The Vanishing Topology of Non Isolated Singularities

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1. Introduction

We consider holomorphic function germs $f : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, 0)$ and allow arbitrary singularities (isolated or non-isolated). We are interested in the topology of this situation, especially the so called vanishing homology.

We first recall the definition of the Milnor fibration. For $\epsilon > 0$ small enough there exist an ϵ-ball B_ϵ in $\mathbb{C}^{n+1}$ and an η-disc D_η in $\mathbb{C}$ such that the restriction:

$$f : \quad E := f^{-1}(D_\eta) \cap B_\epsilon \longrightarrow D_\eta$$

is a locally trivial fibre bundle over $D_\eta \setminus \{0\}$. The fibres, mostly denoted by F, are called Milnor fibres. The groups $H_*(E, F)$ are called the vanishing homology groups.

The Milnor fibre F, its homotopy type and homology are interesting topological objects. So is the monodromy operator

$$\mathbb{T}_* \quad : H_*(F) \to H_*(F)$$

of the fibration.

Well known facts are:

- The Milnor fibre is $2n$-dimensional and has the homotopy type of an n-dimensional CW-complex (Milnor [33]),
- The Milnor fibre is $(n - s - 1)$-connected, where s is equal to the dimension of the singular locus of f (Kato-Matsumoto [26]),
- If f has an isolated singularity then the Milnor fibre has the homotopy type of a bouquet of n-dimensional spheres. The number μ of these spheres is called the Milnor number of the isolated singularity (Milnor [33]),
- the eigenvalues of the monodromy operator are roots of unity. See Griffiths [19] for references to four different proofs.

Isolated singularities have been studied in great detail during the last 30 years. They have wonderful properties, which relate different aspects of the singularity. In this paper we want to discuss especially non-isolated singularities. Although the properties (e.g. the topological type of the Milnor fibre) are more complicated than for isolated singularities, there is a lot of interesting structure available.

Let $\Sigma = \Sigma(f)$ be the singular locus of f. For every point of Σ we can do the Milnor construction. So for every $x \in \Sigma$ we have a (local) Milnor fibration (e.g. a space $E_{(x)}$, the local Milnor fibre $F_{(x)}$ and a Milnor monodromy $\mathbb{T}_{(x)}$). We want to investigate the relation between these objects for all $x \in \Sigma$. This is (near to) the study of the sheaf of vanishing cycles [10].

To be more precise: one could try to define a stratification of Σ in such a way that two points of Σ are in the same stratum if they can be joined by a (continuous) path such that there exits a (continuous) family of Milnor fibrations of constant fibration type.

According to a result of Massey, the constancy of Lê numbers (for definition see the contribution of Gaffney [17] in this Volume) implies constancy of the fibration type under certain dimension conditions (more precisely $s \leq n-2$ for the homotopy-type and $s \leq n-3$ for the diffeomorphism-type). We refer to Massey's monograph [31] for details and for many other related facts.

Let us suppose that we end up with a situation, where we have stratified Σ (according to the above principle):

$$\Sigma = \Sigma^k \cup \cdots \cup \Sigma^1 \cup \Sigma^0 ,$$

where $\Sigma^j \setminus \Sigma^{j-1}$ is j-dimensional and smooth. For every connected component of $\Sigma^j \setminus \Sigma^{j-1}$ we have a monodromy representation of its fundamental group on the homology groups of a typical Milnor fibre at a general point on the statum:

$$\pi_1(\Sigma^j \setminus \Sigma^{j-1}, x) \to \mathrm{Aut}\ [H_*(F_{(x)})] .$$

We call these monodromies"vertical". The vertical monodromies contain a lot of extra information about the singularity. The Milnor monodromy is called "horizontal" and commutes with the vertical monodromies.

We intend to discuss this situation in several examples; paying most attention to the situation where Σ is 1-dimensional, where the stratification is rather simple. We also intend to treat some examples of higher dimension.

This paper is organized as follows. In section 2 we recall some facts about isolated singularities. In particular we discuss the relation between variation mapping, monodromy and intersection form.

In section 3 we treat singularities with a 1-dimensional critical set. We follow first [50] and [51], treat several examples, where vertical and horizontal monodromy play a role, and focus at the end on bouquet decompositions of the Milnor fibre. These seem to occur as soon as we stay near to the case of isolated singularities.

Section 4 is about singular sets of higher dimension. We discuss and summarize recent work.

2. About isolated singularities

The theory of variation mappings plays an important role in our discussions. We first repeat some of the well known facts about isolated singularities. They can be found in the literature on several places, e.g. Milnor [33], Lamotke [27], Arnol'd-Gusein Zade-Varchenko [3], Stevens [55].

In the isolated singularity case there exists a geometric monodromy $h : F \to F$ such that $h|_{\partial F}$ is the identity.
Let $\mathbb{T}_q = h_* : H_q(F) \to H_q(F)$ be the algebraic monodromy. The map $\mathbb{T}_q - \mathbb{I} : H_q(F) \to H_q(F)$ factors over:

$$\mathrm{VAR}_q : H_q(F, \partial F) \to H_q(F)$$

which is defined by

$$\mathrm{VAR}_q[x] = [h(x) - x]$$

We have a commutative diagram:

$$\begin{tikzcd}
H_q(F) \arrow[r, "\mathbb{T}_q-\mathbb{I}"] \arrow[d, "j_*"] & H_q(F) \arrow[d, "j_*"] \\
H_q(F,\partial F) \arrow[r, "\mathbb{T}_q-\mathbb{I}"] \arrow[ur, dashed, "\mathrm{VAR}"] & H_q(F,\partial F)
\end{tikzcd}$$

LEMMA 2.1. $\mathrm{VAR}_q : H_q(F, \partial F) \to H_q(F)$ *is an isomorphism if* $q \neq 0$.

Proof. Consider the exact sequence of the pair (S^{2n+1}, F) and the following isomorphism:

$$H_{q+1}(S^{2n+1}, F) \cong H_q(F, \partial F) \otimes H_1(I, \partial I) \cong H_q(F, \partial F).$$

This gives the following exact variation sequence

$$\ldots \to H_{q+1}(S^{2n+1}) \to H_q(F, \partial F) \stackrel{\mathrm{VAR}_q}{\longrightarrow} H_q(F) \to H_q(S^{2n+1}) \to \ldots$$

The lemma now follows from the fact that $\tilde{H}_q(S^{2n+1}) = 0$ for $q \neq 2n+1$

PROPOSITION 2.2. *For isolated singularities,* Ker $j_* =$ Ker $(\mathbb{T}_n - \mathbb{I})$.

Proof. Let E be the total space of the Milnor fibration $f : E \to \partial D_\eta$. The diagram above relates the variation mapping and j_* to the Wang sequence of the fibration:

$$0 \to H_{n+1}(E) \to H_n(F) \stackrel{\mathbb{T}_n - \mathbb{I}}{\to} H_n(F) \to H_n(E) \to 0$$

REMARK 2.3. The intersection form S on $H_n(F)$ is related by Poincaré-duality [,] to j_* by

$$S(x, y) = [j_* x, y].$$

So: $\mathbb{T}$ has eigenvalue $1 \Leftrightarrow S$ is degenerate.
Let $K = f^{-1}(O) \cap \partial B$. A related fact is that for $n \neq 2$:

$$K \text{ is a topological sphere } \Leftrightarrow \det(\mathbb{T}_n - \mathbb{I}) = \pm 1.$$

Because $E \stackrel{h}{\simeq} S^{2n+1} \backslash K$ and by duality

$$H_{n+1}(E) \cong \tilde{H}^{n-1}(K) \cong H_n(K)$$

$$H_n(E) \cong \tilde{H}^n(K) \cong H_{n-1}(K)$$

the Wang sequence tells:

$$K \text{ is a homology sphere } \Leftrightarrow \det(\mathbb{T}_n - \mathbb{I}) = \pm 1.$$

For the step from homology sphere to topological sphere we refer to Milnor [33].

3. One dimensional singular locus

3.1. Introduction

In this section we consider singularities with a 1-dimensional critical locus (for short: 1-*isolated singularities*) and study the vanishing homology in a full neighbourhood of the origin. In this case the vanishing homology is concentrated on the 1-dimensional set Σ. We can write

$$\Sigma = \Sigma_1 \cup \ldots \cup \Sigma_r$$

where each Σ_i is an irreducible curve.

At the origin we consider the Milnor fibre F of f and on each $\Sigma_i - \{O\}$ a local system of transversal singularities:
Take at any $x \in \Sigma_i - \{O\}$ the germ of a generic transversal section. This gives an isolated singularity whose μ-class is well-defined. We denote a typical Milnor fibre of this transversal singularity by F_i'. On the level of homology we get in this way a local system with fibre $\tilde{H}_{n-1}(F_i')$.

More precisely we consider in the 1-isolated case the following data:

The **Milnor fibre** F. The vanishing homology is concentrated in dimensions $n-1$ and n:

$$\begin{cases} H_n(F) = \mathbb{Z}^{\mu_n}, & \text{which is free.} \\ H_{n-1}(F), & \text{which can have torsion.} \end{cases}$$

The Milnor monodromy acts on the fibre F :

$$\mathbb{T}_n \quad : H_n(F) \to H_n(F)$$

$$\mathbb{T}_{n-1} : H_{n-1}(F) \to H_{n-1}(F)$$

The **transversal Milnor fibres** F_i'. The vanishing homology is concentrated in dimension $n-1$:

$$\tilde{H}_{n-1}(F_i') = \mathbb{Z}^{\mu_i'} \text{ , which is free.}$$

On this group there act two different monodromies:

1. the vertical monodromy (or local system monodromy)

   $$A_i : \tilde{H}_{n-1}(F_i') \to \tilde{H}_{n-1}(F_i')$$

   which is the characteristic mapping of the local system over the punctured disc $\Sigma_i - \{O\}$.

2. the horizontal monodromy (or Milnor monodromy)

$$T_i : \tilde{H}_{n-1}(F_i') \to \tilde{H}_{n-1}(F_i')$$

which is the Milnor fibration monodromy, when we restrict f to a transversal slice through $x \in \Sigma_i - \{0\}$.

In fact A_i and T_i are defined over $(\Sigma_i - \{O\}) \times S^1$, which is homotopy equivalent to a torus. So they commute:

$$A_i T_i = T_i A_i$$

One of the topics of this section is to show how the above data enter into a good description of the topology of a 1-isolated singularity. For details we refer to [50, 51].

EXAMPLE 3.1. D_∞-singularity: $f = xy^2 + z^2$.
Σ is given by $y = z = 0$ and is a smooth line. The transversal type is A_1.
It is known that F is homotopy equivalent to S^2 (cf. [45]).
One can show that:

$$\begin{array}{lll} H_2(F) = \mathbb{Z} & \mathbb{T}_2 = -\mathbb{I} & ; \ H_1(F) = 0 \\ H_1(F') = \mathbb{Z} & T = \mathbb{I} \quad \text{and} & A = -\mathbb{I} \end{array}$$

EXAMPLE 3.2. $T_{\infty,\infty,\infty}$-singularity: $f = xyz$
$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and consists of the three coordinate axes in $\mathbb{C}^3$.
The transversal type is again A_1.
It is known that F is homotopy equivalent to the 2-torus $S^1 \times S^1$ (cf. [46])
One can show that:

$$\begin{array}{ll} H_2(F) = \mathbb{Z} & \mathbb{T}_2 = \mathbb{I} \\ H_1(F) = \mathbb{Z} \oplus \mathbb{Z} & \mathbb{T}_1 = \mathbb{I} \\ H_1(F') = \mathbb{Z} & T_i = \mathbb{I} \ \text{ and } A_i = \mathbb{I} \ \ i = 1, 2, 3. \end{array}$$

EXAMPLE 3.3. Let f be 1-isolated and homogeneous of degree d. In this case one has the relation:

$$A_i = T_i^{-d}$$

We can assume that all the Σ_i's are straight lines through O. We can suppose that Σ_i is the x_0-axis; the formula follows from $f(sx_0, x_1, \cdots, x_n) = s^d f(x_0, s^{-1}x_1, \cdots, s^{-1}x_n)$, cf [54].

3.2. SERIES OF SINGULARITIES

Let again $f : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, 0)$ be a germ of an analytic function. Let f have a 1-dimensional critical locus $\Sigma = \Sigma(f)$. One considers for each $N \in \mathbb{N}$ the series of functions:

$$f_N = f + \epsilon x^N$$

where x is an admissible linear form, which means that $f^{-1}(0) \cap \{x = 0\}$ has an isolated singularity. One calls this series of function germs a Yomdin series of the hypersurface singularity f. Under the above condition all members of the Yomdin series have isolated singularities. Moreover their Milnor numbers can be computed using the so-called Lê-Yomdin formula:

$$\mu(f + \epsilon x^N) = \mu_n(f) - \mu_{n-1}(f) + N e_0(\Sigma).$$

Here μ_n, resp μ_{n-1} are the corresponding Betti-numbers of the Milnor fibre F of the non-isolated singularity f and $e_o(\Sigma)$ is the intersection multiplicity of Σ and $x = 0$. The formula holds for all N sufficiently large. Moreover $e_0(\Sigma) = \sum d_i \mu_i'$, where d_i is the intersection multiplicity of Σ_i (with reduced structure) and x.

The following formula relates the characteristic polynomials of the monodromies of f and f_N. Other ingredients are the horizontal and vertical monodromies. The eigenvalues of the monodromy satisfy Steenbrink's spectrum conjecture, cf [54]. This conjecture was later proved by M. Saito [42], using his theory of Mixed Hodge Modules.

THEOREM 3.4. *Let $f : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, 0)$ have 1-dimensional critical locus $\Sigma = \Sigma_1 \cup \cdots \cup \Sigma_r$ (irreducible components). Let x be an admissible linear form. Let $M(f)(\lambda)$ be the alternating product of the characteristic polynomials of the monodromy $\mathbb{T}$ of f in dimensions n and $n-1$. Let $M(f + \epsilon x^N)(\lambda)$ be the characteristic polynomial of the monodromy of $f + \epsilon x^N$ in dimensions n. For all N sufficiently large*

$$M(f + \epsilon x^N)(\lambda) = M(f)(\lambda) \prod \det(\lambda^{N d_i} I - A_i T_i^{N d_i}),$$

where A_i and T_i are the vertical and horizontal monodromy along the branch Σ_i.

Proof. The idea behind the proof is to use polar methods and to consider the map germ

$$\Phi = (f, x) \ : \mathbb{C}^{n+1} \to \mathbb{C} \times \mathbb{C}.$$

The Milnor fibres F of f, resp F^N of f_N occur as inverse images under Φ of the sets $\{f = t\}$, resp $\{f + \epsilon x^N = t\}$. Next one constructs via a (stratified) isotopy an embedding

$$F \subset F^N.$$

From the corresponding homology sequence one gets the following 4-term exact sequence

$$0 \to H_n(F) \to H_n(F^N) \to H_n(F^N, F) \to H_{n-1}(F) \to 0.$$

The difference $F^N \backslash F$ is (by excision and homotopy equivalence) related to the part of F^N located near the d_i intersection points of Σ_i and F^N. One obtains:

$$H_q(F^N, F) = \oplus_{i=1}^{r} \oplus_{k=1}^{Nd_i} \tilde{H}_{n-1}(F_{i,k}),$$

where each $F_{i,k}$ is a copy of the Milnor fibre of the transversal singularity F_i'. From this one gets

$$b_n(F) - b_{n-1}(F) = b_n(F^N) - N \sum d_i \mu_i'.$$

To obtain the monodromy statement one constructs a geometric monodromy, which acts on the 4-term sequence. One uses Lê's carrousel method. The monodromy on F^N respects the distance function $|x|$ as nearly as possible. The geometric monodromy gets an x-component, which gives rise to the appearance of the vertical monodromy A_i in $\det(\lambda^{Nd_i} I - A_i T_i^{Nd_i})$. For details cf [50].

REMARK 3.5. $M[f]$ is related to Z_f, the zeta function of the monodromy, which is defined by $Z_f(t) = \prod_{q \geq 0}(det(\mathbb{I} - t\mathbb{T})^{(-1)^{q+1}})$, cf [33]. For homogeneous singularities of degree d the formula $Z(t) = (1 - t^d)^{-\frac{\chi(F)}{d}}$ is well known and valid in all generality (without assumptions on the dimension of the critical set), cf eg [13].

REMARK 3.6. The theorem can be used in two ways: computing mononodromies for isolated singularities in the series, but also for computing monodromies of certain non-isolated singularities with one dimensional singular sets.

In the case of a homogeneous polynomial one can compute almost all ingredients in the formula by taking N as the degree d of f.

One gets in this way the formula $Z_f(t) = (1-t^d)^{-\frac{\chi(F_{reg})}{d} \pm \sum \mu_i'}$ (where the μ_i' are transversal Milnor numbers), and $\chi(F_{reg}) = 1+(-1)^n(d-1)^n$, the Euler characteristic of an isolated singularity of degree d. Also (in some cases) generalizations to the quasi-homogeneous case can be obtained. See also [12].

Also some questions about the relative monodromy being of finite order can be treated with the above formula, cf [53].
Generalizations of the method are in Tibăr's work [58].

3.3. Variation mappings

It is not possible to construct a geometric monodromy $h : F \to F$ which is the identity on the whole of the boundary ∂F of the Milnor fibre F. However it is possible to make it the identity on a big part $\partial_1 F$ of ∂F but not on its complement $\partial_2 F$, which is situated near the singular locus Σ. We can suppose $\partial F = \partial_1 F \cup \partial_2 F$ and $\partial_2 F = \partial F \cap T$, where T is a tubular neighborhood within S^{2n+1} of the link $L := \Sigma \cap S^{2n+1}$.

The homology sequence of the pair $(F, \partial_2 F)$ gives the following fundamental sequence:

$$0 \to H_{n+1}(F, \partial_2 F) \to H_n(\partial_2 F) \to H_n(F) \to$$

$$\to H_n(F, \partial_2 F) \to H_{n-1}(\partial_2 F) \to H_{n-1}(F) \to 0$$

The homology groups of $\partial_2 F$ play an important role in this sequence. They are related to the local system monodromies $A_i : F_i' \to F_i'$ in the following way:
$\partial_2 F$ is a disjoint union $\partial_2 F = \bigcup_{i=1}^{r} \partial_2 F_i$ concentrated near the components of Σ. Each $\partial_2 F_i$ is fibered over the circle (which is the neighborhood boundary of Σ_i) with fibre F_i'.
The Wang sequence of this fibration is:

$$0 \to H_n(\partial_2 F_i) \to H_{n-1}(F_i') \xrightarrow{A_i - \mathbb{I}} H_{n-1}(F_i') \to H_{n-1}(\partial_2 F_i) \to 0$$

So:

$$\begin{aligned} H_n(\partial_2 F) &= \bigoplus_{i=1}^{r} \mathrm{Ker}(A_i - \mathbb{I}) \\ H_{n-1}(\partial_2 F) &\cong \bigoplus_{i=1}^{r} \mathrm{Coker}\,(A_i - \mathbb{I}) \end{aligned}$$

The first group is always free, the second can have torsion.

COROLLARY 3.7.

1. $\dim H_n(\partial_2 F) = 0 \Leftrightarrow$ *no vertical monodromy* A_i *has an eigenvalue* $1 \Rightarrow \dim H_{n-1}(F) = 0$, *but* $H_{n-1}(F)$ *can have torsion!*

2. Let λ be an eigenvalue of $\mathbb{T}_{n-1} : H_{n-1}(F) \to H_{n-1}(F)$*, then λ is also an eigenvalue of some* $T_i : H_{n-1}(F_i') \to H_{n-1}(F_i')$*; in fact one of the eigenvalues occurring in* Coker $(A_i - \mathbb{I})$.

EXAMPLE 3.8. If the transversal type of the singularity is A_1 for all the branches then $\mathbb{T}_{n-1}$ can only have eigenvalue 1 if n is even, or eigenvalue -1 if n is odd.

Given a geometric monodromy $h : F \to F$ such that $h|_{\partial_1 F}$ is the identity, we can consider the variation mappings

$$\begin{array}{lll} \mathrm{VAR}^{I} & : & H_*(F, \partial_1 F) \to H_*(F) \\ \mathrm{VAR}^{II} & : & H_*(F, \partial F) \to H_*(F, \partial_2 F) \\ \mathrm{VAR}^{III} & : & H_*(\partial F, \partial_1 F) \to H_*(\partial_2 F). \end{array}$$

One can show that they are isomorphisms near the middle dimensions, along the same lines as in the isolated singularity case. One has a *first variation sequence*

$$\ldots \to H_{q+1}(S^{2n+1} \backslash L) \to H_q(F, \partial_1 F) \stackrel{\mathrm{VAR}_q^I}{\to} H_q(F) \to H_q(S^{2n+1} \backslash L) \to \ldots$$

and a *second variation sequence*:

$$\ldots \to H_{q+1}(S^{2n+1}, L) \to H_q(F, \partial F) \stackrel{\mathrm{VAR}_q^{II}}{\to} H_q(F, \partial_2 F) \to H_q(S^{2n+1}, L) \to \ldots$$

The vanishing of $H_q(S^{2n+1} \backslash L)$ and $H_{q+1}(S^{2n+1}, L)$ determine the range of isomorphisms.

Moreover there are Lefschetz type dualities involved (for manifolds with boundary and corners) between

$$\begin{array}{lll} H_*(F) & \text{and} & H_*(F, \partial F) \\ H_*(F, \partial_1 F) & \text{and} & H_*(F, \partial_2 F) \end{array}$$

See [47] or Dold [16], chapter VII. See also the remark at the end of the paper of Samelson [43].

A lot of homological information about the singularity is contained in a big commutative diagram, which we call the variation ladder. Recall that $\dim \Sigma = 1$. We shall assume $n \geq 3$ in order to avoid special features of low dimensions. In [51] there is a version adapted for the case $n = 2$. Most of the corollaries that follow from the variation ladder are also true in that case.

THEOREM 3.9. [The variation ladder] *The following commutative diagram has as columns the exact sequences of the triple* $(F, \partial F, \partial_1 F)$ *and the pair* $(F, \partial_2 F)$*. The maps $\leftarrow$ are induced by inclusion. The maps VAR are isomorphisms.*

$$
\begin{tikzcd}
0 & = & H_{n+1}(F,\partial_1 F) \arrow[r, shift left, "\mathrm{VAR}^{I}_{n+1}"] \arrow[d] & H_{n+1}(F) \arrow[l, shift left] \arrow[d] & = & 0 \\
\mathbb{Z}^{\mu_{n-1}} & = & H_{n+1}(F,\partial F) \arrow[r, shift left, "\mathrm{VAR}^{II}_{n+1}"] \arrow[d] & H_{n+1}(F,\partial_2 F) \arrow[l, shift left] \arrow[d] & = & \mathbb{Z}^{\mu_{n-1}} \\
\mathbb{Z}^{\alpha} & = & H_n(\partial F,\partial_1 F) \arrow[r, shift left, "\mathrm{VAR}^{III}_{n}"] \arrow[d] & H_n(\partial_2 F) \arrow[l, shift left] \arrow[d] & = & \mathbb{Z}^{\alpha} \\
\mathbb{Z}^{\mu_n} & = & H_n(F,\partial_1 F) \arrow[r, shift left, "\mathrm{VAR}^{I}_{n}"] \arrow[d, "(i_1)_*"] & H_n(F) \arrow[l, shift left, "(j_1)_*"] \arrow[d, "(j_2)_*"] \arrow[dl, "j_*"'] & = & \mathbb{Z}^{\mu_n} \\
\mathbb{Z}^{\mu_n} \oplus T & = & H_n(F,\partial F) \arrow[r, shift left, "\mathrm{VAR}^{II}_{n}"] \arrow[d] & H_n(F,\partial_2 F) \arrow[l, shift left, "(i_2)_*"] \arrow[d] & = & \mathbb{Z}^{\mu_n} \oplus T \\
\mathbb{Z}^{\alpha} \oplus T_2 & = & H_{n-1}(\partial F,\partial_1 F) \arrow[r, shift left, "\mathrm{VAR}^{III}_{n-1}"] \arrow[d] & H_{n-1}(\partial_2 F) \arrow[l, shift left] \arrow[d] & = & \mathbb{Z}^{\alpha} \oplus T_2 \\
\mathbb{Z}^{\mu_{n-1}} \oplus T & = & H_{n-1}(F,\partial_1 F) \arrow[r, shift left, "\mathrm{VAR}^{I}_{n-1}"] \arrow[d] & H_{n-1}(F) \arrow[l, shift left] \arrow[d] & = & \mathbb{Z}^{\mu_{n-1}} \oplus T \\
0 & = & H_{n-1}(F,\partial F) \arrow[r, shift left, "\mathrm{VAR}^{II}_{n-1}"] & H_{n-1}(F,\partial_2 F) \arrow[l, shift left] & = & 0
\end{tikzcd}
$$

where

$$
\begin{aligned}
\mu_n &= \dim H_n(F) \\
\mu_{n-1} &= \dim H_{n-1}(F) \\
\alpha &= \dim H_n(\partial_2 F) = \sum_{i=1}^{r} \dim \ker(A_i - \mathbb{I}) \\
T_2 &= \textit{torsion part of } H_{n-1}(\partial_2 F) \\
T &= \textit{torsion part of } H_{n-1}(F)
\end{aligned}
$$

It follows from the Lefschetz dualities mentioned above that the rows are "opposite dual" to each other. The composition of $\leftarrow$ *and VAR is* $\mathbb{T}_* - \mathbb{I}$.

COROLLARY 3.10. *If all vertical monodromies have only eigenvalues* $\lambda \neq 1$ *then*

1° $(j_2)_*$ *is injective (even bijective over* $\mathbb{Q}$*)*

2° $H_{n+1}(F, \partial F) = 0$ *and* $H_{n-1}(F)$ *is torsion.*

REMARK 3.11. The variation ladder occurs also in the following way. Consider the sheaf $\Phi^\cdot_f$ of vanishing cycles, cf [10]. The stalk of this sheaf at a point is the reduced cohomology over $\mathbb{C}$ of the Milnor fibre of f at the point. Our approach works with homology and with coefficients in $\mathbb{Z}$, and we are also interested in torsion.

Let j denote the inclusion of O in $X = f^{-1}(0)$, i be the inclusion of $X - O$ in X, and let $\mathbf{K}^\cdot$ denote $\Phi^\cdot_f(\mathbb{C})$, Then there exists a distinguished triangle:

$$\begin{tikzcd} j_*j^!\mathbf{K}^\cdot \arrow[rr] & & \mathbf{K}^\cdot \arrow[dl] \\ & Ri_*i^*\mathbf{K}^\cdot \arrow[ul, "{[+1]}"] & \end{tikzcd}$$

The associated stalk cohomology exact sequence at the origin becomes:

$$0 \to H^{n-1}((\mathbf{K}^\cdot)_0) \to \bigoplus_{i=1}^{r} \mathrm{Ker}(A_i - \mathbb{I}) \to H^n((j_*j^!\mathbf{K}^\cdot)_0) \to$$

$$\to H^n((\mathbf{K}^\cdot)_0) \to \bigoplus_{i=1}^{r} \mathrm{Ker}(A_i - \mathbb{I}) \to H^{n+1}((j_*j^!\mathbf{K}^\cdot)_0) \to 0$$

where $H^k((\mathbf{K}^\cdot)_0) = H^k(F, \mathbb{C})$.

Since $H^k((j_*j^!\mathbf{K}^\cdot)_0)$ can be identified with $H^k(F, \partial_2 F; \mathbb{C})$, this sequence is the cohomology sequence of the pair $(F, \partial_2 F)$ and is the cohomology version (over $\mathbb{C}$) of the right hand side of the variation ladder 3.9, at least for $n \geq 3$.

Other work in this direction was done by M. Saito [42] and D. Barlet [5, 6]. They deal with Hodge theoretical aspects.

3.4. Relation between the monodromy and the intersection form

In the isolated singularity case it is known [33] that

$$S \text{ is degenerate} \Leftrightarrow T \text{ has eigenvalue } 1.$$

This follows from $S(x,y) = [j_*x, y]$ by Poincaré Duality and the fact that $\ker j_* = \mathrm{Ker}\ (\mathbb{T}_n - \mathbb{I})$. We discussed this already in section 2. What happens in the case $\dim \Sigma = 1$?
Consider the diagram:

$$\begin{tikzcd}
H_n(F,\partial_1 F) \arrow[rrr, shift left, "\mathrm{VAR}_n^I"] \arrow[dd] & & & H_n(F) \arrow[lll, shift left, "j_{1*}"] \arrow[dd, "j_{2*}"] \\
& H_n(F) \arrow[ul, "j_{1*}"] \arrow[r, "\mathbb{T}_n - \mathbb{I}"] \arrow[dl, dashed, "j_*"] & H_n(F) \arrow[ur, dashed, "="] \arrow[dr, "j_{2*}"] & \\
H_n(F,\partial F) \arrow[rrr, shift left, "\mathrm{VAR}_n^{II}"] & & & H_n(F,\partial_2 F) \arrow[lll, shift left]
\end{tikzcd}$$

So $\mathbb{T}_n - \mathbb{I} = \mathrm{VAR}^I \circ j_{1*}$; $\mathrm{Ker}\ j_{1*} = \mathrm{Ker}\ (\mathbb{T}_n - \mathbb{I})$ and $(j_2)_* \circ (\mathbb{T}_n - \mathbb{I}) = \mathrm{VAR}^{II} \circ j_*$

PROPOSITION 3.12. *T has eigenvalue $1 \Rightarrow S$ is degenerate*
Proof. We have $\mathrm{Ker}\ (\mathbb{T}_n - \mathbb{I}) = \mathrm{Ker}\ j_{1*} \subset \mathrm{Ker}\ j_*$.

In the other direction we have the following two partial results:

PROPOSITION 3.13. *If all vertical monodromies have only eigenvalues $\lambda \neq 1$, then:*

$$\mathbb{T}_n \textit{ has eigenvalue } 1 \Leftrightarrow S \textit{ is degenerate}$$

Proof. Since $(j_2)_*$ is injective we have $\mathrm{Ker}\ (\mathbb{T}_n - \mathbb{I}) = \mathrm{Ker}\ j_*$.

PROPOSITION 3.14.

$$\left.\begin{array}{c} S \textit{ is non-degenerate} \\ H_{n-1}(F,\mathbb{Q}) = 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{l} \mathrm{Ker}\ (\mathbb{T}_n - \mathbb{I}) = 0 \\ \mathrm{Ker}\ (A_i - \mathbb{I}) = 0 \textit{ for all } i. \end{array}\right.$$

EXAMPLE 3.15. We consider the following homogeneous polynomial

$$f = x^2y^2 + y^2z^2 + z^2x^2 - 2xyz(x+y+z)$$

In $\mathbb{P}^2$ this defines a curve, which is known as Zariski's example with 3 cusps. So f has a 1-dimensional singular locus, consisting of 3 complex lines and transversal type A_2.

Since f is homogeneous of degree 4 it is known that $A_i = T_i^{-4}$ where T_i is the monodromy of A_2. Since the eigenvalues of T_i are q and q^2, where $q = e^{2\pi i/6}$ it follows that A_i has eigenvalues q^4 and q^{-4}.
As a corollary: $\begin{cases} \text{Ker } (A_i - \mathbb{I}) = 0 \\ \text{Coker}(A_i - \mathbb{I}) = \mathbb{Z}_3 \text{ torsion} \end{cases}$ and this implies that $H_{n-1}(F)$ is torsion.

The eigenvalues of $\mathbb{T}_n : H_n(F) \to H_n(F)$ can be deduced from 3.4: r, r^2 and r^3 where $r = e^{2\pi i/4}$. So in particular Ker $\mathbb{T}_n - \mathbb{I} = 0$. As a consequence S is non-degenerate.

In this case the fundamental group $\pi_1(F) = \mathbb{Z}_3$, so we have only 3-torsion in $H_1(F) = \mathbb{Z}_3$. *Although $b_1(F) = 0$, we will point out, that the monodromy on $H_1(F)$ is non-trivial!*

The 6-term exact sequence in the variation ladder reduces to:

$$0 \to \mathbb{Z}^3 \to \mathbb{Z}^3 \oplus \mathbb{Z}_3 \to (\mathbb{Z}_3)^3 \to \mathbb{Z}_3 \to 0$$

The action of the monodromy on $H_1(F)$ is induced by the actions on each Coker $(A_i - \mathbb{I}) = \mathbb{Z}_3$, which is multiplication by -1. It follows that action of the monodromy on $H_1(F)$ is non-trivial. This remark is also in [14].

EXAMPLE 3.16. Proposition (3.12) can be used to construct several examples with totally degenerate intersection forms. Dimca gave $x^a y^{d-a} + yz^{d-1}$ and other examples, where the geometric monodromy can be chosen homotopy equivalent to the identity. This implies that Ker $(\mathbb{T}_n - \mathbb{I}) =$ Ker $j_* = H_n(F)$.

In case of homogeneous polynomials in x, y, z of degree d it follows from [12] or [50] that a necessary condition for $\mathbb{T}_n = \mathbb{I}$ is $\chi(F) = 0$. Since in the homogeneous case $\chi(F) = d^3 - 3d^2 + 3d - d\sum \mu_i'$, this is equivalent to

$$\Sigma \mu_i' = d^2 - 3d + 3,$$

where μ_i' are the Milnor numbers of the transversal singularities.

Melle [4] pointed out that this condition is not sufficient. First remark that $\chi(F) = 0$ implies that the zeta function of the monodromy is $Z(t) = (1-t^d)^{-\frac{\chi(F)}{d}} = 1$ (due to homogeneity). It follows that

$\mathbb{T}_2 = 1$ is equivalent to $\mathbb{T}_1 = 1$; so any eigenvalue of $\mathbb{T}_1$ different from 1 has to cancel against an eigenvalue of $\mathbb{T}_2$ and vice versa. In case $\mathbb{T}_2 = 1$ it follows that the 6-term sequence splits into two 3-term sequences, one has

$$0 \to H_n(F, \partial_2 F) \to H_{n-1}(\partial_2 F) \to H_{n-1}(F) \to 0$$

So also the monodromy of the middle term must have only eigenvalues 1.

Melle's example $f = (y^2 + 2xz)(axz + z^2 + ay^2)$ is a polynomial of degree 4, where the corresponding projective curve has only one singularity, which is of type A_7. A straightforward calculation gives $\dim H_{n-1}(\partial_2 F) = \dim \text{Coker } (A_* - \mathbb{I}) = 3$ and the eigenvalues of $\mathbb{T}_*$ on this space are $i, -i, 1$.

Examples of homogeneous singularities with $\mathbb{T} = \mathbb{I}$ are used in [7] in order to construct complex hypersurfaces which are homology $\mathbb{P}^n$'s.

Cancelling of eigenvalues of the monodromy is also the subject of Barlet's studies [5, 6] and Denef's conjectures [11] about the topological zeta function.

3.5. Is ∂F a topological sphere?

In Milnor's book [33], section 8 is entitled: Is K a topological sphere? Remember $K = f^{-1}(0) \cap \partial B$. For isolated singularities Milnor showed:

$$K \text{ is a topological sphere } S^{2n-1} \Leftrightarrow \det \mathbb{T}_n - \mathbb{I} = \pm 1$$

Since ∂F is diffeomorphic to K in this case, we also have

$$\partial F \text{ is a topological sphere } S^{2n-1} \Leftrightarrow \det \mathbb{T}_n - \mathbb{I} = \pm 1$$

In this section we study the 1-isolated case. Since K is not smooth, we only study ∂F. The vertical monodromies play an important role in the final result.

PROPOSITION 3.17. *Let $n > 2$. The following are equivalent:*

1. *∂F is a topological sphere S^{2n-1}*

2. $\begin{cases} (a) & H_{n-1}(F) = 0 \\ (b) & \textit{The intersection form } S \textit{ on } H_n(F) \textit{ has determinant } \pm 1 \end{cases}$

3. $\begin{cases} (a) & \det(A_i - \mathbb{I}) = \pm 1 \textit{ for all } i = 1, \ldots, r \\ (b) & \det(\mathbb{T}_n - \mathbb{I}) = \pm 1 \end{cases}$

NB. For $n = 2$ replace (1) by: ∂F is a homology sphere.

REMARK 3.18. ∂F is a topological sphere if and only if K is a homotopy sphere and $\det(A_i - \mathbb{I}) = \pm 1$.

REMARK 3.19. Let f be 1-isolated with transversal type A_1. We have $A_i = \pm \mathbb{I}$. So in this case ∂F can never be a topological sphere.

REMARK 3.20. Let f be 1-isolated, and moreover be *homogeneous* of degree d; then ∂F is never a homology sphere.

To show this, consider the exact homology sequence from (3.2) in the case $N = d$:

$$0 \to H_n(F) \to H_n(F^d) \to H_n(F^d, F) \to H_{n-1}(F) \to 0.$$

This sequence compares the homology of the Milnor fibre F with the homology of F^d, the Milnor fibre of the function $f_d = f + \epsilon x^d$. This function is also homogeneous of degree d, but has an isolated singularity.

In case of a homology sphere, we have $H_{n-1}(F) = 0$. Moreover the monodromy actions commute with the sequence. The eigenvalues of $\mathbb{T}_n$ on $H_n(F^d)$ and of $\mathbb{T}_n^{rel}$ on $H_n(F^d, F)$ are explicitly known, so we can compute exactly all the eigenvalues of $\mathbb{T}_n$ on $H_n(F)$. It follows that it is not possible to satisfy the condition: $\det(\mathbb{T}_n - \mathbb{I}) = \pm 1$.

QUESTION 3.21. Are there examples where ∂F is a topological spheres ?
Up to now only counter-examples are known.

3.6. Transversal type A_1.

In this section we study germs of holomorphic functions $f : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, 0)$, where the critical locus is 1-dimensional and the transversal singularities at points of $\Sigma - \{O\}$ are of type A_1. References are [45], [46], [47].

We know already that the homology of the Milnor fibre F is concentrated in dimensions $n - 1$ and n. The topology seems to depend partly on properties of the critical set. In [47] we showed the following

PROPOSITION 3.22. *Let $f : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, 0)$ have a 1-dimensional singular set, which is an ICIS (isolated complete intersection singularity) and let f have transversal type A_1 outside the origin. Then the homotopy type of F is a bouquet of spheres:*

– *Case A:* $F \overset{h}{\simeq} S^n \vee \cdots \vee S^n$ *(general case)*

– *Case B:* $F \overset{h}{\simeq} S^{n-1} \vee S^n \vee \cdots \vee S^n$ *(special case)*

The main idea of the proof of this type of theorem is *deformation with constant topology.* This generalizes the concept of Morsification from the case of isolated singularities. While Morsifications are generic deformations in the isolated case, it is easy to destroy the topology in the non-isolated case, e.g. for a Yomdin series one finds a infinite variety of Milnor fibres. One has to be careful and to study special deformations, which deform both f and Σ in a good way [25].

Let us suppose that we have constructed a deformation (f_s, Σ_s) of (f, Σ), defined on the same neighborhood as (a Milnor representative of) f, with properties:

– the critical set of f_s consists of a curve Σ_s and some isolated points $a_1 \cdots a_\sigma$ inside the Milnor ball,

– during the deformation from f to F_s the fibration is of constant fibration type,

In this case one can use the principle of *additivity of the vanishing homology*, which says:

$$H_*(E, F) = \bigoplus_{i=0}^{\sigma} H_*(E_i, F_i) \ ,$$

where $E =$ Milnor ball of f,
$E_0 =$ tube neighborhood along Σ_s
$E_i =$ small Milnor ball at s_i,
$F =$ Milnor fibre of f,
$F_i =$ restriction of nearby fibre to E_i.
$\mu_i =$ the Milnor number at a_i.

From the properties of isolated singularities it follows that:

$$H_n(F) = H_{n+1}(E, F) = H_{n+1}(E_0, F_0) \oplus \mathbb{Z}^{\sum \mu_i} \ ,$$

$$H_{n-1}(F) = H_n(E, F) = H_n(E_0, F_0) \ .$$

The contribution to $H_{n-1}(F)$ comes entirely from $H_n(E_0, F_0)$ and this is the part related to 1-dimensional part of the deformed critical set.

Returning to the A_1-case with Σ an ICIS one can always produce the above type of deformation, even with the extra properties:

1. all isolated critical points of f_s are of type A_1,

2. Σ_s is smooth, the Milnor fibre of Σ,

3. For the non-isolated critical points of f_s we have only two types:

   – type A_∞, local formula $w_0^2+w_1^2+\cdots w_n^2$ (transversal Morse),
   – type D_∞, local formula $w_0w_1^2 + w_2^2 + \cdots w_n^2$ (Whitney umbrella).

Cases A and B in the above proposition are distinguished by $\sharp D_\infty > 0$, in case A, and $\sharp D_\infty = 0$ in case B. In general, $\sharp D_\infty$ is related to the vertical monodromy.

As soon as Σ is not an ICIS it can occur that $b_{n-1}(F) \geq 2$. For example, if $f = xyz$, we have $b_1(F) = 2$.

3.7. Bouquet theorems

In the last few years different types of 'bouquet theorems' have appeared. Some of them deal with germs $f : (X, x) \to (\mathbb{C}, 0)$ where f defines an isolated singularity. In some cases, F has the homotopy type of a bouquet of $(\dim X - 1)$-spheres, for example when X is an ICIS , or X is a complete intersection. Moreover if both (X, x) and f have isolated singularities, then F has a bouquet decomposition

$$F \overset{h}{\simeq} F_0 \vee S^n \vee \cdots \vee S^n,$$

where F_0 is the complex link of (X, x), cf [52]. Later Tibăr proved a more general bouquet theorem for the case when (X, x) is a stratified space and f defines an isolated singularity (in the sense of the stratified spaces) for details, cf [57]. Related results are in [23].

In the case of non-isolated singularities the bouquet situation is no longer standard, e.g. the torus is the Milnor fibre of $f = xyz$. At the other hand in several special cases (eg many cases discussed in this paper), we still encounter bouquets of spheres (sometimes in different dimensions).

Némethi treated as part of his paper [37] the question: *When is a CW-complex a bouquet of spheres ?* A necessary condition is of course that all homology groups are free (over $\mathbb{Z}$). In the 1-connected case he added the condition, that the Hurewicz map (from homomotopy to homology groups) is surjective in all dimensions. These two conditions together are sufficient.

As a special corollary he showed:

PROPOSITION 3.23. *Let* $f : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, 0)$ $(n \geq 3)$ *be a germ of analytic function with a 1-dimensional critical locus. Then its Milnor fibre* F *has the homotopy type of a bouquet of spheres if and only if* $H_*(F, \mathbb{Z})$ *is free.*

REMARK 3.24. The condition $n \geq 3$ is important, the statement does not work in the surface case. For example, $f = xyz$ gives a 2 torus; but $g = xyz + w^2$ (its suspension) gives a bouquet $S^2 \vee S^3 \vee S^3$. In general, the case $n = 2$ is more difficult, due to the influence of the fundamental group. Results in special cases are treated in [45, 46, 47].

REMARK 3.25. It seems that the following question is relevant: If f has a deformation with constant topology, such that the homology splitting discussed above is valid, does there exist space F_0 and a decomposition:

$$F \overset{h}{\simeq} F_0 \vee S^n \vee \cdots \vee S^n,$$

where the spheres correspond exactly to the vanishing cycles at the isolated critical points ? Is the splitting 'forced' by the topology of CW-complexes or by properties of singularities ?

3.8. Other directions

Several developments in the case of 1-dimensional singular sets are not discussed here. We mention them below and give some references.

- The topology of line singularities (smooth 1-dimensional singular set) with transversal type A, D or E studied by De Jong [24]. Several data of the vanishing topology have been recently computed by J. Fernandez de Bobadilla.

- The fundamental studies of Pellikaan about the algebraic aspects of the theory. From his thesis, there originated a series of papers. We mention [39, 40].

- The relation between the deformation theories of weakly normal hypersurface singularities and normal surface singularities given by De Jong and Van Straten [25].

- Van Straten's [56] description of the De Rham complex of 1-isolated singularities and an algebraic description for the highest Betti number. See also [47].

- Jiang's [21, 22] study of functions on an isolated complete intersection with 1-dimensional singular set.

- Relation with Mond's [34, 35] work on map germs from $\mathbb{C}^2$ to $\mathbb{C}^3$ of finite codimension. See also the contribution of Damon [9] in this volume.
- Lê numbers, studied by Massey [31], where polar methods are used to study the structure of the singular set. For the 1-isolated case, cf [32].
- Aleksandrov's study of differential forms and vector fields tangent to a hypersurface germ D. In particular Saito singularities, where the module of vector fields tangent to D is locally free. Also non-isolated singularities occur [1, 2].
- Grandjean's approach to residual discriminants and bifurcation sets [18] for function germs of finite codimension within a given singular set.

4. Higher dimensional singular sets

4.1. 2 DIMENSIONAL SINGULAR SET

We restrict the discussion below to a two dimensional singular set Σ, which is an ICIS and with (generic) transversal type A_1.

The thesis of Zaharia [59] deals with this case; part of it is more general. He considers the situation where f has finite codimension in the space of functions with the set Σ as part of the critical set. This condition is equivalent to having $\Sigma(f) = \Sigma$ and the germ of f at every point of $\Sigma \setminus \{O\}$ equivalent to a so-called $D(k, p)$ singularity [39]. In our two dimensional case we encounter for dimension reasons only:

$D(2,0) : x_2^2 + \cdots + x_n^2 \quad ; \ A_\infty \times \mathbb{C}$,

$D(2,1) : x_1 x_2^2 + x_3^2 + \cdots + x_n^2; \ D_\infty \times \mathbb{C}$

(coordinates are $x_0, \cdots, x_n$).

Outside the origin one only has to deal with two types of local Milnor fibers, with homotopy type S^{n-2}, respectively S^{n-1}. The closure of the set of $D(2,1)$ types forms an ICIS -curve Δ inside Σ. In exceptional cases, Δ can be void. The stratification according to vanishing homology type consists of the following strata:

$$\Sigma, \Sigma \setminus \Delta, \Delta \setminus \{O\}, \{O\}.$$

Zaharia [59, 60] studied especially the topology of the Milnor fibre. This was later improved by Némethi [37]. The following statement shows that the homotopy type is still a bouquet of spheres, indeed of

n-spheres, in certain cases extended by one sphere of dimension $n-1$ or $n-2$. This is very close to the case of isolated singularities.

PROPOSITION 4.1. *Let $f : (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, 0)$ have a 2-dimensional singular set, which is an ICIS , and let f have transversal type A_1 outside a curve Δ in Σ; then the homotopy type of F is a bouquet of spheres:*

- *Case A:* $F \overset{h}{\simeq} S^n \vee \cdots \vee S^n$
- *Case B:* $F \overset{h}{\simeq} S^{n-1} \vee S^n \vee \cdots \vee S^n$
- *Case C:* $F \overset{h}{\simeq} S^{n-2} \vee S^n \vee \cdots \vee S^n$ *(special case).*

If $n = 2$, statement C should read $F \overset{h}{\simeq} S^0 \times (S^2 \vee \cdots \vee S^2)$.

This theorem is due to Némethi [37], especially the homotopy part. The results of Zaharia played a crucial role in the proof.

The idea behind the proof is similar to the 1-dimensional case: construct a deformation with constant topology, which has only certain elementary singularity types as 'building blocks'. More precisely Zaharia constructed a deformation $(f_s, \Sigma_s, \Delta_s)$ of (f, Σ, Δ) with constant topology and if $s \neq 0$:

- the singular set of f_s consists of Σ_s and finitely many points, where f_s is Morse,
- Σ_s and Δ_s are Milnor fibres of Σ, resp Δ (recall: they are both ICIS),
- f_s has only points of type $D(2,0)$ and $D(2,1)$ on Σ_s,
- Δ_s is the $D(2,1)$-locus in Σ_s.

Remark next that above each point of Δ_s one can consider the Milnor fibre of the $D(2,1)$-singularty, which has the homotopy type of S^{n-1}. These induces a monodromy map:

$$\Xi \ : \pi_1(\Delta_s, \star) \to \text{Aut } (H_{n-1}(S^{n-1}, \mathbb{Z})) = \mathbb{Z}_2$$

This local system of vertical monodromies plays an important role in the determination of the toplogy of the Milnor fibre. It seems to be 'deeper' than the vertical monodromy on the Δ-stratum, since the system is defined on its Milnor fibre Δ_s. Then cases A, B and C in the above proposition are distinguished by:

- case A: Δ_s is non void and Ξ non trivial,
- case B: Δ_s is non void and Ξ trivial,
- case C: Δ_s is void.

Zaharia showed that the Euler-characteristic of the Milnor fibre is equal to:

$$1 + (-1)^n(2\mu_\Delta + \mu_\Sigma + \sigma - 1),$$

where σ is the number of A_1 -points which appear in the deformation, μ_Σ is the Milnor number of the ICIS Σ, and μ_Δ is the Milnor number of the ICIS Δ. (We use the convention $\mu_\emptyset = -1$; moreover in case C, if $n = 2$, one has $\chi(F) = 2 + 2\mu_\Sigma$).

4.2. Codimension 1 case

Shubladze [44] studied the case, where the singular set is a hypersurface Σ. Generically there is only one transversal type, which must be an A_k singularity. The function f can be written as $f = h^k g$, where $h = 0$ defines the singular set. Considering the case when f has finite codimension in the set of these functions he showed that if $g(O) = 0$

$$F \simeq S^1 \vee S^n \vee \cdots \vee S^n.$$

NB The case $g(O) \neq 0$ gives k copies of the Milnor fibre of h. The finite codimension condition is equivalent to the conjunction of two conditions (a) g defines an isolated singularity, (b) the pair (g, h) defines an ICIS . Later Némethi [36], unaware of the results of Shubladze, recovered the Shubladze result as a by-product of his theory of composed singularities. The number of n-spheres is related to the Milnor numbers as follows:

$$b_n(F) = (k + 1)\mu(h, g) + k\mu(h)) + \mu(g).$$

4.3. Composed singularities

The method of composed singularities can give rise to non-isolated singularities of codimension 2 or 1. We discuss here the work carried out by A. Némethi in his paper [36]. The situation is as follows. One considers the sequence of mappings:

$$f : \mathbb{C}^{n+1} \overset{(g,h)}{\to} \mathbb{C}^2 \overset{P}{\to} \mathbb{C},$$

where the pair (g, h) defines an ICIS Y and P is any germ. If P has an isolated singularity at the origin, then the singular set $\Sigma(f)$ is exactly the ICIS Y and has dimension $n-1$. The local system of vanishing homology groups is defined over (each component) of $Y - \{O\}$. The transversal type is equal to the singularity type of P.

If P is not reduced then the singular set of f is the inverse image of the non-reduced locus of P and has dimension n. The situation becomes more complicated.

Let D be the (reduced) discriminant locus of the ICIS (g, h).

THEOREM 4.2. *Let $P^{-1}(0) \cap D = \{O\}$ then the Milnor fibre F of the composed mapping f has the homotopy type of the disjoint union of π_0 copies of a bouquet of spheres*

$$S^1 \vee \cdots \vee S^1 \vee S^n \vee \cdots \vee S^n,$$

where π_0 is the number of connected components of the Milnor fibre of P.

The number of 1-spheres in a connected component of F is equal to the Milnor number of P, the number of n-spheres is determined by topological data.

Némethi treats also the 'bad case' where $P^{-1}(0) \cap D$ contains 1-dimensional components. His work has some interesting consequences, e.g. the relation to series of singularities.

The theorem also allows P to be regular. In those cases f has an isolated singularity, at least if the condition $P^{-1}(0) \cap D = \{O\}$ is satisfied. In special cases well known situations occur:

- $P = z$ (a generic coordinate), Lê attaching formula,
- $P = z + w^k$: formula for Iomdin series if f has a 1-dimensional singular set,
- $P = z + w$ and g and h have separate variables and isolated singularities: Sebastiani-Thom formula.

A second theorem in the same paper [36] gives a formula for the zeta function of the monodromy of f as a product of other zeta functions related to the topological data of the composed singularity.

4.4. Other cases

Above we studied more or less the summit of the iceberg of non-isolated singularities, those which seem to be close to isolated singularities. There are natural candidates for further investigation of the full system of vanishing homology: central arrangements of hyperplanes, and discriminant spaces of Coxeter arrangements. In both cases, there is also combinatorial and geometric structure around. For the homology of the Milnor fibre of an arrangement we refer to Orlik-Terao [38]. The zeta function is just $Z(t) = (1 - t^d)^{\chi(M^*)}$, where M^* is the complement of the arrangement, modulo the natural $\mathbb{C}^*$-action. The zeta function of the discriminant hypersurface of a Coxeter arrangement is studied in geometric terms by [15].

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