This is section 4.4 from Duco van Staten's Thesis (1987): Weakly Normal Surface Singularities and Their Improvements

## § 4.4 The First Betti Number of a Smoothing

The following result is due to Greuel & Steenbrink:

**(4.4.1) Theorem :** Let $\mathcal{X} \xrightarrow{f} S$ be a smoothing of a *normal isolated* singularity $X = f^{-1}(0)$. Let $X_t = f^{-1}(t)$, $t \neq 0$, be its Milnor fibre. Then:

$$b_1(X_t) := \dim_{\mathbb{C}} H^1(X_t, \mathbb{C}) = 0$$

For a proof see [G-S].

When one looks for a similar simple statement for non-isolated singularities one runs soon into big trouble. By taking the cone over Zariski's plane sextic with six cusps, we get a surface in $\mathbb{C}^3$. The first Betti number of the Milnor fibre of this surface (which thus appears as a six-fold cover of the complement of the curve) depends on the position of the cusps: when they are on a conic, then $b_1(X_t) = 2$, when they are not, then $b_1(X_t) = 0$. (see [Es]). This shows that $b_1$ is a subtle invariant.

The cone over a curve $\Gamma \subset \mathbb{P}^2$ is weakly normal precisely when $\Gamma$ has only ordinary double points. In that case the first Betti number is independent of the exact position of the double points: one has $b_1(X_t) = r - 1$, where r is the number of irreducible components of $\Gamma$. We are going to proof the following generalization of theorem (4.4.1):

**Theorem :** Let $\mathcal{X} \xrightarrow{f} S$ be a smoothing of a (reduced, equidimensional and) *weakly normal* space (germ) X. Let $X_t = f^{-1}(t)$, $t \neq 0$, be its Milnor fibre and r the number of irreducible components of X. Then:

$$b_1(X_t) \leq r - 1$$

For a hypersurface one has equality.

The proof will be along the lines of [G-S].

### (4.4.2) Notation & Topological description

Let X be a fixed contractible Stein representative of a *reduced* and *equidimensional* germ (X,p).
We consider a smoothing of X over a smooth curve (germ) S:

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow f \\ \{0\} & \longrightarrow & S \end{array}$$

We also assume that $\mathcal{X}$ is contractible and Stein. Remark that in this situation we have that $\mathcal{X}$ is *normal*: $\mathrm{Sing}(\mathcal{X}) \subset \Sigma := \mathrm{Sing}(X)$, so this is of codimension $\geq 2$. Further $\mathrm{depth}_{\Sigma}(X) \geq 1$ , so we have $\mathrm{depth}_{\Sigma}(\mathcal{X}) \geq 2$ .

To study the Milnor fibre $X_t := f^{-1}(t)$, $t \neq 0$, it is convenient to take an *embedded resolution* of X in $\mathcal{X}$. So we get a space $\mathcal{Y}$ together with a proper map $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$ with the following properties:

1) $\mathcal{Y} - \pi^{-1}(\Sigma) \longrightarrow \mathcal{X} - \Sigma$.
2) $Y := (f \circ \pi)^{-1}(0)$ is a normal crossing divisor.
3) $\mathcal{Y}$ is smooth.

After a finite base change we may assume that Y is reduced.
(Semi-stable reduction.)
In Y we find in general three types of divisors:

a) $\tilde{X}$, the strict transform of X.
b) F, a set of non-compact divisors, mapping properly to $\Sigma$.
c) $E = \pi^{-1}(p)$, a compact divisor.

(c.f. with the situation in (2.6.?)).

The Milnor fibre $X_t$ is via $\pi$ isomorphic to $Y_t := (f \circ \pi)^{-1}(t) \subset \mathcal{Y}$.
In a semi-stable family this Milnor fibre $Y_t$ "passes along" every component of Y just once. One can find a "contraction"

$$c: Y_t \longrightarrow Y$$

of the Milnor fibre $Y_t$ on the special fibre Y (see [Cl],[Stee 1]).
Now we can use the Leray spectral sequence for c to find the beginning of an exact sequence:

**Leray:**

$$0 \longrightarrow H^1(Y) \longrightarrow H^1(Y_t) \longrightarrow H^0(\mathbb{C}_{Y^{[0]}}/\mathbb{C}_Y) \longrightarrow H^2(Y) \longrightarrow$$

Here we have used the easily verified formulas:

$$c_*\mathbb{C}_{Y_t} = \mathbb{C}_Y$$
$$R^1c_*\mathbb{C}_{Y_t} = \mathbb{C}_{Y^{[0]}}/\mathbb{C}_Y$$

$Y^{[0]} := \coprod Y_i$ , where $Y_i$ are the irreducible components of $Y$. (The sheaf $\mathbb{C}_{Y^{[0]}}$ is considered on $Y$.)

We note that there are two other exact sequences in which $H^1(Y_t)$ appears:

**Milnor's Wang sequence (see [Mi], p. 67)**

$$0 \longrightarrow H^0(Y_t) \longrightarrow H^1(B) \longrightarrow H^1(Y_t) \xrightarrow{h_*-Id} H^1(Y_t) \longrightarrow \cdots$$

Here $B = \mathcal{X} - X$ , the total space of the Milnor fibration over $S - \{0\}$ and $h_*$ is the monodromy transformation.

**Sequence of the pair** $B = \mathcal{Y} - Y \longrightarrow \mathcal{Y} \approx Y$ ($\approx$ means homotopy equivalence)

$$0 \longrightarrow H^1(Y) \longrightarrow H^1(B) \xrightarrow{\alpha} H^0(Y^{[0]}) \xrightarrow{\beta} H^2(Y) \longrightarrow \cdots$$

Here we have used the isomorphism $H^2(\mathcal{Y}, \mathcal{Y} - Y) \approx H^0(Y^{[0]})$

These three sequences fit into a single big diagram:

$$\begin{array}{cccccccccc}
 & & & & \vdots & & \vdots & & & \\
 & & & & \uparrow & & \uparrow & & & \\
 & & & & H^1(Y_t) & & H^1(Y) & & & \\
 & & & & \uparrow {\scriptstyle h_*-id} & & \uparrow & & & \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(Y_t) & \longrightarrow & H^0(\mathbb{C}_{Y^{[0]}}/\mathbb{C}_Y) & \longrightarrow & H^2(Y) & \longrightarrow \\
 & & \uparrow\wr & & \uparrow & & \uparrow & & \uparrow\wr & \\
0 & \longrightarrow & H^1(Y) & \longrightarrow & H^1(B) & \xrightarrow{\alpha} & H^0(Y^{[0]}) & \xrightarrow{\beta} & H^2(Y) & \longrightarrow \\
 & & & & \uparrow & & \uparrow & & & \\
 & & & & H^0(Y_t) & \xrightarrow{\sim} & H^0(Y) & & &
\end{array}$$

We note that $\dim H^0(Y) = 1$.
From this diagram we draw the following conclusions:

**(4.4.3) Conclusion:** In the above sitution we have:

1) $\dim H^1(B) = \dim H^1(Y) + \dim \ker \beta$.
2) $\dim H^1(Y_t) \geq \dim H^1(B) - 1$, and equality holds if $H^1(Y) = 0$.
3) If $H^1(Y) = 0$, then the monodromy acts trivially on $H^1(Y_t)$.

We now study the parts $H^1(Y)$ and $\ker \beta$ separately.

**(4.4.4) The group $H^1(Y)$.** If X is a plane curve singularity, then one can compute $\dim H^1(Y)$. The result is:

$$\dim H^1(Y) = 2.g + b$$

where g is the sum of the genera of the compact components of Y and b is the number of cycles in the dual graph of Y. (These numbers g and b are invariants of the limit Mixed Hodge Structure on $H^1(X_t)$; one has $b = \dim Gr_0^W Gr_F^1 H^1(Y_t)$, $g = \dim Gr_1^W Gr_F^1(Y_t)$ , see [Stee 1].). By taking $X \times \mathbb{C}$ we can construct (trivial) examples of *irreducible* surfaces with $H^1(X_t)$ arbitrarily high. Only in the case that X is an ordinary double point, one has $H^1(Y) = 0$. It turns out that it is exactly the *weak normality of $X_0$* that forces $H^1(Y)$ to vanish.

**(4.4.5) Proposition :** Let $\mathcal{X} \xrightarrow{f} S$ be a flat deformation of a weakly normal $X = f^{-1}(0)$. Let $\mathcal{Y} \xrightarrow{\pi} \mathcal{X}$ be map such that:

1) $\mathcal{Y} - \pi^{-1}(\Sigma) \longrightarrow \mathcal{X} - \Sigma$, $\Sigma = \mathrm{Sing}(\Sigma)$
2) $\pi_* \mathcal{O}_{\mathcal{Y}} \approx \mathcal{O}_{\mathcal{X}}$.

Then one has: $R^1 \pi_* \mathcal{O}_{\mathcal{Y}} = 0$.

**proof :** This is the crucial point and the argument is the same as in [G-S]. First look at the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathcal{Y}} \xrightarrow{t} \mathcal{O}_{\mathcal{Y}} \longrightarrow \mathcal{O}_Y \longrightarrow 0$$

Here t is a local parameter on S and Y is the fibre over 0. Taking the direct image of the above sequence gives a diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \pi_*\mathcal{O}_{\mathcal{Y}} & \xrightarrow{t} & \pi_*\mathcal{O}_{\mathcal{Y}} & \longrightarrow & \pi_*\mathcal{O}_Y & \longrightarrow & R^1\pi_*\mathcal{O}_{\mathcal{Y}} & \xrightarrow{t} & R^1\pi_*\mathcal{O}_{\mathcal{Y}} \longrightarrow \\ & & \uparrow & & \uparrow & & \uparrow & & & & \\ 0 & \longrightarrow & \mathcal{O}_{\mathcal{X}} & \xrightarrow{t} & \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & & \end{array}$$

By assumption $\pi_*\mathcal{O}_{\mathcal{Y}} \approx \mathcal{O}_{\mathcal{X}}$ . From this it follows that the sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y \longrightarrow R^1\pi_*\mathcal{O}_{\mathcal{Y}} \xrightarrow{t} R^1\pi_*\mathcal{O}_{\mathcal{Y}}$$

is also exact. We claim that $\mathcal{O}_X \approx \pi_*\mathcal{O}_Y$. Note that by condition 2) we have that the fibres of $\pi$ are *connected*. Consider a section $g \in \pi_*\mathcal{O}_Y$, or what amounts to the same, a funcion on Y. As the $\pi$-fibres are compact and connected, this function is *constant* along the $\pi$-fibres. Hence g can be considered as a *continous* function on X , which is holomorphic on $Y - \pi^{-1}(\Sigma) \longrightarrow X - \Sigma$. Because we assumed X to be weakly normal, $g \in \mathcal{O}_X$ . So we have indeed $\mathcal{O}_X \longrightarrow \pi_*\mathcal{O}_Y$. Because the map $\pi$ is an isomorphism outside $\Sigma$, the coherent sheaf $R^1\pi_*\mathcal{O}_{\mathcal{Y}}$ has as support a set contained in $\Sigma$. By the last exact sequence t acts *injectively* $R^1\pi_*\mathcal{O}_{\mathcal{Y}}$ . As t vanishes on $\Sigma$ ($\subset X$) we conclude that $R^1\pi_*\mathcal{O}_{\mathcal{Y}} = 0$. ∎

**(4.4.6) Corollary :** Let C be a weakly normal curve singularity and X the total space of a flat deformation $X \longrightarrow S$ of C. Then X is weakly rational. This was stated as (2.5.7).

**(4.4.7) Proposition :** With the notation of (4.4.2) we have:

$$X \text{ weakly normal} \Rightarrow H^1(Y) = 0$$

**proof :** The embedded resolution map $\mathcal{Y} \longrightarrow \mathcal{X}$ clearly fulfils condition 1) of (4.4.5). It also fulfils condition 2), because $\mathcal{X}$ is normal, hence $\mathcal{O}_{\mathcal{X}} \longrightarrow i_*\mathcal{O}_{\mathcal{X}-\Sigma}$ where $i: \mathcal{X} - \Sigma \longrightarrow \mathcal{X}$ is the inclusion map. Because $\mathcal{Y} - \pi^{-1}(\Sigma) \longrightarrow \mathcal{X} - \Sigma$, it follows that $\pi_*\mathcal{O}_{\mathcal{Y}} \approx \mathcal{O}_{\mathcal{X}}$ . Now according to (4.4.5) we have $R^1\pi_*\mathcal{O}_{\mathcal{Y}} = 0$, in other words :

$$H^1(\mathcal{O}_{\mathcal{Y}}) = 0$$

From the exponential sequence

$$0 \longrightarrow \mathbb{Z}_{\mathcal{Y}} \longrightarrow \mathcal{O}_{\mathcal{Y}} \longrightarrow \mathcal{O}^*_{\mathcal{Y}} \longrightarrow 0$$

and the similar sequence for $\mathcal{X}$ and the fact that $\mathcal{O}_{\mathcal{X}} \approx \pi_* \mathcal{O}_{\mathcal{Y}}$ it then follows that:

$$H^1(\mathcal{Y}, \mathbb{Z}_{\mathcal{Y}}) = 0$$

As $\mathcal{Y}$ is contractible to Y, we have $H^1(Y, \mathbb{Z}) = 0$ ∎

**(4.4.8) The kernel of β.**

In the big diagram of (4.4.2) there was a map β

$$H^0(Y^{[0]}, \mathbb{Z}) \xrightarrow{\beta} H^2(\mathcal{Y}, \mathbb{Z}) \quad (= H^2(Y, \mathbb{Z}))$$

This map works as follows: Elements of the first groups can be considered as *divisors* $\sum a_i . Y_i$ with support on Y. (The $Y_i$ are the irreducible components of Y.) Then one has:

$\beta(\sum n_i . Y_i)$ = first chern class of the line bundle determined by the divisor $\sum n_i . Y_i$

So the map β factorizes over the map ψ which associates to a divisor its line bundle:

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & H^1(\mathcal{O}_{\mathcal{Y}}) & \longrightarrow & H^1(\mathcal{O}^*_{\mathcal{Y}}) & \longrightarrow & H^2(\mathcal{Y}, \mathbb{Z}) & \longrightarrow & \ldots \\ & & & & \psi \uparrow & \nearrow \beta & & & \\ & & & & H^0(Y^{[0]}, \mathbb{Z}) & & & & \end{array}$$

We first study the map ψ. Note that if $H^1(\mathcal{O}_{\mathcal{Y}}) = 0$, then we have $\ker \psi = \ker \beta$.

**(4.4.9) Definition :** Let $(\mathcal{X}, p)$ be a germ of a *normal* analytic space. The *local class group* is the group

$$Cl_p(X) = \{\text{(germs of) Weil divisors}\}/\{\text{(germs of) principal divisors}\}$$

**(4.4.10) Proposition :** With the notation as in (4.4.2) there is a diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \ker \psi & \xrightarrow{\sim} & \ker \gamma & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(F^{[0]}) & \longrightarrow & H^0(Y^{[0]}) & \longrightarrow & H^0(X^{[0]}) & \longrightarrow & 0 \\
 & & \downarrow \wr & & \psi \downarrow & & \gamma \downarrow & & \\
0 & \longrightarrow & H^0(F^{[0]}) & \longrightarrow & H^1(\mathcal{O}^*_{\mathcal{Y}}) & \longrightarrow & Cl_p(\mathcal{X}) & \longrightarrow & 0
\end{array}
$$

Here $F^{[0]} = Y^{[0]} \setminus \tilde{X}$ (so it contains the components F and E of (4.4.2)) and $X^{[0]} = \coprod X_i$ , where the $X_i$ are the irreducible components of X. The maps are the obvious ones.

**proof :** The surjection of $H^1(\mathcal{O}^*_{\mathcal{Y}})$ (or better of $R^1\pi_*\mathcal{O}^*_{\mathcal{Y}}$) to the local class group is obvious: pulling back a Weil divisor on $\mathcal{X}$ gives a Cartier divisor on $\mathcal{Y}$ (hence a line bundle) that maps down to the original Weil divisor, as the map $\pi$ is a modification in codimension $\geq 2$ (c.f. [Mu]). The main point is to show that the kernel of the bottom row is not bigger, or what amounts to the same, that $\ker \psi \approx \ker \gamma$. Let $A = \sum a_i . Y_i$ be in the kernel of $\psi$. We may assume that $a_i \geq 0$ . Hence there is a function $g \in H^0(\mathcal{O}_{\mathcal{Y}})$ with $(g) = A$. By the normality of $\mathcal{X}$ we have $\mathcal{O}_{\mathcal{X}} = \pi_*\mathcal{O}_{\mathcal{Y}}$ , so g can be considered as holomorphic on $\mathcal{X}$, having of course as divisor on $\mathcal{X}$ just (the image) of that part of A that does not involve $F^{[0]}$. This gives the map $\ker \psi \longrightarrow \ker \gamma$. This map is injective because if the divisor of g (on $\mathcal{X}$) would be zero, g would be a unit, hence A = 0. Surjectivity follows by pulling back functions. ∎

The use of (4.4.10) is that we get rid of the global object $\mathcal{Y}$ In (4.4.2) we used a base change to arrive at a semi-stable family. The kernel of the map $\gamma$ is essentially independent of this base change:

**(4.4.11) Lemma :** Consider a normal space $\mathcal{X}$ and a reduced principal divisor $X \subset \mathcal{X}$. Let $X^{[0]} = \coprod X_i$ , where the $X_i$ are the irreducible components of X. Let $\tilde{\mathcal{X}}$ be obtained from $\mathcal{X}$ by taking a d-fold cyclic cover branched along X. Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl_p(\mathcal{X})$ , $\tilde{\gamma} : H^0(X^{[0]}) \longrightarrow Cl_p(\tilde{\mathcal{X}})$ be the obvious maps. Then $\ker(\gamma) \otimes \mathbb{Q} = \ker(\tilde{\gamma}) \otimes \mathbb{Q}$

**proof :** Excercise. ∎

We summarize the above results in one theorem:

**(4.4.12) Theorem :** Let $\mathcal{X} \xrightarrow{f} S$ be (a contractible Stein representative of) a smoothing of a reduced germ (X,p). Let $X_t = f^{-1}(t)$, $t \neq 0$, be its Milnor fibre.
Let $X^{[0]} = \coprod X_i$ , where the $X_i$ are the irreducible components.
Let $\gamma : H^0(X^{[0]}) \longrightarrow Cl_p(\mathcal{X})$ be the obvious map.
Then one has:

1) $b_1(X_t) \geq \text{rank } (\ker \gamma) - 1$.
2) If X is weakly normal, then one has equality:

$$b_1(X_t) = \text{rank } (\ker \gamma) - 1.$$

In particular, when X is a *hypersurface*, rank ker $\gamma$ is equal to the number of irreducible components of X.

**(4.4.13) Remark :** For a hypersurface germ X in $\mathbb{C}^3$ with a *complete intersection* as singular locus and transversal type $A_1$ it is known that the first Betti number $b_1(X_t)$ is zero or one (see [Sie 2], [Str]). So the number of irreducible components of X is one or two. To put it in another way, the singular locus of a weakly normal hypersurface in $\mathbb{C}^3$ which has more than three components is *never* a complete intersection.

**(4.4.14) Question :** J. Stevens has shown that all degenerate cusps are smoothable (private communication). What is the first Betti number for these smoothings? Is the first Betti number an invariant of X? (Probably not, but at this moment I do not have computed any non-trivial example.)