. Springer-Verlag, Berlin, 1988.

[Gr] Ph.A. Griffiths, *Complex differential and integral geometry and curvature integrals associated to singularities of complex analytic varieties*, Duke Math. J. 45 (1978), no. 3, 427–512.

[Ku] N. H. Kuiper *Minimal total curvature for immersions*, Inventiones Math 10 (1970) 209–238.

[La1] R. Langevin, *Courbure et singularités complexes*, Comment. Math. Helv. 54 (1979), no. 1, 6–16.

[La2] R. Langevin, *Classe moyenne d'une sous-variété d'une sphère ou d'un espace projectif*, Rend. Circ. Mat. Palermo (2) 28 (1979), no. 2, 313–318.

[La3] R. Langevin, *Courbures, feuilletages et surfaces*, Dissertation, Université Paris-Sud, Orsay, 1980. Publications Mathématiques d'Orsay 80 , 3. Université de Paris-Sud, Département de Mathématique, Orsay, 1980.

[Lau] G. Laumon, *Degré de la variété duale d'une hypersurface á singularités isolées*, Bull. Soc. Math. France 104 (1976), no. 1, 51–63.

[Mi] J. Milnor, *On the total curvature of knots*, Ann. of Math. (2) 52, (1950). 248–257.

[Ri] J-J Risler, *On the curvature of the real Milnor fiber*, Bull. London Math. Soc. 35 (2003), no. 4, 445–454.

[ST] D. Siersma, M. Tibăr, *Singularity exchange at the frontier of the space*, math.AG/0401396.

[Te1] B. Teissier, *Cycles évanescents, sections planes et conditions de Whitney*, Singularités à Cargèse (1972), pp. 285–362. Asterisque, Nos. 7 et 8, Soc. Math. France, Paris, 1973.

[Te2] B. Teissier, *Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces*, Invent. Math. 40 (1977), no. 3, 267–292.

[Te3] B. Teissier, *Varietés polaires 2: Multiplicités polaires, sections planes et conditions de Whitney*, Géométrie Algèbrique à la Rabida, Springer L.N.M. **961** (1981), pp. 314–491.

[Th] A. Thorup, *Generalized Plücker formulas*, in: Recent progress in intersection theory (Bologna, 1997), 299–327, Trends Math., Birkhäuser Boston, Boston, MA, 2000.

[Ti1] M. Tibăr, *Topology at infinity of polynomial mappings and Thom regularity condition*, Compositio Math. 111 (1998), no. 1, 89–109.

[Ti2] M. Tibăr, *Asymptotic equisingularity and topology of complex hypersurfaces*, Internat. Math. Res. Notices 1998, no. 18, 979–990.

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