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# METRIC PROPERTIES OF CONFLICT SETS

LEV BIRBRAIR AND DIRK SIERSMA

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Abstract. In this paper we show that the tangent cone of a conflict set in $\mathbb{R}^n$ is a linear affine cone over a conflict set of a smaller dimension and has dimension $n-1$. Moreover we give an example where the conflict sets is not normally embedded and not locally bi-Lipschitz equivalent to the corresponding tangent cone.

## 1. Introduction

Let $M$ be a metric space and let $X = \{X_1, \ldots, X_k\}$ be a finite collection of closed disjoint nonempty subsets of $M$. We define a *territory* of a subset $X_i \in X$ with respect to the space $M$ and the collection $X$ in the following way:

$$Ter_M(X_i, X) = \{x \in M \quad \text{such that} \quad \forall j \quad d(x, X_i) \leq d(x, X_j)\}.$$

We define a *conflict set* of the collection $X$ with respect to $M$ as follows:

$$Conf_M(X) = \{x \in Ter_M(X_i, X) \cap Ter_M(X_j, X) \quad \text{for some} \quad i \neq j\}.$$

Singularities of conflict sets of collections of disjoint subsets of $\mathbb{R}^n$ is one of natural objects of Singularity Theory. The investigation of conflict sets was initiated by Y. Yomdin [7]. J. Damon [3], P. Giblin and V.M. Zakalyukin [5] and others pointed out that the theory of conflict sets is closely related to other important objects in Singularity Theory: cut loci, medial axes, wave fronts. The results of [3], [6], [8] are devoted to differential geometry of conflict sets.

Here we study properties of general (not necessary generic) singularities of conflict set from a metric view-point. Our restriction is so-called ”definability”. We suppose that all the sets appearing in our investigation are definable in some o-minimal structure [2]. If a reader is not familiar with ”o-minimal” language

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he can suppose that all the sets are semialgebraic or subanalytic. If the sets $X_i$ are definable then the same is true for their conflict sets. Thus, all the "good" topological properties hold and tangent cones are well defined.

The main statement of this paper is the structural theorem about the tangent cone of a conflict set. We prove that it is a linear affine cone over a certain conflict set on $S^{n-1}$ (Theorem 2.2). As a corollary of this result we obtain that, for each set $X_i$ of the collection $\{X_1, X_2, \dots, X_k\}$ in $\mathbb{R}^2$, the set $Ter\, X_i$ does not have the cusp-like regions. Moreover, we show that the tangent cone to $Ter\, X_i$ has a dimension of the ambient space and the tangent cone of $Conf\,(X)$ has a dimension $n-1$. A natural question is the following. Is it true or not that conflict sets have "metrically conic" structure near a singular point? More specifically, a conflict set is locally bi-Lipschitz equivalent to the corresponding tangent cone?

The answer is NO: In section 3, we present an example of a collection of sets $\{X_1, X_2\} \subset \mathbb{R}^3$ such that the conflict set of $X_1$ and $X_2$ is not locally homeomorphic to its tangent cone and thus not normally embedded in $\mathbb{R}^3$.

## 2. Tangent cones of conflict sets

Let $A$ be an o-minimal structure over $\mathbb{R}$. Let $X$ be a collection of definable subsets of a definable set $Y \subset \mathbb{R}^n$ in an o-minimal structure $A$. Then $Conf_Y(X)$ and $Ter_Y(X_i, X)$ are definable in $A$ sets.

In this paper we are going to suppose that the space $\mathbb{R}^n$ is equipped with the Euclidean metric.

**Proposition 2.1.** *Let $X = \{X_1, X_2, X_3, \dots, X_k\}$ be a finite collection of closed and definable in an o-minimal structure $A$ subsets of $\mathbb{R}^n$. Then* $\dim Conf_{\mathbb{R}^n}(X) = n-1$.

Proof. Let us first show that if $x_0 \in Conf_{\mathbb{R}^n}(X)$ then there exists a number $j$ such that $x_0 \in \partial Ter_M(X_j, X)$. Let $r_0 = \min d(x_0, X_i)$. Let $\widetilde{X} = \{\widetilde{X}_1, \widetilde{X}_2, \dots, \widetilde{X}_k\}$ be a collection of sets (called "supports") defined as follows: $\widetilde{X}_i = X_i \cap S_{x_0, r_0}$. Since $x_0 \in Conf_{\mathbb{R}^n}(X)$ we can suppose that there exist two numbers $j_1$ and $j_2$ such that $\widetilde{X}_{j_1}$ and $\widetilde{X}_{j_2}$ are nonempty. Let $x_{j_1} \in \widetilde{X}_{j_1}$ and $x_{j_2} \in \widetilde{X}_{j_2}$. Then the half-open segment $[x_{j_1}, x_0)$ belongs to $Ter_{\mathbb{R}^n}(X_{j_1}, X)$ and does not belong to $Ter_{\mathbb{R}^n}(X_{j_2}, X)$, the segment $(x_{j_2}, x_0)$ belongs to $Ter_{\mathbb{R}^n}(X_{j_2}, X)$ and does not belong to $Ter_{\mathbb{R}^n}(X_{j_1}, X)$. Hence, $x_0$ is a boundary point of $Ter_{\mathbb{R}^n}(X_{j_1}, X)$ and of $Ter_{\mathbb{R}^n}(X_{j_2}, X)$. This argument also proves that the sets $Int(Ter_{\mathbb{R}^n}(X_i, X))$ are disjoint. Since the sets $Ter_{\mathbb{R}^n}(X_i, X)$ are definable in the o-minimal structure $A$, we have:$Conf_{\mathbb{R}^n}(X) \subset \bigcup_i \partial(Ter_{\mathbb{R}^n}(X_i, X))$. That is why $\dim Conf_{\mathbb{R}^n}(X) \leq n-1$

(see [2]). From the other hand, $\mathbb{R}^n = \bigcup_i Int(Ter_{\mathbb{R}^n}(X_i, X))) \bigcup Conf_{\mathbb{R}^n}(X)$. Since the sets $Int(Ter_{\mathbb{R}^n}(X_i, X))$ are disjoint, we obtain that $\dim Conf_{\mathbb{R}^n}(X) \geq n-1$ . This proves the proposition. $\square$

A similar statement is true for collections of definable subsets of $S^n$ and is not true for collections of definable subsets of $\mathbb{R}^n$ equipped with New York metric.

Let $M$ be a subset of $\mathbb{R}^n$. The cone over $M$ with respect to $x_0$ (notation: $C_{x_0}M$) is a union of all rays connecting $x_0$ with all the points $y \in M$.

Let $Y \subset \mathbb{R}^n$ be definable in an o-minimal structure $A$. A tangent cone $T_{x_0}X$ at a point $x_0 \in X$ is the set of all tangent vectors $\frac{d}{d\gamma}\big|_{t=0}\gamma(t)$ of all definable in $A$ arcs $\gamma\colon [0,\varepsilon) \to Y$ such that $\gamma(0) = x_0$ ( see [4], [1]). If $x_0$ is a smooth point of $Y$ then we obtain the definition of the tangent space.

We are going to use another definition of the tangent cone (see also [4]). Let $N_\varepsilon(Y)$ be a set defined as follows:

$$N_\varepsilon(Y) = \frac{1}{\varepsilon}\big[Y \bigcap S_{x_0,\varepsilon} - x_0\big] + x_0.$$

We use the notations: $S_{x_0,\varepsilon}$, for the sphere centered at $x_0$ with the radius $\varepsilon$; $B_{x_0,\varepsilon}$, for the closed ball.

Then the following statement is true:

$$T_{x_0}Y = C_{x_0}(\lim_{\varepsilon\to 0} Hausdorff N_\varepsilon(Y)).$$

Let $X = \{X_1, \ldots, X_k\}$ be a family of closed and disjoint sets on $S^{n-1}$. Then there is the following relation:

$$Conf_{\mathbb{R}^n}(X) = C_{o\in\mathbb{R}^n} Conf_{S^{n-1}}(X).$$

Here we use the standard geodesic metric on $S^{n-1}$. The proof of this statement is straightforward.

The main result of this paper is the following statement.

**Theorem 2.2.** *Let $X = \{X_1, \ldots, X_k\}$ be a collection of definable in $A$ closed subsets of $\mathbb{R}^n$ such that $X_i \cap X_j = \emptyset$, for $i \neq j$. Let $x_0 \in Conf_{\mathbb{R}^n}(X)$ and let $r_0 = \min_i d(x_0, X_i)$. Let $\widetilde{X} = \{\widetilde{X}_1, \widetilde{X}_2, \ldots, \widetilde{X}_k\}$ be a collection of sets ( called "the supports" ) defined as follows: $\widetilde{X}_i = X_i \cap S_{x_0,r_0}$. Then the following identities hold:*

(1) $T_{x_0}(Ter_{\mathbb{R}^n}(X_i, X)) = C_{x_0}(Ter_{S_{x_0,r_0}}(\widetilde{X}_i, \widetilde{X}))$.
(2) $T_{x_0}(Conf_{\mathbb{R}^n}(X)) = C_{x_0}(Conf_{S_{x_0,r_0}}(\widetilde{X}))$.

PROOF. Observe that the statement 2 follows immediately from the statement 1 by the definition of conflict sets. Now we are going to show that the germs of $Ter_{\mathbb{R}^n}(X_i, X)$ and of $Conf_{\mathbb{R}^n}(X)$ at $x_0$ do not change if we cut the sets $X_i$ by balls of the radius bigger than $r_0$ centered at $x_0$. Namely, we prove the following statement:

**Lemma 2.3.** *Let $X^\varepsilon = \{X_1^\varepsilon, X_2^\varepsilon, \dots, X_k^\varepsilon\}$ be a collection of the sets defined as follows: $X_i^\varepsilon = X_i \cap \bar{B}_{x_0, r_0+\varepsilon}$. Then, for all $i$, the germ of $Ter_{\mathbb{R}^n}(X_i, X)$ at $x_0$ is the same as the germ of the set $Ter_{\mathbb{R}^n}(X_i^\varepsilon, X^\varepsilon)$ at $x_0$.*

PROOF. Let $z$ be a point of $X_j$ such that $d(x_0, z) = d(x_0, X_j)$. Take $\delta = \varepsilon/3$. Let $x \in B_{x_0,\delta}$. Let $y$ be a point of $X_j$ such that $d(x, y) = d(x, X_j)$. We are going to show that $y$ actually belong to $X_j^\varepsilon$.

First: $d(x, y) = d(x, X_j) \leq d(x, z) \leq d(x, x_0) + d(x_0, z) \leq \delta + r_0$. Second: suppose that $y \notin X_j^\varepsilon$. Then $d(x, y) > d(x_0, y) - d(x, x_0) = r_0 + \varepsilon - \delta$. This is a contradiction. So, on $B_{x_0,\delta}$ one have: $d(x, X_j) = d(x, X_j^\varepsilon)$ and, therefore, $Ter_{\mathbb{R}^n}(X_i, X) \cap B_{x_0,\delta} = Ter_{\mathbb{R}^n}(X_i^\varepsilon, X^\varepsilon) \cap B_{x_0,\delta}$. □

We consider "polar coordinates" $(\rho, \phi)$ near the point $x_0$ defined as follows. Let $x \in \mathbb{R}^n$ be a point. Set $\phi(x) = \frac{x - x_0}{||x - x_0||}$ and $\rho(x) = d(x, x_0)$.

**Lemma 2.4** (Shadow lemma)**.** *Let $X = \{X_1, \dots, X_n\}$ be a family of definable sets such that, for all $j$, $\phi(X_j) = \phi(\widetilde{X}_j)$ (we say that $X$ lies in the shadow of the support of $X$ at $x_0$). Then the germs at $x_0$ of the sets $Ter_{\mathbb{R}^n}(X_j, X)$ and $Ter_{\mathbb{R}^n}(\widetilde{X}_j, \widetilde{X})$ are equal, with $\widetilde{X}_j$ as in Theorem 2.2.*

PROOF. Let $x_1 \in B_{x_0, r_0/3}$. Then $d(x_1, X_j) = d(x_1, \widetilde{X}_j)$. That is why

$$Ter_{\mathbb{R}^n}(\widetilde{X}_j, \widetilde{X}) \cap B_{x_0, r_0/3} = Ter_{\mathbb{R}^n}(X_j, X) \cap B_{x_0, r_0/3}.$$

□

Using polar coordinates we define sets $Y_i^\varepsilon$ in the following way: $y \in Y_i^\varepsilon$ if, and only if, $y \in B_{x_0, r_0+\varepsilon}$, $r_0 \leq \rho(y) \leq r_0 + \varepsilon$ and there exists $x \in X_i^\varepsilon$ s.t. $\phi(x) = \phi(y)$. Clearly, $X_i^\varepsilon \subset Y_i^\varepsilon$. Let $Y^\varepsilon = \{Y_1^\varepsilon, Y_2^\varepsilon, \dots, Y_k^\varepsilon\}$. Let $Z^\varepsilon = \{Z_1^\varepsilon, Z_2^\varepsilon, \dots, Z_k^\varepsilon\}$ be a collection of sets defined as follows: $Z_i^\varepsilon = Y_i^\varepsilon \cap S_{x_o, r_0}$. The Hausdorff limits for $\varepsilon$ tending to zero of the families $Y_i^\varepsilon$ and $Z_i^\varepsilon$ are equal to $\widetilde{X}_i$. If $x \in B_{x_o, r_0/3}$ and $d(x, Y_i^\varepsilon) = d(x, y)$, for some $y \in Y_i^\varepsilon$, then $y \in Z_i^\varepsilon$. Clearly, for small $\varepsilon$, we have: $Y_i^\varepsilon \cap Y_j^\varepsilon = \emptyset$, for $i \neq j$, and, thus, $Z_i^\varepsilon \cap Z_j^\varepsilon = \emptyset$. Since all the sets $Z_i^\varepsilon$ belong to $S_{x_0, r_0}$, then $Ter_{\mathbb{R}^n}(Z_i^\varepsilon, Z^\varepsilon) = C_{x_0} Ter_{S_{x_0, r_0}}(Z_i^\varepsilon, Z^\varepsilon)$. Moreover, the germ of the set $Ter_{\mathbb{R}^n}(Y_i^\varepsilon, Y^\varepsilon)$ at the point $x_0$ is the same as the germ of $Ter_{\mathbb{R}^n}(Z_i^\varepsilon, Z^\varepsilon)$ [Shadow Lemma].

Let us consider the following collection of sets $W^\varepsilon = \{W_1^\varepsilon, W_2^\varepsilon, \dots, W_k^\varepsilon\}$ and $V^\varepsilon = \{V_1^\varepsilon, V_2^\varepsilon, \dots, V_k^\varepsilon\}$ where $W_j^\varepsilon = Y_j^\varepsilon$, for $i \neq j$, and $W_i^\varepsilon = \widetilde{X}_i$, $V_j^\varepsilon = \widetilde{X}_j$, for $i \neq j$, and $V_i^\varepsilon = Y_i^\varepsilon$. Since $\widetilde{X}_s \subset X_s^\varepsilon \subset Y_s^\varepsilon$, for any $s$, we obtain the following inclusions:

$$\begin{aligned} &Ter_{\mathbb{R}^n}(X_i^\varepsilon, X^\varepsilon) \subset Ter_{\mathbb{R}^n}(Y_i^\varepsilon, V^\varepsilon), \\ &Ter_{\mathbb{R}^n}(\widetilde{X}_i, W^\varepsilon) \subset Ter_{\mathbb{R}^n}(X_i^\varepsilon, X^\varepsilon). \end{aligned} \tag{1}$$

Let $\widetilde{W}^\varepsilon$ and $\widetilde{V}^\varepsilon$ be the collections of sets defined as follows: $\widetilde{W}^\varepsilon = W^\varepsilon \cap S_{x_0,r_0}$ and $\widetilde{V}^\varepsilon = V^\varepsilon \cap S_{x_0,r_0}$. Note, that the germs of the sets $Ter_{\mathbb{R}^n}(V_j^\varepsilon, V^\varepsilon)$ and $Ter_{\mathbb{R}^n}(\widetilde{V}_j^\varepsilon, \widetilde{V}^\varepsilon)$ at $x_0$ are equal and the germs of $Ter_{\mathbb{R}^n}(W_j^\varepsilon, W^\varepsilon)$ and $Ter_{\mathbb{R}^n}(\widetilde{W}_j^\varepsilon, \widetilde{W}^\varepsilon)$ are also equal (Shadow lemma). The sets $Ter_{\mathbb{R}^n}(\widetilde{V}_i^\varepsilon, \widetilde{V}^\varepsilon)$ and $Ter_{\mathbb{R}^n}(\widetilde{W}^\varepsilon, \widetilde{W})$ are purely conic., i.e.

$$Ter_{\mathbb{R}^n}(\widetilde{V}_i^\varepsilon, \widetilde{V}^\varepsilon) = C_{x_0} Ter_{S_{x_0,r_0}}(\widetilde{V}_i^\varepsilon, \widetilde{V}^\varepsilon)$$

and

$$Ter_{\mathbb{R}^n}(\widetilde{W}_i^\varepsilon, \widetilde{W}^\varepsilon) = C_{x_0,r_0} Ter_{S_{x_0,r_0}}(\widetilde{W}_i^\varepsilon, \widetilde{W}^\varepsilon).$$

Hence,

$$T_{x_0} Ter_{\mathbb{R}^n}(\widetilde{V}_i^\varepsilon, \widetilde{V}^\varepsilon) = Ter_{\mathbb{R}^n}(\widetilde{V}_i^\varepsilon, \widetilde{V}^\varepsilon)$$

and

$$T_{x_0} Ter_{\mathbb{R}^n}(\widetilde{W}_i^\varepsilon, \widetilde{W}^\varepsilon) = Ter_{\mathbb{R}^n}(\widetilde{W}_i^\varepsilon, \widetilde{W}^\varepsilon).$$

Using the inclusions (1) we obtain:

$$\begin{aligned} T_{x_0}(Ter_{\mathbb{R}^n}(X_i, X)) = T_{x_0}(Ter_{\mathbb{R}^n}(X_i^\varepsilon, X^\varepsilon)) \subset Ter_{\mathbb{R}^n}(\widetilde{V}_i^\varepsilon, \widetilde{V}^\varepsilon), \\ Ter_{\mathbb{R}^n}(\widetilde{W}_i^\varepsilon, \widetilde{W}^\varepsilon) \subset T_{x_0}(Ter_{\mathbb{R}^n}(X_i^\varepsilon, X^\varepsilon)) = T_{x_0}(Ter_{\mathbb{R}^n}(X_i, X)). \end{aligned} \tag{2}$$

Taking the Hausdorff limit in the inclusions (2) we obtain

$$T_{x_0}(Ter_{\mathbb{R}^n}(X_i, X)) \subset C_{x_0}(Ter_{S_{x_0,r_0}}(\widetilde{X}_i, \widetilde{X}))$$

and

$$C_{x_0}(Ter_{S_{x_0,r_0}}(\widetilde{X}_i, \widetilde{X})) \subset T_{x_0}(Ter_{\mathbb{R}^n}(X_i, X)).$$

This proves the theorem. □

**Proposition 2.5.** *Let $X = \{X_1, \dots, X_k\}$ be a collection of definable in $A$ sets in $\mathbb{R}^n$. Then*

1. *$T_{x_0}(Ter_{\mathbb{R}^n}(X_i, X))$ has a nonempty interior, for all $i$ such that $x_0 \in Ter(X_i, X)$.*
2. *If $x_0 \in Conf_{\mathbb{R}^n}(X)$ then $\dim(T_{x_0} Conf_{\mathbb{R}^n}(X)) = n - 1$.*

1. If $\widetilde{X}_i \neq \emptyset$ then $Ter_{S^{n-1}}(\widetilde{X}_i, \widetilde{X})$ has a nonempty interior. Thus, by Theorem 2.2, $T_{x_0}(Ter_{\mathbb{R}^n}(X_i, X))$ has a nonempty interior. [2] By Proposition 2.1 (see also the remark), $\dim Conf_{S^{n-1}}(\widetilde{X}) = n-2$. Hence, $\dim(T_{x_0} Conf_{\mathbb{R}^n}(X)) = n-1$. □

**Theorem 2.6** ( "no cusp" property in $\mathbb{R}^2$)**.** *Let $X = \{X_1, \dots, X_k\}$ be a collection of definable in A sets on $\mathbb{R}^2$. Let $Y = Conf_{\mathbb{R}^2}(X)$. Let $y_0 \in Y$. Then the germ of $Y$ at $y_0$ is a collection of definable in A differentiable arcs $\{\gamma_1, \gamma_2, \dots, \gamma_s\}$ such that $y_0$ belongs to each $\gamma_i$ and the unit tangent vectors of $\gamma_i$ and $\gamma_j$ are different, for $i \neq j$.*

Proof. Let $\gamma_1 \colon [0, \varepsilon) \to Conf_{\mathbb{R}^2}(X)$ be a definable nonconstant arc such that $\gamma_1(0) = y_0$ and $|\gamma_1(t) - y_0| = t$. Since the sets $X_i$ are definable in the o-minimal structure $A$ we can find two sets $X_1, X_2 \in X$ such that $\gamma_1 \subset Conf_{\mathbb{R}^2}(\bar{X})$ where $\bar{X} = \{X_1, X_2\}$. Let $l$ be the tangent ray to $\gamma_1$ at $y_0$. By Theorem 2.2, $l$ is a bisector ray of the angle defined by points $x_1 \in \widetilde{X}_1$, $x_2 \in \widetilde{X}_2$ and $y_0$, where $x_1, x_2$ are boundary points on $S_{y_0,r_0}$ of the supporting sets $\widetilde{X}_1$ and $\widetilde{X}_2$ on $S_{y_0,r_0}$.

Let $\delta > 0$ be a sufficiently small number such that the sets $X_1^\delta$ and $X_2^\delta$ - their radial projections to the supporting circle are disjoint. Let $\gamma_2 \colon [0, \varepsilon) \to Conf_{\mathbb{R}^2}(\bar{X})$ be another definable in $A$ arc such that $|\gamma_2(t) - y_0| = t$ and the germs at $y_0$ of the sets $\Gamma_1 = \gamma_1([0, \varepsilon))$ and $\Gamma_2 = \gamma_2([0, \varepsilon))$ are different and $\Gamma_2$ is also tangent to $l$ at $y_0$. By Arc Selection Lemma (see [2]), there exist two pairs of definable in $A$ arcs $\alpha_1, \alpha_2 \colon [0, \varepsilon) \to X_1^\delta$ and $\beta_1, \beta_2 \colon [0, \varepsilon) \to X_2^\delta$ such that $d(\gamma_1(t), X_1) = |\gamma_1(t) - \alpha_1(t)|$, $d(\gamma_2(t), X_1) = |\gamma_2(t) - \alpha_2(t)|$, $d(\gamma_1(t), X_2) = |\gamma_1(t) - \beta_1(t)|$ and $d(\gamma_2(t), X_2) = |\gamma_2(t) - \beta_2(t)|$. If $\alpha_1(t) = \alpha_2(t) = x_1$ and $\beta_1(t) = \beta_2(t) = x_2$, then the germs of $\gamma_1$ and $\gamma_2$ at $y_0$ are equal to the germ of $l$ at $y_0$. Thus, we can suppose that $\alpha_1(t) \neq Const$, for small $t$, and that, for small $t \neq 0$ we have $d(\gamma_1(t), X_1) > d(\gamma_2(t), X_1)$.

Take $t > 0$ sufficiently small. The segment connecting $\gamma_1(t)$ and $\beta_1(t)$ intersects the arc $\gamma_2(t)$. Let $z$ be an intersection point. Observe that $\beta_1(t)$ realizes the shortest distance between $z$ and $X_1$. Since $t$ is small and $\beta_1$ is a definable in $A$ arc, then we can suppose that $\beta_1(t)$ is a smooth point.

Consider now the circle $S_{\gamma_1(t), |\gamma_1(t) - \beta_1(t)|}$ and the circle $S_{z, |z - \beta_1(t)|}$. These circles are tangent at the point $\beta_1(t)$. That is why the ball with the center at $z$ and the radius $|z - \beta_1(t)|$ does not contain any point of $X_1$. But it means that $z$ does not belong to $Conf_{\mathbb{R}^2}(\bar{X})$. It is a contradiction. □

## 3. An example of not normally embedded conflict set

Here we are going to construct an example of a family of sets $X_1, X_2 \in \mathbb{R}^3$ satisfying the following conditions:

a) $Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ is not normally embedded in $\mathbb{R}^3$.
b) There exists a point $x_0 \in Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ such that, for small $r$, we have that $B_{x_0,r} \cap Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ is not homeomorphic to $T_{x_0}(Conf_{\mathbb{R}^3}(\{X_1, X_2\}))$.

**Example 3.1.** *Consider the space $\mathbb{R}^3$ with coordinates $(x_1, x_2, x_3)$. Let $X_1 \subset \mathbb{R}^3$ be a union of the hyperplanes: $x_3 = 1$ and $x_3 = -1$. Let $X_2$ be a union of the points: $a_1 = (1, 0, 0)$ and $a_2 = (-1, 0, 0)$.*

**Theorem 3.2.** *The set $Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ satisfies the conditions a) and b) described above.*

Proof. The set $Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ can be obtained as follows. Let $Y_1 \subset \mathbb{R}^2$ be the conflict set af the point $a_1 = (1, 0)$ and the union of straight lines $x_3 = 1$ and $x_3 = -1$. Observe, that here we consider $\mathbb{R}^2$ with coordinates $(x_1, x_3)$. The set $Y_1$ is a union of a part of the parabola defined by the point $a_1$ and the line $x_3 = 1$ situated above the line $x_3 = 0$ and a part of the parabola defined by the same point $a_1$ and the line $x_3 = -1$ situated below the line $x_3 = 0$.

Let $Y_2 \subset \mathbb{R}^2$ be the conflict set of the point $a_2 = (-1, 0)$ and the union of the lines $x_3 = 1$ and $x_3 = -1$. The set $Y_2$ can be obtained from $Y_1$ by the transformation: $(x_1, x_3) \to (-x_1, x_3)$. The set $Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ can be obtained as a union of the revolution surface of $Y_1$ with respect to the straight line $x_1 = 1, \quad x_2 = 0$ and the revolution surface of $Y_2$ with respect to the line $x_1 = -1, \quad x_2 = 0$. The intersection of the set $Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ with the plane $x_3 = 0$ is a union of two metric copies of $S^1$. These circles are tangent at the origin. That is why the set $Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ is not normally embedded. The tangent cone $T_0 Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ is a union of two planes intersecting transversally. The germ of the set $Conf_{\mathbb{R}^3}(\{X_1, X_2\})$ is homeomorphic to the quotient space of the disjoint union of two copies of $\mathbb{R}^2$ by the identification of the two origins. □

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Departamento de Matemática, Universidade Federal do Ceará (UFC), Campus do Picici, Bloco 914, Cep. 60455-760. Fortaleza-Ce, Brasil

*E-mail address*: birb@ufc.br

Department of Mathematics, Utrecht University, P.O.Box 80.010, 3508 TA Utrecht, The Netherlands

*E-mail address*: siersma@math.uu.nl