Local Embeddings of Lines in Singular Hypersurfaces

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Abstract

Lines on hypersurfaces with isolated singularities are classified. New normal forms of simple singularities with respect to lines are obtained. Several invariants are introduced.

Key Words and Phrases: classification, line, singular hypersurface, embedding.

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Introduction

Simple surface singularities have been known for long time since they occur in several different situations. They have at least fifteen characterizations [4]. One often describes them by the normal forms of their defining equations in certain coordinate system. This has been done by Arnol'd [1] and Siersma [8, 9]. The idea is to classify all the surfaces that contain the origin 0 under the right equivalence defined by the group $\mathcal{R}$, consisting of all the coordinate transformations preserving 0. The equivalence classes without parameters in their normal forms are called the simple surface singularities, or the $A-D-E$ singularities.

The observation by Gonzalez-Springberg and Lejeune-Jalabert [5] that on a singular surface X in $\mathbb{C}^3$ one can not always find a smooth line passing through the singular point was our starting point for classifying functions f, where the corresponding hypersurface $X = f^{-1}(O)$ contains a smooth curve.

We take the following set-up. Take a line L as the x-axis of the coordinate system in $\mathbb{C}^{n+1}$, classify all the hypersurfaces that contain L under the equivalence defined by the subgroup $\mathcal{R}_L$ of $\mathcal{R}$, consisting of all the coordinate transformations preserving L. If we compare this with the usual $\mathcal{R}$-classification of surfaces, the result classifies the lines contained in X. Our work is in fact about the pair (X, L).

We obtain the following results:

- The $\mathcal{R}_L$-classification is finer than the $\mathcal{R}$-classification, i.e. there are different 'positions' of lines on the same hypersurface.

- There exist an $\mathcal{R}_L$ invariant λ , which counts the maximal number of Morse points on the line over all Morsifications.
- λ is invariant under sections with generic hyperplanes containing L.
- Another proof of the non-existence of lines on E_8 surface.

We also introduce a sequence of higher order $\mathcal{R}_L$ invariants which gives more information about lines on hypersurfaces. It turns out that our work tells how lines are embedded in a hypersurface.

What brought this problem to our attention was the study of functions with non-isolated singularities on singular spaces [7]. Natural questions are: given a singular hypersurface X with isolated singularity, does there exist smooth curves on X passing through the singular point of X? If there exist smooth curves on X, how many of them? etc.

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1 Lines contained in singular spaces

1.1 Let L be a smooth curve in $\mathbb{C}^{n+1}$. Since L is locally biholomorphic to a line, we often say that L is a line. Let X be a singular space embedded in $\mathbb{C}^{n+1}$. If for a singular point $O \in X_{\text{sing}}$, there exists a line (smooth curve) L in $\mathbb{C}^{n+1}$ such that $O \in L$ and $L \setminus \{O\} \subset X_{\text{reg}}$, we say that X *contains (or has) smooth curve* passing through O. Very often, we say that X *contains (or has) a line* passing through O.

1.2 On a singular surface X in $\mathbb{C}^3$ one can not always find a line passing through (not contained in) the singular locus of X. Gonzalez-Sprinberg and Lejeune-Jalabert [5], by using resolutions of singularities, have proved a criterion for the existence of lines on any (two dimensional) surface, and given a natural partition of the set of smooth curves on $(X, 0)$ into families. Especially, on surfaces with A_k, $D_k (k \geq 4)$, E_6 or E_7 type singularity, there exist lines passing through the singular point. In fact one can find easily lines on this kind of surfaces simply by looking at the defining equations. But it is not easy to tell how many different lines there are on a surface, and what kind of invariants can be used to tell the differences. The hardest thing is to prove the non-existence of smooth curve on certain surfaces (e.g. E_8 type surface *loc.cit.*). There exists similar question for hypersurfaces with other dimensions.

1.3 Isolated hypersurface singularities Let X be a hypersurface germ with isolated singularity at the origin. The singularity of X is *simple* if and only if X can be defined in a neighborhood of the origin in $\mathbb{C}^{n+2}$ by the following equations (see [1] and [8, 9]):

$$\begin{aligned}
A_k: &\quad x^{k+1} + y^2 + z_1^2 + \cdots + z_n^2 = 0 \quad (k \geq 1),\\
D_k: &\quad x^2y + y^{k-1} + z_1^2 + \cdots + z_n^2 = 0 \quad (k \geq 4),\\
E_6: &\quad x^4 + y^3 + z_1^2 + \cdots + z_n^2 = 0,\\
E_7: &\quad x^3y + y^3 + z_1^2 + \cdots + z_n^2 = 0,\\
E_8: &\quad x^5 + y^3 + z_1^2 + \cdots + z_n^2 = 0.
\end{aligned}$$

The left hand sides of the equations are called the normal forms of the simple (or A-D-E) singularities.

The following equations define the *simple elliptic hypersurface singularities* (see [2] and [8, 9]):

$$\begin{aligned}
\tilde{E}_6: &\quad x^2y + z_1^3 + \alpha y^2 z_1 + \beta y^3 + z_2^2 + \cdots + z_n^2 = 0, \quad 4\alpha^3 + 27\beta^2 \neq 0;\\
\tilde{E}_7: &\quad x^4 + y^4 + \alpha x^2y^2 + z_1^2 + \cdots + z_n^2 = 0, \quad \alpha \neq \pm 2;\\
\tilde{E}_8: &\quad x^3 + \alpha xy^4 + \beta y^6 + z_1^2 + \cdots + z_n^2 = 0, \quad 4\alpha^3 + 27\beta^2 \neq 0.
\end{aligned}$$

2 $\mathcal{R}_L$-equivalence

2.1 Let $\mathcal{O}_{n+1}$ (or $\mathcal{O}$) be the stalk at 0 of the holomorphic structure sheaf $\mathcal{O}_{\mathbb{C}^{n+1}}$, $\mathfrak{m}$ be the maximal ideal of $\mathcal{O}$. Let $\mathcal{R}$ be the group of all the holomorphic automorphisms of $\mathbb{C}^{n+1}$ that preserve 0. Take L as the x-axis in $\mathbb{C}^{n+1}$, then the defining ideal of L is $\mathfrak{g} := (y_1, \ldots, y_n)$, and

$$\mathcal{R}_L := \{\phi \in \mathcal{R} \mid \phi^* \mathfrak{g} \subset \mathfrak{g}\}.$$

It has an action on $\mathfrak{mg}$ from the right hand side. This defines an equivalence relation on $\mathfrak{mg}$.

We recall some notations from [10]. Two germs $f, g \in \mathfrak{mg}$ are called $\mathcal{R}_L$-*equivalent* if there exists a $\phi \in \mathcal{R}_L$ such that $f = g \circ \phi$. A germ $f \in \mathfrak{mg}$ is called $k - \mathcal{R}_L$-*determined* in $\mathfrak{mg}$ if for each $g \in \mathfrak{mg}$ with $f - g \in \mathfrak{m}^{k+1} \cap \mathfrak{g} = \mathfrak{m}^k\mathfrak{g}$, f and g are $\mathcal{R}_L$-equivalent.

2.2 Theorem If

$$\mathfrak{m}^k\mathfrak{g} \subset \mathfrak{m}\tau_{\mathfrak{g}}(f) + \mathfrak{m}^{k+1}\mathfrak{g},$$

then f is $k - \mathcal{R}_L$-determined, where

$$\tau_{\mathfrak{g}}(f) := \mathfrak{m}\left(\frac{\partial f}{\partial x}\right) + \mathfrak{g}\left(\frac{\partial f}{\partial y_1}, \dots, \frac{\partial f}{\partial y_n}\right)$$

is the tangent space at f of the $\mathcal{R}_L$-orbit $\mathcal{R}_L(f)$ of f.

Proof Sincce the proof is a standard argument in singularity theory and very similar to [10], we give a brief proof. For any function $g \in \mathfrak{m}^k\mathfrak{g}$, define a one parameter family of functions $F := f + t(g - f)$. Let $\tilde{\tau}_{\mathfrak{g}}(F)$ be the collection of all the differentials of F by the one-parameter families of vector fields preserving 0 and L. By the assumption and Nakayama lemma, one finds that there exists a vector field δ of the above mentioned type, such that $g - f = \delta(F)$. The flow of δ gives an $\mathcal{R}_L$-equivalence between f and g by continuous induction. □

2.3 Definitions For a germ $h \in \mathfrak{m}\mathfrak{g}$, the $\mathcal{R}_L$-codimension of h is defined by

$$c_L = c_L(h) := \dim_{\mathbb{C}} \frac{\mathfrak{g}\mathfrak{m}}{\tau_{\mathfrak{g}}(h)}$$

Moreover there exists the next $\mathcal{R}_L$-invariant

$$\lambda = \lambda(h) := \dim_{\mathbb{C}} \frac{\mathcal{O}}{J(h) + \mathfrak{g}},$$

where $J(h)$ is the Jacobian ideal of h. Note that

$$1 \le \lambda(h) \le \mu(h),$$

where $\mu(h)$ is the Milnor number of h. The number λ is equal to the torsion number defined in [7], see also §2.4.

One can always write h in the following form

$$h = \sum A_j x y_j + \sum B_{jk} y_j y_k,$$

where A_j are functions of one variable x: $A_j \in \mathcal{O}_1$, and $B_{jk} \in \mathcal{O}$. With this notation we have the following simple formula

$$\lambda = \dim_{\mathbb{C}} \frac{\mathcal{O}_1}{x(A_1, \dots, A_n)}$$

2.4 Torsion number We recall the definition of the torsion number from [7]. This section plays no role in the rest of the paper.
Let L and X be analytic spaces in $\mathbb{C}^{n+1}$ defined by the ideal $\mathfrak{g}$ and $\mathfrak{h}$ respectively.

If $L \subset X$, then $\mathfrak{h} \subset \mathfrak{g}$. As a subspace, L can be defined by the ideal $\bar{\mathfrak{g}}$, the image of $\mathfrak{g}$ in $\mathcal{O}_X := \mathcal{O}/\mathfrak{h}$. The $\mathcal{O}_L := \mathcal{O}/\mathfrak{g}$-module

$$M := \frac{\bar{\mathfrak{g}}}{\bar{\mathfrak{g}}^2} \cong \frac{\mathfrak{g}}{\mathfrak{g}^2 + \mathfrak{h}}$$

is called the conormal module of $\bar{\mathfrak{g}}$. Denote by $T(M)$ the torsion submodule of M, and by $N := M/T(M)$ the torsionless factored module. We have the following exact sequence

$$0 \longrightarrow T(M) \longrightarrow M \longrightarrow N \longrightarrow 0.$$

In case L is a line on an isolated complete intersection singularity X, this sequence splits. The dimension of $T(M)$, as a $\mathbb{C}$ vector space, is called the *torsion number* of L and X, denoted by

$$\varrho(L, X) := \dim_{\mathbb{C}} T(M)$$

When L is a line on a hypersurface X with isolated singuarity, then $\lambda = \varrho(L, X)$. In fact, the defining equation h of X can be written in the form $h \equiv xA_1(x)y_1 \mod \mathfrak{g}^2$ by a coordinate transformation preserving L, which has been taken to be the x-axis. A calculation shows (see [7] for details) that

$$M = \frac{\mathfrak{g}}{(xA_1(x)y_1) + \mathfrak{g}^2}, \quad T(M) = \frac{(y_1)}{(xA_1(x)y_1) + (y_1^2) + y_1(y_2, \ldots, y_n)}.$$

Hence

$$\dim T(M) = \dim \frac{\mathcal{O}_1}{(xA_1(x))} = \lambda.$$

2.5 Semi-continuity of $\lambda(h)$ For $h \in \mathfrak{gm}$ with isolated singularity at 0, there exist deformations of h in $\mathfrak{gm}$ such that for generic parameters, the deformed function has only A_1 type critical points. Is $\lambda(h)$ a constant with respect to the deformation? The answer is negative in general, as easy examples show.

In order to prove a semi-continuity result we consider a map

$$H : (\mathbb{C}^{n+1} \times \mathbb{C}^\sigma, 0) \longrightarrow (\mathbb{C} \times \mathbb{C}^\sigma, 0)$$

defined by $H(z, s) = (h_s(z), s)$, where h_s is a deformation of h in $\mathfrak{mg}$. The critical locus of H is the germ at 0 of

$$C_H := \left\{ (z, s) \in \mathbb{C}^{n+1} \times \mathbb{C}^\sigma \mid \frac{\partial h_s}{\partial z_j} = 0, j = 0, \ldots n \right\}$$

Let $\tilde{\mathcal{O}}$ be the stalk at 0 of the structure sheaf of $\mathbb{C}^{n+1} \times \mathbb{C}^\sigma$. We consider $\mathcal{O}$ as a subring of $\tilde{\mathcal{O}}$. Denote

$$J(h_s) := \left(\frac{\partial h_s}{\partial z_0}, \ldots, \frac{\partial h_s}{\partial z_n} \right) \tilde{\mathcal{O}}.$$

Obviously, $C_H \cap (L \times \mathbb{C}^\sigma)$ can be defined by the ideal $J(h_s) + \mathfrak{g}\tilde{\mathcal{O}}$. Let

$$\tilde{\Lambda} := \frac{\tilde{\mathcal{O}}}{J(h_s) + \mathfrak{g}\tilde{\mathcal{O}}}, \quad \pi : (\mathbb{C}^{n+1} \times \mathbb{C}^\sigma, 0) \longrightarrow (\mathbb{C}^\sigma, 0), \quad \pi(z,s) = s.$$

Denote $p := \pi \mid_{C_H \cap (L \times \mathbb{C}^\sigma)}$.

Theorem Let h define an isolated singularity at 0.

1) p is finite analytic map;

2) there exist representatives of all the germs considered such that for all $s \in \mathbb{C}^\sigma$

$$\lambda(h) \geq \sum_{(P,s) \in p^{-1}(s)} \lambda(h_{s,P})$$

where $h_{s,P}$ denotes the germ of h_s at point P.

Proof By [6] theorem 5(d) and $\mu(h) < \infty$, p is finite.

Next we prove 2). Still we denote by $\tilde{\Lambda}$ the sheaf defined by $\tilde{\Lambda}$. Since p is finite, $p_*(\tilde{\Lambda})$ is again coherent. For any $s \in \mathbb{C}^\alpha$, we have

$$(p_*(\tilde{\Lambda}))_s \cong \bigoplus_{(P,s) \in p^{-1}(s)} \tilde{\Lambda}_P.$$

Note that $p^{-1}(0) = \{0\}$ and the minimal generating set of $\tilde{\Lambda}_0$ consists of $\lambda(h)$ elements. For s near 0, let λ_s be the number of the elements in the minimal generating set of $(p_*(\tilde{\Lambda}))_s$. Then $\lambda_s \leq \lambda(h)$. Hence

$$\lambda(h) = \dim \left(\frac{\tilde{\Lambda}_0}{p^*(\mathfrak{m}_{(\mathbb{C}^\sigma,0)})\tilde{\Lambda}_0} \right) \geq \sum_{(P,s) \in p^{-1}(s)} \dim \left(\frac{\tilde{\Lambda}_P}{p^*(\mathfrak{m}_{(\mathbb{C}^\sigma,s)})\tilde{\Lambda}_P} \right) = \sum_{(P,s) \in p^{-1}(s)} \lambda(h_{s,P}).$$

□

Remark that there always exist deformations of h in $\mathfrak{m}\mathfrak{g}$ with only A_1 points, called Morsifications. An easy calculation shows that there exist Morsifications h_s of h such that the number of A_1 points of h_s sitting on L is exact $\lambda(h)$.

Conclusion $\lambda(h)$ is semi-continuous under deformations and equal to the maximal number of A_1 points of h_s sitting on L, where h_s is a Morsification of h in $\mathfrak{g}\mathfrak{m}$.

□

3 Classification of lines contained in hypersurfaces

The aim of this section is to classify lines on surfaces in $\mathbb{C}^3$ (see §3.3), but first (for completeness) we classify the lines contained in curves in $\mathbb{C}^2$. The equivalence is defined by $\mathcal{R}_L$-action.

3.1 Theorem A curve $X \subset \mathbb{C}^2$ with an isolated simple or elliptic simple singularity contains a line passing through its singular point if and only if it is of type

$$A_{2k-1} \quad (k \geq 1), \quad D_k \quad (k \geq 4), \quad E_7, \quad \tilde{E}_7 \quad \text{or} \quad \tilde{E}_8.$$

If we choose the singular point as the origin and the line as the x-axis of the local coordinate system, the defining equation of X is $\mathcal{R}_L$-equivalent to one of the equations in table 1.

Table 1 : Curves containing x-axis

Type of X	Equations	λ	Name	c_L
A_{2k-1} $(k \geq 1)$	$x^k y + y^2 = 0 \quad (k \geq 1)$	k	$A_{2k-1,k}$	$k-1$
D_k $(k \geq 4)$	$x^2 y + y^{k-1} = 0 \quad (k \geq 4)$	2	$D_{k,2}$	$k-2$
	$x^l y + xy^2 = 0 \quad (k = 2l, \quad l \geq 3)$	l	$D_{k,l}$	l
E_7	$x^3 y + y^3 = 0$	3	$E_{7,3}$	4
$\tilde{E}_7$	$xy(x+y)(x+ty) = 0 \quad (t \neq 0,1)$	3	$\tilde{E}_{7,3}$	6
$\tilde{E}_8$	$x^2y^2 + \alpha x^4 y + y^3 = 0 \quad (\alpha \neq 0, \frac{1}{4})$	4	$\tilde{E}_{8,4}$	6

Sketch of the proof One considers the 2-jet $j^2 h$ of a function $h \in (xy, y^2)$. Let r be the rank of the coefficient matrix of $j^2 h$. If $r = 2$, then h is $\mathcal{R}_L$-equivalent to $A_{1,1}$. If $r = 1$, one finds that h is $\mathcal{R}_L$-equivalent to $A_{2k-1,k}$ $(k > 1)$ or y^2.

If $r = 0$, one needs to consider $j^3 h = ax^2 y + bxy\ 2 + cy^3$.

In case $a \neq 0$, $j^3 h$ is $\mathcal{R}_L$-equivalent to $D_{4,2}$ if $b^2 - 4ac \neq 0$ and to $x^2 y$ if $b^2 - 4ac = 0$. In the latter case, one considers the higher jets and finds the $D_{k,2}$ singularities.

In case $a = 0, b \neq 0$, one finds the $D_{2l,l}$'s by considering the higher jets of h.

In case $a = b = 0, c \neq 0$, one finds that $j^4 h$ is $\mathcal{R}_L$-equivalent to $E_{7,3}$ or $y^3 + x^2 y^2$ or y^3. If $j^4 h \overset{\mathcal{R}_L}{\leftrightsquigarrow} y^3 + x^2 y^2$ (where $\overset{\mathcal{R}_L}{\leftrightsquigarrow}$ means "to be $\mathcal{R}_L$-equivalent to "), then $j^5 h \overset{\mathcal{R}_L}{\leftrightsquigarrow} y^3 + x^2 y^2 + \alpha x^4 y \overset{\mathcal{R}_L}{\leftrightsquigarrow} \tilde{E}_{8,4}$ $(\alpha \neq 0, \frac{1}{4})$. One stops here since the singularity of h is neither simple nor elliptic simple if $j^4 h \overset{\mathcal{R}_L}{\leftrightsquigarrow} y^3$ or $j^5 h \overset{\mathcal{R}_L}{\leftrightsquigarrow} y^3 + x^2 y^2 + \alpha x^4 y$ $(\alpha = 0, \frac{1}{4})$.

It remains to study the case $j^3 h = 0$. In the generic case, $j^4 h$ is $\mathcal{R}_L$-equivalent to $\tilde{E}_{7,3}$. In the non-generic case, the singularity of h is neither simple nor elliptic simple. □

Proposition If $h \in \mathfrak{m}\mathfrak{g}$ defines an isolated singularity and $n = 1$, then

$$c_L = \mu - \lambda.$$

Proof It is an easy exercise in algebra to show that for any $n \geq 1$

$$c_L = \mu - \lambda + \nu.$$

where $\nu = \dim \frac{(\mathfrak{m}J(h))\cap\mathfrak{m}\mathfrak{g}}{\tau_{\mathfrak{g}}(h)}$.
And if $n = 1$, we have $\nu = 0$. $\square$

3.2 A class of singularities S (with respect to an equivalence) is *adjacent to* a class T (notation: $T \longleftarrow S$) if every function $f \in S$ can be deformed into a function of T by an arbitarily small perturbation (cf. [3]).

For example, since $x^2y + y^6 + cy^5$ is $\mathcal{R}_L$-equivalent to $D_{6.2}$, hence $D_{6.2} \longleftarrow D_{7.2}$.

In the following diagram, we record some adjacencies from the proof of the theorem above, called the classification tree. Note that it does not contain all the adjacencies.

Classification Tree 1

$$\begin{tikzcd}
c_L & & & & & \\
0 & A_{1,1} & & & & \\
1 & A_{3,2} \arrow[u] & & & & \\
2 & A_{5,3} \arrow[u] & D_{4,2} \arrow[ul] & & & \\
3 & A_{7,4} \arrow[u] & D_{5,2} \arrow[u] & D_{6,3} \arrow[ul] & & \\
4 & A_{9,5} \arrow[u] & D_{6,2} \arrow[u] & D_{8,4} \arrow[u] & E_{7,3} \arrow[ul] & \\
5 & A_{11,6} \arrow[u] & D_{7,2} \arrow[u] & D_{10,5} \arrow[u] & & \\
6 & A_{13,7} \arrow[u] & D_{8,2} \arrow[u] & D_{10,5} \arrow[u] & \tilde{E}_{8,4} \arrow[uu] & \tilde{E}_{7,3} \arrow[uul] \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & &
\end{tikzcd}$$

3.3 Theorem A surface $X \subset \mathbb{C}^3$ with a simple isolated singularity contains a line passing through its singular point if and only if it is of type:

$$A_k \quad (k \geq 1), \qquad D_k \quad (k \geq 4), \qquad E_6, \quad \text{or} \quad E_7.$$

Moreover, if we choose the singular point of X as the origin and the line L on X as the x-axis in $\mathbb{C}^3$, that is, L is defined by the ideal $\mathfrak{g} = (y, z)$, then the defining equation of X is $\mathcal{R}_L$-equivalent to one of the equations in table 2.

Table 2 : Simple Surface Singularities Passing Through x-axis

Type of X	Equations	λ	Name	c_L
A_k $(k \geq 1)$	$xy + z^{k+1} = 0$	1	$A_{k,1}$	$k-1$
	$x^l y + x^s z^2 + yz = 0$ $(k = 2l + s - 1, l \geq 2, s \geq 0)$	l	$A_{k,l}$	$k-1$
D_k $(k \geq 4)$	$x^2 y + y^{k-1} + z^2 = 0$	2	$D_{k,2}$	$k-1$
	$x^2 y + xz^2 + y^2 = 0$	2	$D^*_{5,2}$	4
	$x^l y + xy^2 + z^2 = 0 \quad (k = 2l, l \geq 3),$	l	$D_{k,l}$	$k-1$
	$x^l y + xz^2 + y^2 = 0 \quad (k = 2l+1, l \geq 3)$	l	$D_{k,l}$	$k-1$
E_6	$x^2 z + y^3 + z^2 = 0$	2	$E_{6,2}$	5
E_7	$x^3 y + y^3 + z^2 = 0$	3	$E_{7,3}$	6

Sketch of the proof It is just classifying the functions in $(y, z)\mathfrak{m}$ by choosing all the coordinate transformations in $\mathcal{R}_L$. One starts from the 2-jet of $h \in (y, z)\mathfrak{m}$:

$$j^2 h = 2a_{12}xy + 2a_{13}xz + a_{22}y^2 + 2a_{23}yz + a_{33}z^2, \quad a_{ij} = a_{ji}$$

Let r be the rank of the coefficient matrix (a_{ij}) of $j^2 h$.

If $r = 3$, then $(a_{12}, a_{13}) \neq 0$. We can assume $a_{12} \neq 0$. Then

$$j^2 h \overset{\mathcal{R}_L}{\leftrightsquigarrow} xy + az^2 \overset{\mathcal{R}_L}{\leftrightsquigarrow} xy + z^2 \overset{\mathcal{R}}{\leftrightsquigarrow} A_1$$

where we still denote by x, y, z the new coordinates by abusing of notations.

Suppose $r = 2$. If $(a_{12}, a_{13}) \neq 0$, then $j^2 h \overset{\mathcal{R}_L}{\leftrightsquigarrow} xy$. The higher order jet $j^{k+1} h \overset{\mathcal{R}_L}{\leftrightsquigarrow} xy + az^{k+1}$ which, by theorem 3.2, is $(k+1)$-$\mathcal{R}_L$-determined in $(y, z)\mathfrak{m}$, and $\mathcal{R}_L$-equivalent to $xy + z^{k+1}$ if $a \neq 0$.

If $(a_{12}, a_{13}) = 0$, $j^2 h \overset{\mathcal{R}_L}{\leftrightsquigarrow} yz$. For $l \geq 2$, $j^{l+1} h \overset{\mathcal{R}_L}{\leftrightsquigarrow} yz + ax^l y + bx^l z$

- $\underline{ab \neq 0}$: $j^{l+1} h \overset{\mathcal{R}_L}{\leftrightsquigarrow} yz + x^l y + z^2$, which is $A_{2l-1,l}$.

- $\underline{ab = 0, a \neq 0}$: $j^{l+1} h \overset{\mathcal{R}_L}{\leftrightsquigarrow} yz + x^l y$ which is not finitely $\mathcal{R}_L$-determined in $(y, z)\mathfrak{m}$. But $j^{l+s+1} h \overset{\mathcal{R}_L}{\leftrightsquigarrow} yz + x^l y + \alpha x^s z$ which, if $\alpha \neq 0$, is $(l + s + 1)$-$\mathcal{R}_L$-determined in $(y, z)\mathfrak{m}$, $\mathcal{R}_L$-equivalent to $yz + x^l y + x^s z$.

Suppose $r = 1$. We may assume $j^2h = z^2$. A calculation shows that

$$j^3h \overset{\mathcal{R}_L}{\leftrightsquigarrow} z^2 + axy^2 + bx^2y + cx^2z + ey^3$$

which is $\mathcal{R}_L$-equivalent to $z^2 + x^2y + ey^3$ if $b \neq 0$. Further study of this case, gives $D_{k,2}$ $(k \geq 4)$. If $b = 0$, we have

- $\underline{ac \neq 0}$: h is $\mathcal{R}_L$-equivalent to $D^*_{5,2}$.
- $\underline{a \neq 0, c = 0}$: by studying the higher jet of h, we obtain $D_{2l,l}$ and $D_{2l+1,l}$ $(l \geq 3)$.
- $\underline{a = 0, ce \neq 0}$: we obtain $E_{6,2}$.
- $\underline{a = c = 0, e \neq 0}$: we can write $j^3 = z^2 + y^3$, and $j^4h \overset{\mathcal{R}_L}{\leftrightsquigarrow} z^2 + y^3 + \alpha x^3y + \beta x^3z + \gamma x^2y^2$. A calculation shows that h is $\mathcal{R}_L$-equivalent to $E_{7,3}$ if $\alpha \neq 0$. And in case $\alpha = 0$, there are no simple germs with 4-jet $\mathcal{R}_L$-equivalent to $z^2 + y^3 + \beta x^3z + \gamma x^2y^2$.
- $\underline{a = e = 0}$: there are no simple germs with 3-jet $\mathcal{R}_L$-equivalent to $z^2 + cx^2z$.

If $r = 0$, also there are no simple germs with 2-jet $\mathcal{R}_L$-equivalent to 0.

E_8 does not appear in the list. An intuitive reason for this is that one can bring the 4-jet of the E_k $(k \geq 7)$ singularities into the normal form: $z^2 + y^3 + ax^3y$. In case $a \neq 0$, one has E_7, otherwise, there is no term x^4, so one can never get E_8.

The list of $\mathcal{R}$-simple singularities is now exhausted.

One can easily calculate the $\lambda(h)$: $\lambda(h) = 1, 2$, up to $\left[\frac{k+1}{2}\right]$ when h defines an A_k surface, $\lambda(h) = 2$ or $\left[\frac{k}{2}\right]$ when h defines a D_k surface, and so on. □

3.4 Corollary 1) There does not exist smooth curves on surface with E_8 type isolated singularity;

2) All the normal forms of the germs in table 2 are also $\mathcal{R}_L$-simple. And all the $\mathcal{R}_L$ simple germs of $(y, z)\mathfrak{m}$ are contained in table 2.

Proof 1) follows from the proof of the theorem in §3.3.
2) A germ $h \in (y, z)\mathfrak{m}$ is $\mathcal{R}_L$-simple if $c_L(h) < \infty$ and any small deformation $\tilde{h}$ of h in $(y, z)\mathfrak{m}$ cuts only finite number of the $\mathcal{R}_L$-orbits of $(y, z)\mathfrak{m}$. It follows from the classification procedure that all the germs in table 2 are $\mathcal{R}_L$-simple and that they are the only ones. All $\mathcal{R}_L$-simple germs in $(y, z)\mathfrak{m}$ are also $\mathcal{R}$-simple germs, which are all contained in table 2 except E_8. □

3.5 Remarks For a hypersurface X defined by $h = 0$, two lines L_1 and L_2 in X are in one class if there exists a $\varphi \in \mathcal{R}$ such that $\varphi(L_1) = L_2$ and $h \circ \varphi = h$.

The results of [5] say that there exist k, 3, 2 and 1 families of smooth curves on A_k $(k \geq 1)$, D_k $(k \geq 4)$, E_6 and E_7 surfaces respectively. In our classification of smooth curves on $A-D-E_6-E_7$ surfaces above, there exist $\left[\frac{k+1}{2}\right]$, 2, 1 and 1 class(es) smooth curves on A_k $(k \geq 1)$, D_k $(k \geq 4)$, E_6 and E_7 surfaces respectively. The reason for this is the symmetry property of the simple singularities. Especially, there is only one class of smooth curves on A_1, A_2, D_4, E_6 and E_7 surface. There exist two different classes of smooth curves on D_5 surface with the same $\lambda = 2$. On A_k $(k \geq 3)$ and D_k $(k \geq 6)$ surfaces the number λ tells the differences of the smooth curves on these surfaces.

3.6 The classification tree We record some adjacencies in the proof of the theorem in §3.3 in the classification tree 2. Note that this table contains only some of the adjacencies. (cf. §3.2)

<u>The Classification Tree 2</u>

$$
\begin{tikzcd}
c_L & & & & & & & & \\
0 & A_{1,1} & & & & & & & \\
1 & A_{2,1} \arrow[u] & & & & & & & \\
2 & A_{3,1} \arrow[u] & A_{3,2} \arrow[ul] & & & & & & \\
3 & A_{4,1} \arrow[u] & A_{4,2} \arrow[u] & & & & D_{4,2} \arrow[ullll] & & \\
4 & A_{5,1} \arrow[u] & A_{5,2} \arrow[u] & A_{5,3} \arrow[ul] & & & D_{5,2} \arrow[u] & D^*_{5,2} \arrow[ul] & \\
5 & A_{6,1} \arrow[u] & A_{6,2} \arrow[u] & A_{6,3} \arrow[u] & & & D_{6,2} \arrow[u] & D_{6,3} \arrow[u] & E_{6,2} \arrow[ul] \\
6 & A_{7,1} \arrow[u] & A_{7,2} \arrow[u] & A_{7,3} \arrow[u] & A_{7,4} \arrow[ul] & & D_{7,2} \arrow[u] & D_{7,3} \arrow[u] & E_{7,3} \arrow[u] \\
7 & A_{8,1} \arrow[u] & A_{8,2} \arrow[u] & A_{8,3} \arrow[u] & A_{8,4} \arrow[u] & & D_{8,2} \arrow[u] & D_{8,4} \arrow[u] & \\
8 & A_{9,1} \arrow[u] & A_{9,2} \arrow[u] & A_{9,3} \arrow[u] & A_{9,4} \arrow[u] & A_{9,5} \arrow[ul] & D_{9,2} \arrow[u] & D_{9,4} \arrow[u] & \\
\vdots & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}
$$

3.7 Lines on surfaces with simple elliptic singularities If we continue the calculations for the case $r = 1, 0$ in the proof of theorem 3.3 and record the adjancencies appeared, we obtain the following theorem and classifcation tree.

Theorem A surface X with simple elliptic singularity at O has a line L passing through O. And if one chooses L as the $x-$axis of the local coordinate system, the defining equation of X is $\mathcal{R}_L$-equivalent to one of the equations in table 3.

Table 3 : Simple Elliptic Surface Singularities Containing x-axis

Type of X	Equations	λ	Name	c_L
$\tilde{E}_6$	$x^2y + d_0xz^2 + d_1y^3 + d_2y^2z + d_3yz^2 + d_4z^3 = 0$ (For generic d_i $(0 \leq i \leq 4)$, especially $\Delta_1 \neq 0$)	2	$\tilde{E}_{6,2}$	7
$\tilde{E}_7$	$x^2z + x^3y + bx^2y^2 + cxy^3 + dy^4 + z^2 = 0$ $(\theta \neq 0, 1, \quad \Delta_2 \neq 0)$	2	$\tilde{E}_{7,2}$	8
	$xy(x-y)(x-\nu y) + z^2 = 0 \quad (\nu \neq 0, 1)$	3	$\tilde{E}_{7,3}$	8
$\tilde{E}_8$	$x^3z + \gamma x^2y^2 + \delta x^4y + y^3 + z^2 = 0 \quad (D \neq 0)$	3	$\tilde{E}_{8,3}$	9
	$\alpha x^4y + x^2y^2 + y^3 + z^2 = 0 \quad (\alpha \neq 0, \frac{1}{4})$	4	$\tilde{E}_{8,4}$	9

We give the explanation of the notations in table 3.

$$\Delta_1 := \det \begin{bmatrix} 2d_4 & 0 & 0 & -4d_0d_1 & -3d_0d_2 & -2d_0d_3 \\ c & 0 & -6d_1 & -4d_2 & -2d_3 & 0 \\ 0 & -12d_1 & -8d_2 & -4d_3 & 0 & -c^2 \\ 0 & 0 & 4d_1 & 3d_2 & 2d_3 & d_4 \\ 0 & d_2 & 2d_3 & 3d_4 & -c^2 & 0 \\ 0 & 0 & d_2 & 2d_3 & 3d_4 & -c^2 \end{bmatrix}.$$

$$\Delta_2 := \det \begin{bmatrix} 2 & 3 & 2b & c & 0 & 0 \\ c & -1 & 3 & 2b & c & 0 \\ 0 & 0 & 3+2b & 2b+3c & c+4d & 0 \\ 0 & 3+2b & 2b+3c & c+4d & 0 & 0 \\ 0 & 0 & -1 & 3 & 2b & c \\ 0 & 0 & 0 & 3+2b & 2b+3c & c+4d \end{bmatrix}.$$

θ is the cross ratio of the four lines defined by

$$-\frac{1}{4}x^4 + x^3y + bx^2y^2 + cxy^3 + dy^4 = 0.$$

and

$$D := \frac{9}{4} \pm 6\gamma\delta - \frac{4}{3}\gamma^3 + \frac{16}{3}\delta^3 - \frac{4}{3}\gamma^2\delta^2.$$

Classification Tree 3

$$\begin{tikzcd}
D^*_{5,2} & & \tilde{E}_{6,2} \arrow[ll] & & \\
& E_{6,2} \arrow[ul] & \tilde{E}_{7,2} \arrow[l] & \tilde{E}_{7,3} \arrow[l] & \\
& & E_{7,3} \arrow[ul] & \tilde{E}_{8,3} \arrow[l] & \tilde{E}_{8,4} \arrow[l]
\end{tikzcd}$$

3.8 Remark For $\tilde{E}_{6,2}$, there are two $\mathcal{R}_L$ moduli. $\tilde{E}_{7,2}$, $\tilde{E}_{7,3}$, $\tilde{E}_{8,3}$ and $\tilde{E}_{8,4}$ are one $\mathcal{R}_L$ modular.

3.9 Invariance of λ under sections with hyperplanes through L Let L be a line contained in a hypersurface $X \subset \mathbb{C}^{n+1}$ with isolated singularity, H a generic hyperplane in the pencil of hyperplanes passing through L. Then $X \cap H$ has also isolated singularity and contains L. A straightforward computation shows:

Proposition $\lambda(X \cap H) = \lambda(X)$.

In fact we can continue cutting until we are in the curve case. If we apply this on the surfaces defined by the equation in table 2, we have

X	$A_{k,l}$	$D_{k,l}$	$D^*_{5,2}$	$E_{6,2}$	$E_{7,3}$
$X \cap H$	$A_{2l-1,l}$	$A_{2l-1,l}$	$A_{3,2}$	$A_{3,2}$	$A_{5,3}$

Obviously we have in these examples $\mu(X \cap H) = 2\lambda - 1$. This reminds very much to Milnor's double point formula

$$\mu = 2\delta + 1 - r,$$

where δ is the maximal number of double points on a deformed curve, and r is the number of the branches of the curve. But this is a coincidence, which holds since $X \cap H$ is of type A_{odd}. Accordining to the above proposition

$$\lambda = \lambda(X) = \lambda(X \cap H) \leq \delta(X \cap H)$$

Inspection shows $\lambda(A_{\text{odd}}) = \delta(A_{\text{odd}})$. However for X with $\tilde{E}_{6,2}$ type singularity, this is not the case. $X \cap H$ is of $D_{4,2}$ type singularity and $\delta = 3$, $\lambda = 2$ and $r = 3$.

3.10 Remark For all the germs in table 2 and 3, we have $c_L = \mu - 1$, especially $c_L(h)$ does not depend on λ. This means that for those surfaces we can not move from one position of a line into another position. This was a surprise to us; in other dimensions this is different (see also question 4.4).

4 Lines on hypersurfaces in $\mathbb{C}^4$

About the existence of lines on hypersurfaces in $\mathbb{C}^4$, we mention first that on corank 2 singularities there always exist lines due to the suspension $xy + g(z, w)$. This gives lines on, for example, the E_8 hypersurface in $\mathbb{C}^4$. And $\lambda = 1$ all the time. For corank 3 singularities one can consider the suspension $g(x, y, z) + w^2$. If g contains the x-axis, so does this suspension with the same λ. But there are examples of corank 3 singularities which do not contain any lines. Here is one of them:

$$h = w^2 + x^{11} + y^{13} + z^{15} = 0.$$

It is natural to consider the above classifications with four variables. We give here the beginning of the classification for simple singularities. The proof is very similar to the cases considered in §3. Also here we record some of adjancencies in the classification tree. We distinguish between $\lambda = 1$ and $\lambda > 1$.

4.1 Theorem Let $X \subset \mathbb{C}^4$ be hypersurfaces with isolated simple singularity. If X contains x-axis and $\lambda = 1$, then the defining equation of X is $\mathcal{R}_L$-equivalent to one of the equations in table 4. □

Table 4

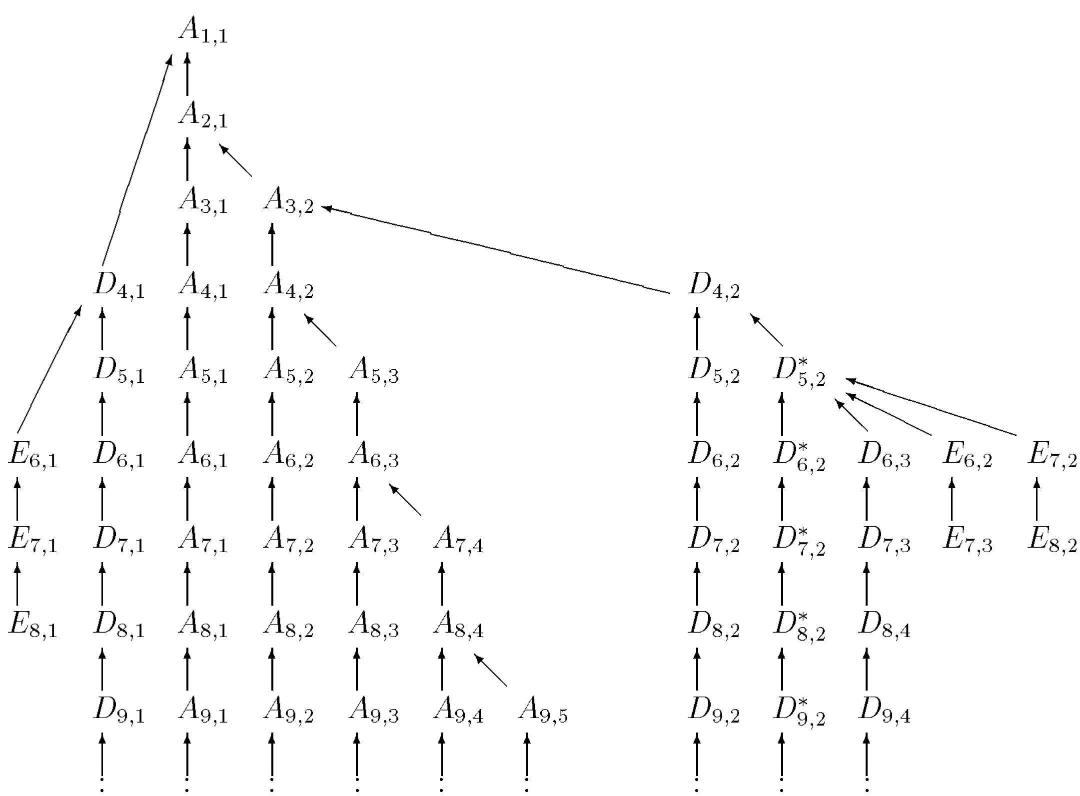
Type of X	Equations	λ	Name	c_L
A_k $(k \geq 1)$	$xy + z^{k+1} + w^2 = 0$	1	$A_{k,1}$	$k-1$
D_k $(k \geq 4)$	$xy + z^2w + w^{k-1} = 0$	1	$D_{k,1}$	$k-1$
E_6	$xy + z^4 + w^3 = 0$	1	$E_{6,1}$	5
E_7	$xy + z^3w + w^3 = 0$	1	$E_{7,1}$	6
E_8	$xy + z^5 + w^3 = 0$	1	$E_{8,1}$	7

4.2 Theorem Let $X \subset \mathbb{C}^4$ be hypersurfaces with simple isolated singularity. If X contains x-axis, $\lambda > 1$ and $c_L(X) \leq 8$, then the defining equation of X is $\mathcal{R}_L$ equivalent to one of the equations in table 5. □

Table 5

Type of X	Equations	λ	Name	c_L
A_k $(k \geq 1)$	$x^l y + x^s z^2 + yz + w^2 = 0$ $(k = 2l + s - 1, l \geq 2, s \geq 0)$	l	$A_{k,l}$	$3l + s - 3$
D_k $(k \geq 4)$	$x^2 y + y^{k-1} + z^2 + w^2 = 0$	2	$D_{k,2}$	k
	$x^2 z + x^{l-3} w + xy^2 + wz = 0 \quad (l \geq 5)$	2	$D^*_{l,2}$	l
	$x^l y + xy^2 + z^2 + w^2 = 0 \quad (l \geq 3)$	l	$D_{2l,l}$	$3l - 2$
	$x^l z + xy^2 + z^2 + w^2 = 0 \quad (l \geq 3)$	l	$D_{2l+1,l}$	$3l - 1$
E_6	$x^2 y + y^2 + z^3 + w^2 = 0$	2	$E_{6,2}$	6
E_7	$x^2 z + y^3 + xyw + zw = 0$	2	$E_{7,2}$	7
	$x^3 y + y^3 + z^2 + w^2 = 0$	3	$E_{7,3}$	8
E_8	$x^2 z + y^3 + xw^2 + zw = 0$	2	$E_{8,2}$	8

The Classification Tree 4



4.3 Remark 1) One may compare table 4 with the list of simple hypersurface singularities in §1.3;

2) If we compare with the surface case, we find that on E_8 hypersurface there exist two classes of lines. Also on other hypersufaces, there exists more lines than in the case of surfaces in $\mathbb{C}^3$.

4.4 Question As claimed by the proposition in §3.1, all the known examples show the following remarkable formulae:

$$c_L = \mu + (n-2)\lambda - n + 1.$$

Is it true for any hypersurfaces with isolated singularities in $\mathbb{C}^{n+1}$?

4.5 Higher order invariants For $h \in \mathfrak{m}\mathfrak{g}$ defining an isolated singularity, there exists the following filtration:

$$J(h) + \mathfrak{g} \supset J(h) + \mathfrak{g}^2 \supset \cdots \supset J(h) + \mathfrak{g}^k \supset \cdots \supset J(h).$$

Define

$$\Lambda_k = \Lambda_k(h) := \frac{\mathcal{O}}{J(h) + \mathfrak{g}^k}, \quad k = 1, 2, \cdots .$$

and the higher order invariants:

$$\lambda_k = \lambda_k(h) := \dim_{\mathbb{C}} \Lambda_k, \quad k = 1, 2, \cdots .$$

Note that there exists a k_0 such that

$$\lambda = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{k_0} = \mu.$$

Denote

$$\kappa = \kappa(h) = [\lambda_1, \lambda_2, \ldots, \lambda_{k_0} = \mu],$$

where k_0 is the minimal k such that $\lambda_k = \mu$, called the length of the sequence $\kappa(h)$.

The $\kappa(h)$ give more information about how a line is embedded into a singular hypersurface. Recall that in table 2 and 5, we have $D_{l,2}$ and $D^*_{l,2}$ $(l \geq 5)$ with the same λ.

Easy calculations show the following: For surfaces in $\mathbb{C}^3$, $\kappa(D_{5,2}) = [2,3,4,5]$ with length 4, and $\kappa(D^*_{5,2}) = [2,4,5]$ with length 3. For the D_k type hypersurfaces in $\mathbb{C}^4$ we have $\kappa(D_{l,2}) = [2, \ldots, l]$ $(l \geq 5)$ with length $l-1$, $\kappa(D^*_{5,2}) = [2,4,5]$ with length 3. For $l \geq 3$, $\kappa(D^*_{2l,2}) = [2,5,7,\ldots,2l-1,2l]$ with length l, and $\kappa(D^*_{2l+1,2}) = [2,5,7,\ldots,2l-1,2l+1,]$ with length l.

So, higher order invariants λ_k's distinguish $D_{l,2}$ and $D^*_{l,2}$.

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