Deformations of polynomials and their zeta functions *

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Abstract

For an analytic in $\sigma \in (\mathbb{C}, 0)$ family P_σ of polynomials in n variables there is defined a monodromy transformation h of the zero level set $V_\sigma = \{P_\sigma = 0\}$ for $\sigma \neq 0$ small enough. The zeta function of this monodromy transformation is written as an integral with respect to the Euler characteristic of the corresponding local data. This leads to a study of deformations of holomorphic germs and their zeta functions. We show some examples of computations with the use of this technique.

1 Introduction

A complex polynomial P in n variables defines a map from $\mathbb{C}^n$ to $\mathbb{C}$ (also denoted by P). This map is a (C^∞) locally trivial fibration over the complement to a finite subset of the target $\mathbb{C}$ — the bifurcation set: [11]. Topological study of polynomial maps was started in [1] and continued, in particular, in [7, 8]. One is interested in a description of the bifurcation set of a polynomial, its generic level set, degeneration of the level set for singular values, monodromies around singular values, ... The level sets $\{P = \sigma\}$ of the polynomial P are the zero level sets of the family of polynomials $P_\sigma = P - \sigma$. It is natural to study the behaviour of the zero level sets not of this particular family, but of a general family P_σ of polynomials (say, analytic in σ from a neighbourhood of zero in the complex line $\mathbb{C}_\sigma$). In this setting one can also study change of the map $P_\sigma : \mathbb{C}^n \to \mathbb{C}$ in the family. Such a study was started in [9, 10].

Let $P_\sigma(x)$ be an analytic in $\sigma \in (\mathbb{C}_\sigma, 0)$ family of polynomials in n variables $x = (x_1, \ldots, x_n)$: a deformation of the polynomial $P = P_0$. The polynomial P_σ

*Keywords: deformations of polynomials, zeta function. AMS Math. Subject Classification: 32S30, 14D05, 58K10.

†Partially supported by the grants RFBR–04–01–00762, NSh–1972.2003.1, NWO-RFBR 047.011.2004.026

defines a map $P_\sigma : \mathbb{C}^n \to \mathbb{C}$. Let $\mathbb{V}$ be the hypersurface in $\mathbb{C}^{n+1} = \mathbb{C}^n \times \mathbb{C}_\sigma$ defined by the equation $P_\sigma(x) = 0$ ($x \in \mathbb{C}^n$, $\sigma \in \mathbb{C}_\sigma$). The projection $p : \mathbb{V} \to \mathbb{C}_\sigma$ to the second factor is a fibre bundle over a punctured neighbourhood of the origin in $\mathbb{C}_\sigma$ (see, e.g., [11]). Let $V_\sigma = \{P_\sigma = 0\} \subset \mathbb{C}^n$, $\sigma \neq 0$ small enough, be its fibre (the zero level set of a generic member of the family) and let $h : V_\sigma \to V_\sigma$ be a monodromy transformation of the described fibration (it is well defined up to isotopy).

For a transformation $h : Z \to Z$ of a space Z, its zeta function $\zeta_h(t)$ is the rational function

$$\prod_{q\geq 0} \{\det(id - t \cdot h_{*|H_q(Z;\mathbb{R})})\}^{(-1)^q}.$$

The degree of the zeta function (the degree of the numerator minus the degree of the denominator) is equal to the Euler characteristic $\chi(Z)$ of the space Z. The zeta function of the monodromy transformation $h : V_\sigma \to V_\sigma$ will be called the zeta function of the family P_σ.

For $P_\sigma = P + \sigma$, V_σ is a generic level set of the polynomial P and $\zeta_P(t) = \zeta_h(t)$ is the zeta function of the classical monodromy transformation of the polynomial function P around the zero value. In [5], there is a formula which expresses the zeta function $\zeta_P(t)$ (and thus the Euler characteristic of a generic level set of the polynomial P) as an integral with respect to the Euler characteristic of the corresponding local data at each point of the compactification $\mathbb{CP}^n$ of the affine space $\mathbb{C}^n$. This "localization" appeared to be effective for computing these invariants in a number of cases.

Here we formulate a general localization formula for the zeta function and specialise it for families of polynomials. For that we describe local data corresponding to the problem of computing the zeta function of a family of polynomials. These are deformations of germs of (complex analytic) functions on the affine space $\mathbb{C}^n$ or on the affine space $\mathbb{C}^n$ with a distinguished hyperplane ("boundary") $\mathbb{C}^{n-1}$ and their zeta functions. We show some examples of application of the localization formula.

2 The localization principle

Let X be a compact complex analytic (generally speaking singular) variety and let Y be a (compact) subvariety of X. Let L be a line bundle over X and let s_σ be an analytic in $\sigma \in (\mathbb{C}_\sigma, 0)$ family of sections of the line bundle L. Let $\mathbb{V} \subset X \times \mathbb{C}_\sigma$ be defined by the equation $s_\sigma(x) = 0$. The restriction of the projection $p : X \times \mathbb{C}_\sigma \to \mathbb{C}_\sigma$ to the second factor to the complement $\mathbb{V} \setminus (Y \times \mathbb{C}_\sigma)$ is a locally trivial fibration over a punctured neighbourhood of the origin in $\mathbb{C}_\sigma$: [11]. Let $V_\sigma = p^{-1}(\sigma) \cap (\mathbb{V} \setminus (Y \times \mathbb{C}_\sigma))$ ($\sigma \neq 0$ small enough) be its fibre (the zero set in $X \setminus Y$ of a generic section from the family) and let $h : V_\sigma \to V_\sigma$ be a monodromy transformation of the fibration (it is well defined up to isotopy). Let $\zeta_{s_\sigma}(t)$ be the zeta function of the monodromy transformation h of the described fibration.

Now let's describe local versions of these objects. Pay attention that over a neighbourhood of a point a line bundle is trivial and therefore its sections can be considered as functions. Let $(X, x_0) \subset (\mathbb{C}^N, x_0)$ be the germ of a complex analytic variety and let (Y, x_0) be a subvariety of it (possibly the empty one). Let s_σ be an analytic in $\sigma \in (\mathbb{C}_\sigma, 0)$ deformation of a germ s of a function on (X, x_0), i.e. $s_\sigma = S(\cdot, \sigma)$ where S is a germ of a holomorphic function on $(X \times \mathbb{C}_\sigma, (x_0, 0))$, $s_0 = s$. Let $\mathbb{V}_{x_0} \subset (X \times \mathbb{C}_\sigma, (x_0, 0))$ be the germ of the variety defined by the equation $s_\sigma(x) = 0$. Let positive ε be small enough so that all the strata of a Whitney stratification of the pair $(X, \{s = 0\})$ are transversal to the sphere $S_{\varepsilon'}(x_0)$ of radius ε' centred at the point x_0 in $\mathbb{C}^N$ for any positive $\varepsilon' \leq \varepsilon$. Let $B_\varepsilon(x_0)$ be the ball of radius ε centred at the point x_0. The restriction of the projection $X \times \mathbb{C}_\sigma \to \mathbb{C}_\sigma$ to the second factor to the complement $(\mathbb{V}_{x_0} \cap (B_\varepsilon(x_0) \times \mathbb{C}_\sigma)) \setminus (Y \times \mathbb{C}_\sigma)$ is a locally trivial fibration over a punctured neighbourhood of the origin in $\mathbb{C}_\sigma$: [6]. Let V_{σ, x_0} and $\zeta_{s_\sigma}(t)$ be the fibre (the local zero set in $(X \setminus Y, 0)$ of a generic function from the family) and the zeta function of a monodromy transformation of this fibration.

For a constructible function Ψ on a constructible set Z with values in an Abelian group A, there exists a notion of the integral $\int_Z \Psi d\chi$ of the function Ψ over the set Z with respect to the Euler characteristic (see, e.g., [12]). For example, if A is the multiplicative group of non-zero rational functions in the variable t, $Z = \bigcup \Xi$ is a finite stratifiction of Z (without any regularity conditions) such that the function $\Psi_x = \Psi(x)$ is one and the same for all points x of each stratum Ξ and is equal to Ψ_Ξ there, then by definition

$$\int_Z \Psi d\chi = \prod_\Xi (\Psi_\Xi(t))^{\chi(\Xi)}.$$

Theorem 1 *One has*

$$\zeta_{s_\sigma}(t) = \int_{\{x \in X : s(x) = 0\}} \zeta_{s_{\sigma,x}}(t) d\chi,$$

and therefore

$$\chi(V_\sigma) = \int_{\{x \in X : s(x) = 0\}} \chi(V_{\sigma,x}) d\chi.$$

Proof. The monodromy transformation h can be supposed to respect a Whitney stratification of the pair (X, Y). Because of the multiplicativity of the zeta function, it is sufficient to proof the statement only for one stratum. Using the induction one can suppose that the statement is already proved for strata of lower dimension. Resolving singularities of the variety, we reduce the problem to the case when X is smooth. In this situation the proof is essentially the same as in [3]. □

Remark. This statement can be considered as a generalization of the localization principles described, in particular, in [3, 5] for particular cases. It also can be deduced from general statements described in [2].

3 Localization for families of polynomials

Let P_σ be an analytic in $\sigma \in (\mathbb{C}_\sigma, 0)$ family of polynomials and let d be the degree of a generic polynomial of the family (i.e., the degree of the polynomial P_σ for $\sigma \neq 0$ small enough; $\deg P_0 \leq d$). Let $X = \mathbb{CP}^n$ be the (standard) compactification of the affine space $\mathbb{C}^n$ and let $Y = \mathbb{CP}^{n-1}_\infty$ be its infinite hyperplane. The family of polynomials P_σ can be considered as a family of sections of the line bundle $\mathcal{O}(-d)$ over the projective space $\mathbb{CP}^n$ (if the degree of the polynomial P_0 is smaller than d, then the corresponding section vanishes on the whole infinite hyperplane $\mathbb{CP}^{n-1}_\infty$).

Thus we are in a situation described in the previous section. For a point $x \in \mathbb{CP}^{n-1}_\infty \subset \mathbb{CP}^n$, let $p_{\sigma,x}$ be the germ of a holomorphic function (section) P_σ at this point (in fact $p_{\sigma,x}$ is a polynomial in an affine chart there), let $\hat{p}_{\sigma,x} = p_{\sigma,x|(\mathbb{CP}^{n-1}_\infty,x)} : (\mathbb{CP}^{n-1}_\infty, x) \to \mathbb{C}$, and let $\hat{\zeta}_{p_{\sigma,x}}(t) := \zeta_{p_{\sigma,x}}(t)/\zeta_{\hat{p}_{\sigma,x}}(t)$. Also let $\hat{\chi}(\{p_{\sigma,x} = 0\}) := \chi(\{p_{\sigma,x} = 0\}) - \chi(\{\hat{p}_{\sigma,x} = 0\})$. Let $\widetilde{P}_\sigma(x_0, x_1, \ldots, x_n)$ be the homogenization of degree d of the polynomial $P_\sigma(x_1, \ldots, x_n)$:

$$\widetilde{P}_\sigma(x_0, x_1, \ldots, x_n) = x_0^d P_\sigma(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0})$$

(if $\deg P_0 < d$, the polynomial $\widetilde{P}_0$ is not the usual homogenization of the polynomial P_0, but differs from it by the factor $x_0^{d-\deg P_0}$). Theorem 1 gives the following.

Theorem 2 *One has*

$$\zeta_{P_\sigma}(t) = \int_{\{x\in\mathbb{C}^n: P_0(x)=0\}} \zeta_{P_{\sigma,x}}(t) d\chi \ \cdot \int_{\{x\in\mathbb{CP}^{n-1}_\infty: \widetilde{P}_0(x)=0\}} \hat{\zeta}_{p_{\sigma,x}}(t) d\chi$$

and therefore

$$\chi(V_\sigma) = \int_{\{x\in\mathbb{C}^n: P_0(x)=0\}} \chi(V_{\sigma,x}) d\chi \ + \int_{\{x\in\mathbb{CP}^{n-1}_\infty: \widetilde{P}_0(x)=0\}} \hat{\chi}(\{p_{\sigma,x} = 0\}) d\chi.$$

Remarks. **1.** If $\deg P_0 = d_0 < d$, then $\{x \in \mathbb{CP}^{n-1}_\infty : \widetilde{P}_0(x) = 0\} = \mathbb{CP}^{n-1}_\infty$ and at a generic point of the infinite hyperplane $\mathbb{CP}^{n-1}_\infty$ one has $\widetilde{\zeta}_{p_{\sigma,x}}(t) = 1 - t^{d-d_0}$, $\widetilde{\chi}(\{p_{\sigma,x} = 0\}_x = d - d_0$.

2. Theorem 2 reduces computation of the zeta function of a family of polynomials to computation of the zeta functions of families of holomorphic germs. An interesting case is a linear family of polynomials and respectively linear families of holomorphic germs. In somewhat other terms this case was treated in the study of meromorphic germs elaborated for study of polynomial maps: [4, 5]. Let $s_\sigma = f + \sigma g$ be a linear family of holomorphic germs (the family of the zero level sets of s_σ is a pencil). Then, modulo the indeterminacy locus $\{f = g = 0\}$, the general local level set and the monodromy transformation of of the family s_σ coincides with the zero Milnor fibre $\mathcal{M}^0_\varphi$ and the corresponding monodromy transformation h^0_φ of the meromorphic germ $\varphi = \frac{f}{g}$ as they are defined in [3]. If $g(0) \neq 0$, the indeterminacy locus is empty and the indicated objects coincide (and coincide with the usual Milnor fibre of the germ f and its classical monodromy transformation). If $f(0) = g(0) = 0$, the indeterminacy locus is (locally) contractible and the monodromy transformation may be supposed to be identity on it. Therefore $\chi(V_{\sigma,0}) = \chi(\mathcal{M}^0_\varphi) + 1$, $\zeta_{s_\sigma,0}(t) = \zeta^0_\varphi(t)(1-t)$. This permits to apply methods elaborated for meromorphic germs to linear families of holomorphic germs. In particular, there is a Varchenko type formula which expresses the zeta function $\zeta^0_\varphi(t)$ in terms of the Newton diagrams of f and g (in the case when the meromorphic germ φ is non-degenerate with respect to the pair of the Newton diagrams): [4].

3. Let $P_\sigma(x) = f(x) + \sigma g_d(x)$ where f is a polynomial of degree d such that the projective closure of any fibre of f and its intersection with the hyperplane at infinity have isolated singularities (in fact isolated boundary singularities) and g is a sufficiently general homogeneous polynomial of degree d, such that the compactified fibres of $f + \sigma g_d$ have transversal intersections with infinity as soon as $\sigma \neq 0$ (e.g. $g = \ell^d$ for a generic linear function ℓ). In [9, section 7] it is shown that the zeta-function of these deformation is equal to

$$\zeta_{P_\sigma}(t) = (1-t)^{\chi(V_0)} \prod \zeta_{Z_i}(t)$$

where V_0 is the fibre $\{f = 0\}$ (supposed to be smooth) and ζ_{Z_i} are the zeta functions of the monodromy transformations of the boundary singularities mentioned above. This formula is now an immedeate corrolary of Theorem 2.

4 Examples

1. Let $P_\sigma(x, y) = x^{d_0} + \sigma(x^d + y^d)$ (n=2). There are 3 different cases.

1) $d_0 > d$. In this case the set $\{\widetilde{P}_0 = 0\} \subset \mathbb{CP}^2$ is the closure of the line $\{x = 0\}$. There are 3 types of points in it:

a) The origin $0 = (0, 0)$. The Varchenko type formula gives

$$\zeta_{P_{\sigma,0}}(t) = (1-t)(1-t^{d_0-d})^{1-d}.$$

b) Other points of the affine line $\{x = 0\}$, i.e. $y \neq 0$. At such a point one has

$\zeta_{P_\sigma,x}(t) = (1-t^{d_0})$. However, the Euler characteristic of the set of these points is equal to zero and therefore this stratum gives no impact to $\zeta_{P_\sigma}(t)$.
c) The infinite point of the line $\{x=0\}$. One can easily see that $\widetilde{\zeta}_{p_\sigma,x}(t) = 1$.

Integrating these local data one gets

$$\zeta_{P_\sigma}(t) = (1-t)(1-t^{d_0-d})^{1-d}.$$

One can say that this zeta function essentially originates from the origin in $\mathbb{C}^2$.
2) $d_0 = d$. This case is not interesting: simple computations (or considerations) give $\zeta_{P_\sigma}(t) = (1-t)$.
3) $d_0 < d$. In this case the set $\{\widetilde{P}_0 = 0\} \subset \mathbb{CP}^2$ is the union of the line $\{x=0\}$ in the affine plane and the infinite line $\mathbb{CP}^1_\infty$. There are 5 types of points in it:
a) The origin $0 = (0,0)$. One has (e.g., from the Varchenko type formula) $\zeta_{P_\sigma,0}(t) = (1-t)$.
b) Other points of the affine line $\{x=0\}$. Again $\zeta_{P_\sigma,x}(t) = (1-t^{d_0})$, but the Euler characteristic of this stratum is equal to zero.
c) The infinite point of the line $\{x=0\}$: $\widetilde{\zeta}_{p_\sigma,x}(t) = 1$.
d) Intersection points of the infinite line $\mathbb{CP}^1_\infty$ with the closure of the curve $\{x^d+y^d = 0\}$; there are d of them. One can easily see that $\widetilde{\zeta}_{p_\sigma,x}(t) = 1$ at them.
e) Finally we have all other (generic) points of the infinite line. The Euler characteristic of this stratum is equal to $2-(1+d) = 1-d$. As it was explaned in the Remark at the end of Section 3, $\widetilde{\zeta}_{p_\sigma,x}(t) = 1 - d^{d-d_0}$.

Integrating these local data one gets

$$\zeta_{P_\sigma}(t) = (1-t)(1-t^{d-d_0})^{1-d}$$

(almost as in the case 1). One can say that this zeta function essentially originates from the open stratum of the infinite line.
2. Let $P_\sigma(x_1,x_2,x_3) = x_1^4 + x_2^3 + x_3^2 + \sigma x_2^4$ (in this example we obeserve some speical changes at the behaviour at infinity). The projective variety $\{\widetilde{P}_0 = 0\} \subset \mathbb{CP}^3$ intersects the infinite plane $\mathbb{CP}^2_\infty = \{x_0 = 0\}$ along the line $\{x_1 = 0\}$ and consists of the following (smooth) strata.
a) The origin $\{0\}$ in $\mathbb{C}^3$. Here we have an equisingular deformation of the (surface) singularity E_6 and therefore $\zeta_{P_\sigma,0}(t) = 1-t$.
b) The set $\{P_0 = 0\} \setminus \{0\}$. One has $\zeta_{P_\sigma,x}(t) = 1-t$ at all points x of this stratum, however the Euler characteristic of the stratum itself is equal to zero and thus it does not contribute to the zeta function of the family P_σ.
c) The "distinguished point" $(0:0:0:1)$ on the infinite projective line $\{x_0 = x_1 = 0\}$. With the help of the Varchenko type formula ([4]; see Fig. 1) one gets $\zeta_{P_\sigma,x}(t) = \frac{(1-t^4)^2}{1-t^2}$ for this point.
d) The affine line $\{x_0 = x_1 = 0\} \setminus \{(0:0:0:1)\}$. Its Euler characteristic is equal to 1. The variety $\widetilde{P}_0(0) = 0$ is non-singular at points of it, however, its intersection

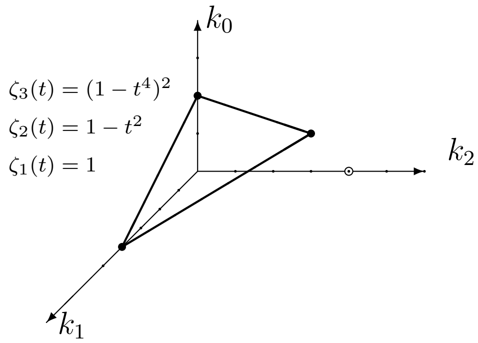

Figure 1: The Dynkin diagrams of the summands corresponding to deformation at the point $(0:0:0:1)$ (marked by $\bullet$ and $\circ$ respectively).

with the infinite plane consists of a line of multiplicity 4 which, for $\sigma \neq 0$, splits into 4 different lines intersecting each other at one point $(0:0:0:1)$. Therefore, for points of this stratum, one has $\zeta_{P_\sigma,x}(t) = \frac{1-t}{1-t^4}$.
Combining all these local data one gets $\zeta_{P_\sigma}(t) = \frac{(1-t^4)(1-t)^2}{1-t^2}$.
3. Let $P_\sigma(x) = f_{d_0}(x) + \sigma g_d(x)$ where $x = (x_1, ..., x_n) \in \mathbb{C}^n$, f_{d_0} is a non-degenerate homogeneous polynomial of degree d_0 (i.e., it has an isolated critical point at the origin), g_d is a generic homogeneous polynomial of degree d, i.e., the polynomial g_d is non-degenerate and the hypersurfaces $\{f_{d_0=0}\}$ and $\{g_d = 0\}$ in $\mathbb{CP}^{n-1}_\infty$ intersect transversally.
1) $d_0 > d$. There are 3 types of points in the set $\{\widetilde{P}_0 = 0\} \subset \mathbb{CP}^n$:
a) The origin in $\mathbb{C}^n$. The Varchenko type formula gives

$$\zeta_{P_\sigma,0}(t) = (1-t) \cdot \left(1 - t^{d_0-d}\right)^{(-1)^{n-1}((d_0-1)^n-(d-1)^n)/(d_0-d)}.$$

b) Other points of the hypersurface $\{P_0 = 0\}$ in the affine space $\mathbb{C}^n$. For these point $\zeta_{P_\sigma,x}(t) = (1-t)$. The Euler characteristic of the set of these points is equal to zero.
c) Infinite points of the set $\{\widetilde{P}_0 = 0\}$, i.e., points of $\{f_{d_0=0}\} \subset \mathbb{CP}^{n-1}_\infty$. For these points one has $\hat{\zeta}_{p_\sigma,x}(t) = 1$.
Combining the local data one gets

$$\zeta_{P_\sigma}(t) = \zeta_{P_\sigma,0}(t) = (1-t) \cdot \left(1 - t^{d_0-d}\right)^{(-1)^{n-1}((d_0-1)^n-(d-1)^n)/(d_0-d)}.$$

2) $d_0 = d$. Obviously $\zeta_{P_\sigma}(t) = 1-t$.
3) $d_0 < d$. The set $\{\widetilde{P}_0 = 0\}$ is the union $\{x \in \mathbb{C}^n : P_0 = 0\} \cup \mathbb{CP}^{n-1}_\infty$.
a) For all points of the set $\{x \in \mathbb{C}^n : P_0 = 0\}$ one has $\zeta_{P_\sigma,x}(t) = (1-t)$. The Euler characteristic of this set is equal to 1.
b) At the points of $\{f_{d_0} = 0\} \subset \mathbb{CP}^{n-1}_\infty$ and at the points of $\{g_d = 0\} \subset \mathbb{CP}^{n-1}_\infty$ one has $\hat{\zeta}_{p_\sigma,x}(t) = 1$.
c) At the points of the complement $\mathbb{CP}^{n-1}_\infty \setminus (\{f_{d_0} = 0\} \cup \{g_d = 0\})$ one has $\hat{\zeta}_{p_\sigma,x}(t) =$

$1 - t^{d-d_0}$. The Euler characteristic of this set is equal to $(-1)^{n-1}((d-1)^n - (d_0 - 1)^n)/(d-d_0)$.

Combying the local data one gets

$$\zeta_{P_\sigma}(t) = (1-t) \cdot \left(1 - t^{d-d_0}\right)^{(-1)^{n-1}((d-1)^n-(d_0-1)^n)/(d-d_0)}.$$

Though for $d_0 > d$ and $d_0 < d$ the answers are similar, as in Example 1 one can say that the origins of the zeta function (and/or of the vanishing cycles) in these cases are different: the origin in $\mathbb{C}^n$ and the infinite hyperplane $\mathbb{CP}^{n-1}_\infty$ respectively.

4. Let $P_\sigma(x) = f_{d_0}(x) + \sigma(\ell(x))^d$ where $x = (x_1, ..., x_n) \in \mathbb{C}^n$, f_{d_0} is a non-degenerate homogeneous polynomial of degree d_0, ℓ is a generic (homogeneous) linear function, i.e., the hypersurfaces $\{f_{d_0=0}\}$ and $\{\ell = 0\}$ in $\mathbb{CP}^{n-1}_\infty$ intersect transversally. Considerations similar to those of Example 3 give:

$$\zeta_{P_\sigma}(t) = \begin{cases} (1-t) \cdot (1-t^{d_0-d})^{(1-d_0)^{n-1}} & \text{for } d_0 > d, \\ 1-t & \text{for } d_0 = d, \\ (1-t) \cdot (1-t^{d-d_0})^{(1-d_0)^{n-1}} & \text{for } d > d_0. \end{cases}$$

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