The following result is due to Greuel & Steenbrink:

\begin{center}
(4.4.1) Theorem : \ Let \( X \xrightarrow{f} S \) be a smoothing of a normal isolated singularity \( X = f^{-1}(0) \). Let \( X_t = f^{-1}(t) \), \( t \neq 0 \), be its Milnor fibre. Then:
\[ b_1(X_t) := \dim_{\mathbb{C}} H^1(X_t, \mathbb{C}) = 0 \]
\end{center}

For a proof see [G-S].

When one looks for a similar simple statement for non-isolated singularities one runs soon into big trouble. By taking the cone over Zariski's plane sextic with six cusps, we get a surface in \( \mathbb{C}^3 \). The first Betti number of the Milnor fibre of this surface (which thus appears as a six-fold cover of the complement of the curve) depends on the position of the cusps: when they are on a conic, then \( b_1(X_t) = 2 \), when they are not, then \( b_1(X_t) = 0 \). (see [Es]).

This shows that \( b_1 \) is a subtle invariant.

The cone over a curve \( \Gamma \subset \mathbb{P}^2 \) is weakly normal precisely when \( \Gamma \) has only ordinary double points. In that case the first Betti number is independent of the exact position of the double points: one has \( b_1(X_t) = r - 1 \), where \( r \) is the number of irreducible components of \( \Gamma \). We are going to prove the following generalization of theorem (4.4.1):

\begin{center}
Theorem : \ Let \( X \xrightarrow{f} S \) be a smoothing of a (reduced, equidimensional and) weakly normal space (germ) \( X \). Let \( X_t = f^{-1}(t) \), \( t \neq 0 \), be its Milnor fibre and \( r \) the number of irreducible components of \( X \). Then:
\[ b_1(X_t) \leq r - 1 \]
\end{center}

For a hypersurface one has equality.

The proof will be along the lines of [G-S].
Let $X$ be a fixed contractible Stein representative of a reduced and equidimensional germ $(X,p)$. We consider a smoothing of $X$ over a smooth curve (germ) $S$:

$$
\begin{array}{c}
X \\
\downarrow \\
\{0\}
\end{array} 
\begin{array}{c}
x \\
\downarrow f \\
S
\end{array}
$$

We also assume that $X$ is contractible and Stein. Remark that in this situation we have that $X$ is normal: $\text{Sing}(X) \subseteq \Xi := \text{Sing}(X)$, so this is of codimension $\geq 2$. Further $\text{depth}_p(X) \geq 1$, so we have $\text{depth}_p(X) \geq 2$.

To study the Milnor fibre $X_t := f^{-1}(t)$, $t \neq 0$, it is convenient to take an embedded resolution of $X$ in $X$. So we get a space $Y$ together with a proper map $Y \xrightarrow{\pi} X$ with the following properties:

1) $\mathcal{Y} = \pi^{-1}(\Xi) \longrightarrow X - \Xi$.
2) $Y := (f \circ \pi)^{-1}(0)$ is a normal crossing divisor.
3) $\mathcal{Y}$ is smooth.

After a finite base change we may assume that $Y$ is reduced.

(Semi-stable reduction.)

In $Y$ we find in general three types of divisors:

a) $\tilde{X}$, the strict transform of $X$.

b) $F$, a set of non-compact divisors, mapping properly to $\Xi$.

c) $E = \pi^{-1}(p)$, a compact divisor.

(c.f. with the situation in (2.6.?).)

The Milnor fibre $X_t$ is via $\pi$ isomorphic to $Y_t := (f \circ \pi)^{-1}(t) \subset Y$. In a semi-stable family this Milnor fibre $Y_t$ "passes along" every component of $Y$ just once. One can find a "contraction"

$$c: Y_t \longrightarrow Y$$

of the Milnor fibre $Y_t$ on the special fibre $Y$ (see [Cl],[Stee 1]). Now we can use the Leray spectral sequence for $c$ to find the beginning of an exact sequence:
Leray:

\[ 0 \longrightarrow H^1(Y) \longrightarrow H^1(Y_t) \longrightarrow H^0(\mathcal{C}_Y^{[0]}/\mathcal{C}_Y) \longrightarrow H^2(Y) \longrightarrow \]

Here we have used the easily verified formulas:

\[ c_* \mathcal{C}_Y = \mathcal{C}_Y \]
\[ R^1 c_* \mathcal{C}_Y = \mathcal{C}_Y^{[0]}/\mathcal{C}_Y \]

\( Y^{[0]} := \bigsqcup Y_i \), where \( Y_i \) are the irreducible components of \( Y \). (The sheaf \( \mathcal{C}_Y^{[0]} \) is considered on \( Y \).)

We note that there are two other exact sequences in which \( H^1(Y_t) \) appears:

**Milnor's Wang sequence** (see [Mi], p. 67)

\[ 0 \longrightarrow H^0(Y_t) \longrightarrow H^1(B) \longrightarrow H^1(Y_t) \xrightarrow{h_* - \text{id}} H^2(Y_t) \longrightarrow \cdots \]

Here \( B = X - X \), the total space of the Milnor fibration over \( S - \{0\} \) and \( h_* \) is the monodromy transformation.

**Sequence of the pair** \( B = Y - Y \longrightarrow Y \approx Y(\approx \text{means homotopy equivalence})\)

\[ 0 \longrightarrow H^1(Y) \longrightarrow H^1(B) \xrightarrow{\alpha} H^0(Y^{[0]}) \xrightarrow{\beta} H^2(Y) \longrightarrow \cdots \]

Here we have used the isomorphism \( H^2(Y, Y - Y) \approx H^0(Y^{[0]}) \).

These three sequences fit into a single big diagram:
We note that \( \dim H^0(Y) = 1 \).
From this diagram we draw the following conclusions:

(4.4.3) Conclusion: In the above situation we have:
1) \( \dim H^1(B) = \dim H^1(Y) + \dim \ker \beta \).
2) \( \dim H^1(Y_t) \geq \dim H^1(B) - 1 \), and equality holds if \( H^1(Y) = 0 \).
3) If \( H^1(Y) = 0 \), then the monodromy acts trivially on \( H^1(Y_t) \).

We now study the parts \( H^1(Y) \) and \( \ker \beta \) separately.

(4.4.4) The group \( H^1(Y) \). If \( X \) is a plane curve singularity, then one can compute \( \dim H^1(Y) \). The result is:
\[
\dim H^1(Y) = 2 \cdot g + b
\]
where \( g \) is the sum of the genera of the compact components of \( Y \) and \( b \) is the number of cycles in the dual graph of \( Y \). (These numbers \( g \) and \( b \) are invariants of the limit Mixed Hodge Structure on \( H^1(X_t) \); one has \( b = \dim \Gr^W_0 \Gr^W_F H^1(Y_t) \), \( g = \dim \Gr^W_1 \Gr^W_F (Y_t) \), see [Stee 1].) By taking \( X \subseteq \mathbb{C} \) we can construct (trivial) examples of irreducible surfaces with \( H^1(X_t) \) arbitrarily high. Only in the case that \( X \) is an ordinary double point, one has \( H^1(Y) = 0 \). It turns out that it is exactly the weak normality of \( X_0 \) that forces \( H^1(Y) \) to vanish.

(4.4.5) Proposition: Let \( X \xrightarrow{f} S \) be a flat deformation of a weakly normal \( X = f^{-1}(0) \). Let \( y \xrightarrow{\pi} x \) be map such that:
1) \( y - \pi^{-1}(\xi) \rightarrow x - \xi, \xi = \text{Sing}(\xi) \)
2) \( \pi \circ \phi_y \approx \phi_x \).
Then one has:
\[
R^1\pi_\ast \phi_y = 0.
\]

Proof: This is the crucial point and the argument is the same as in [G-S]. First look at the exact sequence:
\[
0 \rightarrow \mathcal{O}_y \xrightarrow{T} \mathcal{O}_y \rightarrow \mathcal{O}_x \rightarrow 0
\]
Here \( t \) is a local parameter on \( S \) and \( Y \) is the fibre over \( 0 \). Taking
the direct image of the above sequence gives a diagram:

\[
\begin{array}{ccccccccc}
0 & \to & \pi_*\mathcal{O}_Y & \xrightarrow{t} & \pi_*\mathcal{O}_Y & \to & R^1\pi_*\mathcal{O}_Y & \to & R^1\pi_*\mathcal{O}_Y \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \to & \mathcal{O}_X & \xrightarrow{t} & \mathcal{O}_X & \to & \mathcal{O}_X & \to & 0
\end{array}
\]

By assumption \(\pi_*\mathcal{O}_Y \approx \mathcal{O}_X\). From this it follows that the sequence

\[
0 \to \mathcal{O}_X \to \pi_*\mathcal{O}_Y \to R^1\pi_*\mathcal{O}_Y \to R^1\pi_*\mathcal{O}_Y
\]

is also exact. We claim that \(\mathcal{O}_X \approx \pi_*\mathcal{O}_Y\). Note that by condition 2) we have that the fibres of \(\pi\) are connected. Consider a section \(g \in \pi_*\mathcal{O}_Y\), or what amounts to the same, a function on \(Y\). As the \(\pi\)-fibres are compact and connected, this function is constant along the \(\pi\)-fibres. Hence \(g\) can be considered as a continuous function on \(X\), which is holomorphic on \(Y - \pi^{-1}(\xi) \to X - \xi\).

Because we assumed \(X\) to be weakly normal, \(g \in \mathcal{O}_X\). So we have indeed \(\mathcal{O}_X \to \pi_*\mathcal{O}_Y\). Because the map \(\pi\) is an isomorphism outside \(\xi\), the coherent sheaf \(R^1\pi_*\mathcal{O}_Y\) has as support a set contained in \(\xi\).

By the last exact sequence \(t\) acts injectively \(R^1\pi_*\mathcal{O}_Y\). As \(t\) vanishes on \(\xi (\subset X)\) we conclude that \(R^1\pi_*\mathcal{O}_Y = 0\).

\section*{(4.4.6) Corollary}

Let \(C\) be a weakly normal curve singularity and \(X\) the total space of a flat deformation \(X \to S\) of \(C\). Then \(X\) is weakly rational. This was stated as (2.5.7).

\section*{(4.4.7) Proposition}

With the notation of (4.4.2) we have:

\[X\text{ weakly normal} \implies H^1(Y) = 0\]

\textbf{Proof}: The embedded resolution map \(Y \to X\) clearly fulfils condition 1) of (4.4.5). It also fulfils condition 2), because \(X\) is normal, hence \(\mathcal{O}_X \to i_*\mathcal{O}_{X-\xi}\) where \(i: X - \xi \to X\) is the inclusion map. Because \(Y - \pi^{-1}(\xi) \to X - \xi\), it follows that \(\pi_*\mathcal{O}_Y \approx \mathcal{O}_X\). Now according to (4.4.5) we have \(R^1\pi_*\mathcal{O}_Y = 0\), in other words:

\[H^1(\mathcal{O}_Y) = 0\]

From the exponential sequence

\[
0 \to \mathbb{Z}_Y \to \mathcal{O}_Y \to \mathcal{O}_Y^* \to 0
\]
and the similar sequence for $x$ and the fact that $\mathcal{O}_x \cong \pi_\ast \mathcal{O}_y$ it
then follows that:

$$H^1(y,\mathbb{Z}_y) = 0$$

As $y$ is contractible to $Y$, we have $H^1(Y,\mathbb{Z}) = 0$.

(4.4.8) The kernel of $\beta$.

In the big diagram of (4.4.2) there was a map $\beta$

$$H^0(Y^{[0]},\mathbb{Z}) \xrightarrow{\beta} H^2(y,\mathbb{Z}) (= H^2(Y,\mathbb{Z}))$$

This map works as follows: Elements of the first groups can be
considered as divisors $\Sigma a_i Y_i$ with support on $Y$. (The $Y_i$ are the
irreducible components of $Y$.) Then one has:

$$\beta(\Sigma n_i Y_i) = \text{first chern class of the line bundle determined by}
\text{the divisor } \Sigma n_i Y_i$$

So the map $\beta$ factorizes over the map $\psi$ which associates to a
divisor its line bundle:

$$\ldots \rightarrow H^1(\mathcal{O}_y) \xrightarrow{\psi} H^1(\mathcal{O}_y) \rightarrow H^2(y,\mathbb{Z}) \rightarrow \ldots$$

We first study the map $\psi$. Note that if $H^1(\mathcal{O}_y) = 0$, then we have
$\ker \psi = \ker \beta$.

(4.4.9) Definition: Let $(X,p)$ be a germ of a normal analytic
space. The local class group is the group

$$\text{Cl}_p(X) = \{(\text{germs of) Weil divisors})/\{(\text{germs of) principal divisors})$$

(4.4.10) Proposition: With the notation as in (4.4.2) there
is a diagram with exact rows and
columns:
Here $F^{[0]} = Y^{[0]} \setminus X$ (so it contains the components $F$ and $E$ of (4.4.2)) and $X^{[0]} = \bigsqcup X_i$, where the $X_i$ are the irreducible components of $X$. The maps are the obvious ones.

**proof:** The surjection of $H^2(\mathcal{O}_Y)$ (or better of $R^1\pi_*\mathcal{O}_Y$) to the local class group is obvious: pulling back a Weil divisor on $X$ gives a Cartier divisor on $Y$ (hence a line bundle) that maps down to the original Weil divisor, as the map $\pi$ is a modification in codimension $\geq 2$ (c.f. [Mu]). The main point is to show that the kernel of the bottom row is not bigger, or what amounts to the same, that $\ker \psi \cong \ker \tau$. Let $A = \sum a_i Y_i$ be in the kernel of $\psi$. We may assume that $a_i \geq 0$. Hence there is a function $g \in H^0(\mathcal{O}_Y)$ with $(g) = A$. By the normality of $X$ we have $\mathcal{O}_X = \pi_* \mathcal{O}_Y$, so $g$ can be considered as holomorphic on $X$, having of course as divisor on $X$ just (the image) of that part of $A$ that does not involve $F^{[0]}$. This gives the map $\ker \psi \twoheadrightarrow \ker \tau$. This map is injective because if the divisor of $g$ (on $X$) would be zero, $g$ would be a unit, hence $A = 0$. Surjectivity follows by pulling back functions.

The use of (4.4.10) is that we get rid of the global object $Y$. In (4.4.2) we used a base change to arrive at a semi-stable family. The kernel of the map $\gamma$ is essentially independent of this base change:

**Lemma:** Consider a normal space $X$ and a reduced principal divisor $X \subset X$. Let $X^{[0]} = \bigsqcup X_i$, where the $X_i$ are the irreducible components of $X$. Let $\tilde{X}$ be
obtained from \( X \) by taking a \( d \)-fold cyclic cover branched along \( X \).

Let \( \tilde{\iota} : H^0(X) \to Cl_p(X) \), \( \tilde{\tau} : H^0(X) \to Cl_p(X) \) be the obvious maps. Then \( \ker(\tilde{\iota}) \otimes \mathbb{Q} = \ker(\tilde{\tau}) \otimes \mathbb{Q} \).

**proof:** Excercise.

We summarize the above results in one theorem:

\[(4.4.12) \quad \text{Theorem:} \quad \text{Let } X \xrightarrow{f} S \text{ be (a contractible Stein representative of) a smoothing of a reduced germ } (X,p). \text{ Let } X_t = f^{-1}(t), \ t \neq 0, \text{ be its Milnor fibre.} \]

Let \( X^{[1]} = \bigsqcup X_i \), where the \( X_i \) are the irreducible components.

Let \( \tilde{\tau} : H^0(X^{[1]}) \to Cl_p(X) \) be the obvious map.

Then one has:

1) \( b_1(X_t) \geq \text{rank } (\ker \tilde{\tau}) - 1. \)

2) If \( X \) is weakly normal, then one has equality:

\[ b_1(X_t) = \text{rank } (\ker \tilde{\tau}) - 1. \]

In particular, when \( X \) is a **hypersurface**, rank \( \ker \tilde{\tau} \) is equal to the number of irreducible components of \( X \).

\[(4.4.13) \quad \text{Remark:} \quad \text{For a hypersurface germ } X \text{ in } \mathbb{C}^3 \text{ with a complete intersection as singular locus and transversal type } A_1 \text{ it is known that the first Betti number } b_1(X_t) \text{ is zero or one (see [Sie 2], [Str]). So the number of irreducible components of } X \text{ is one or two. To put it in another way, the singular locus of a weakly normal hypersurface in } \mathbb{C}^3 \text{ which has more than three components is never a complete intersection.} \]

\[(4.4.14) \quad \text{Question:} \quad \text{J. Stevens has shown that all degenerate cusps are smoothable (private communication). What is the first Betti number for these smoothings? Is the first Betti number an invariant of } X? \text{ (Probably not, but at this moment I do not have computed any non-trivial example.)} \]