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Fourier Transforms
Wavelets
Theory and Applications

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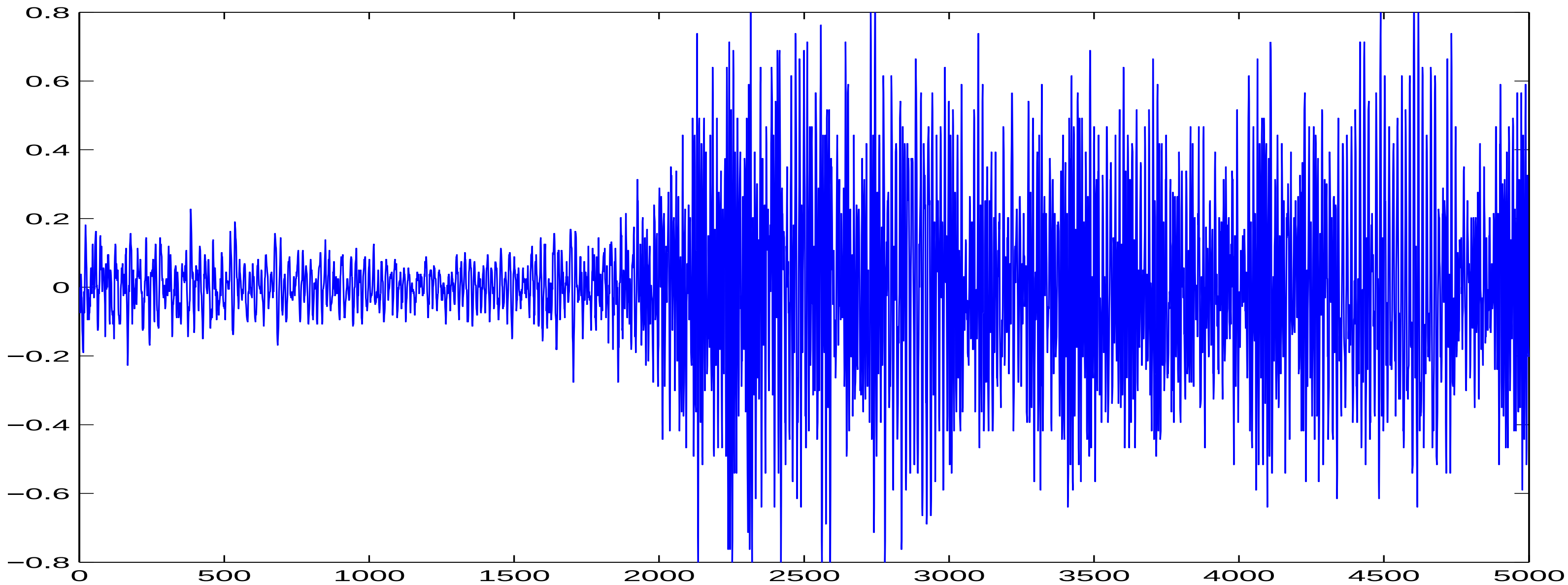
We are interested in real– or complex valued functions f.

f can be defined on $\mathbb{R}$ or on an interval $[a, b]$.

Example. $f(t)$ is the difference $\rho(t) - \rho_0$ of the air pressure $\rho(t)$ at time t at some location (your ear) and the average air pressure ρ_0: f is an acoustic sound.



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$f(t)$ is the voltage difference at time t at the speaker output of an acoustic amplifier.

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Example. Consider a building (bridge) that swings in the wind. $f(t)$ is the distance from some point of the building to its position at rest.

Earthquake, heartbeat, . . .

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Complicated functions, but there is some corrolation in the behaviour of f at $[t, t + \Delta t]$ and at $[t', t' + \Delta t]$ $(t' > t)$.

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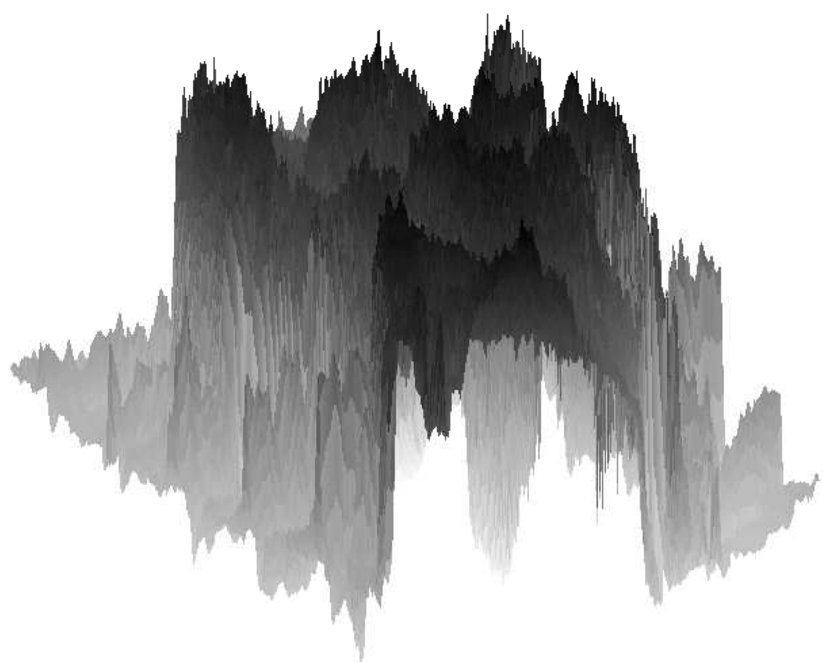

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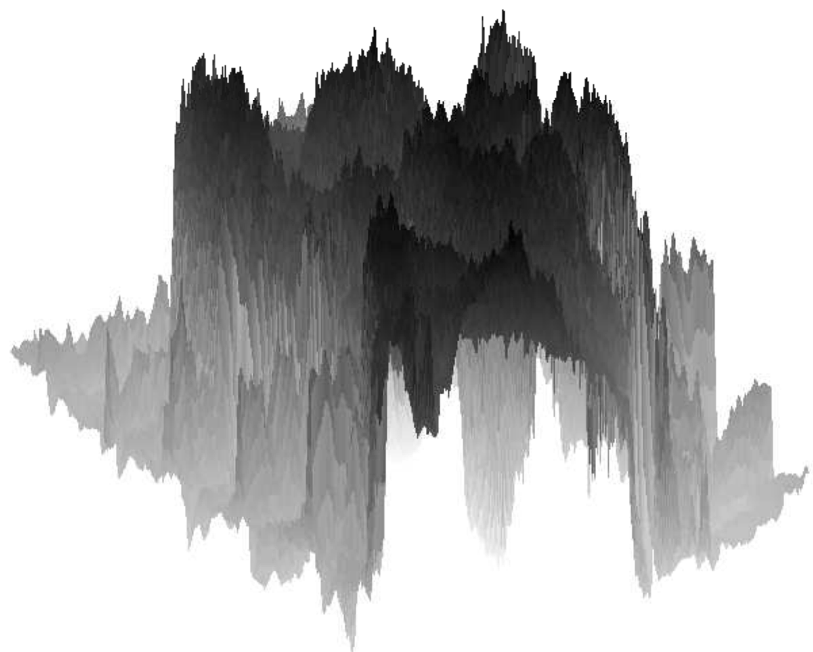

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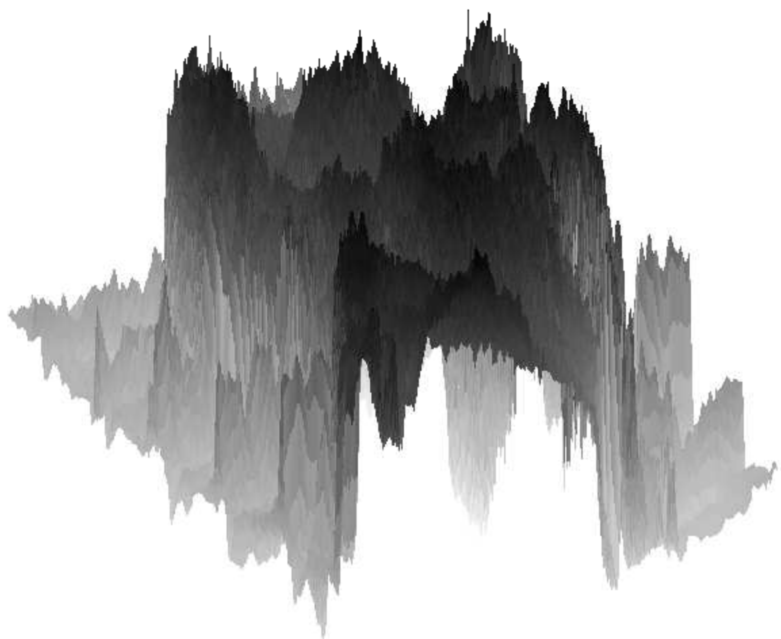

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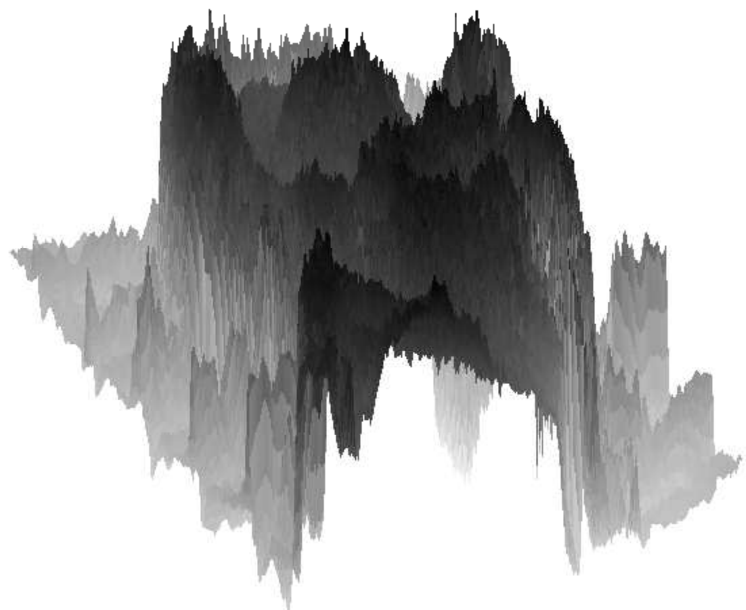

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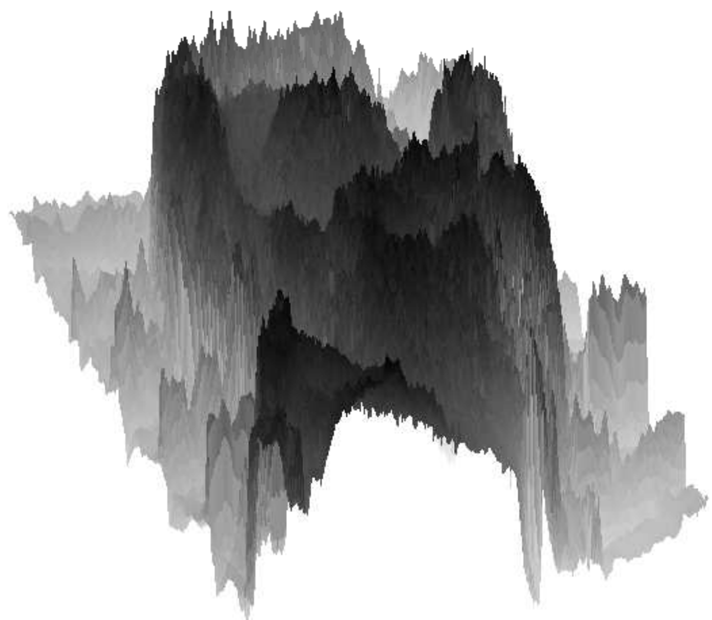

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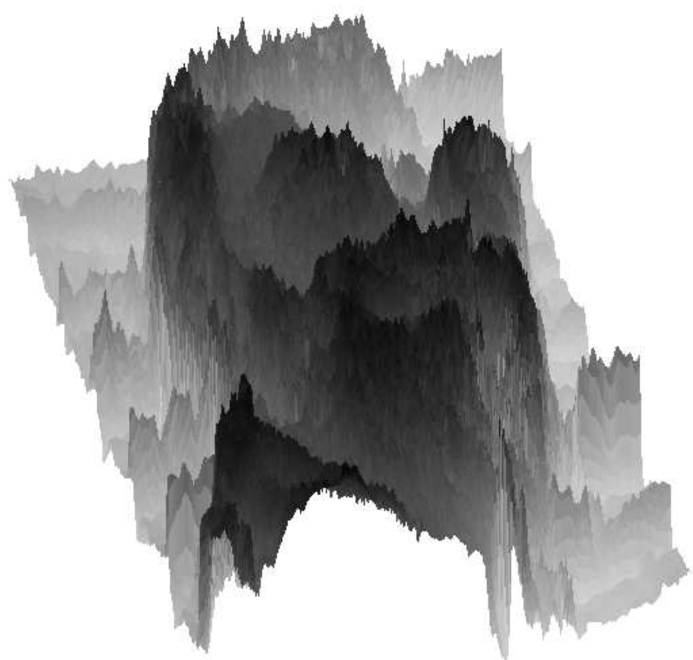

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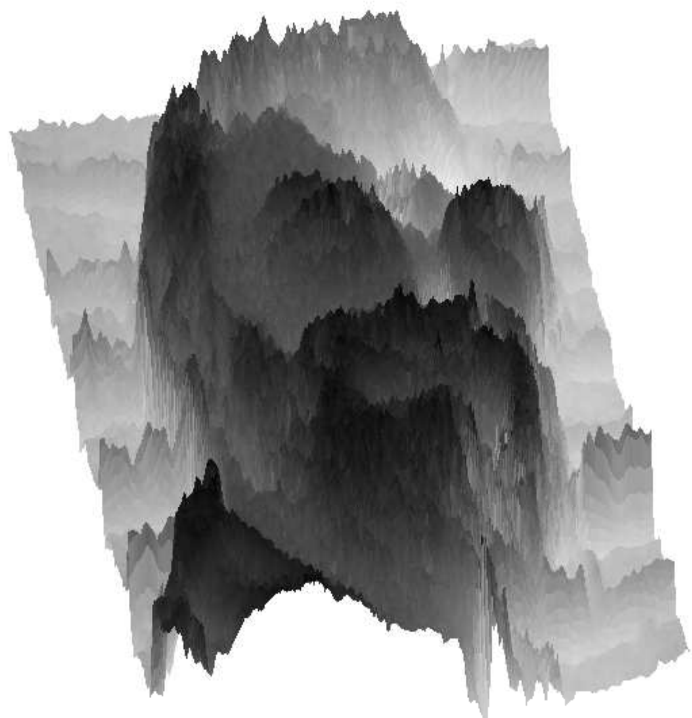

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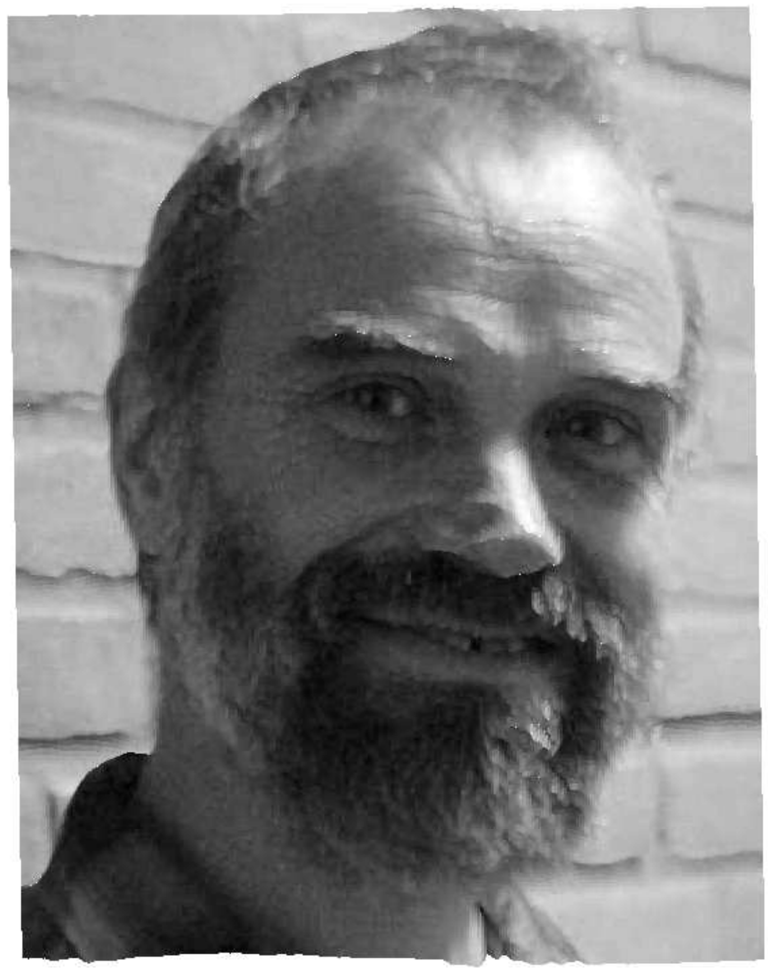

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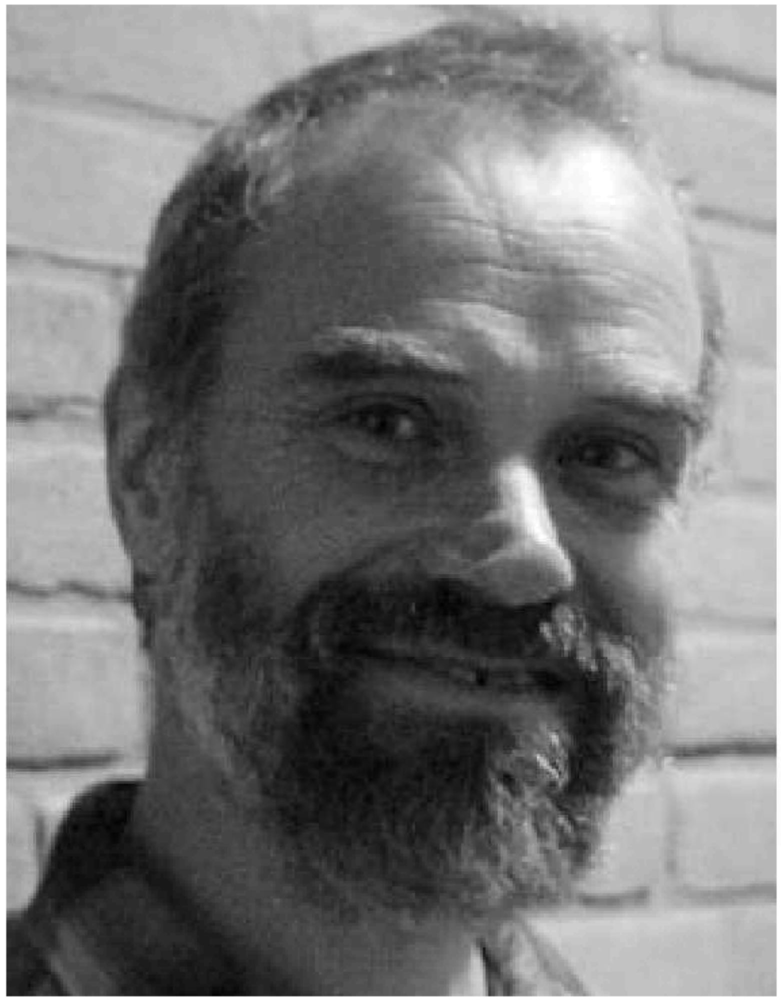

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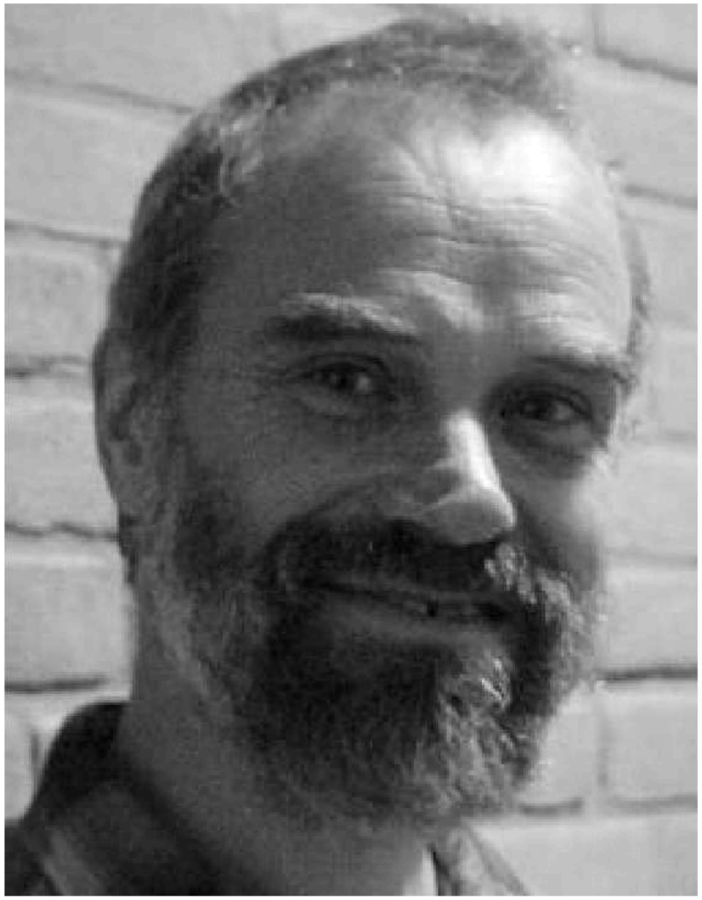
Example. Pictures in a gray scale.

$f(x, y)$ is the gray-value of the picture at position (x, y).



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Example. Pictures in a gray scale.

$f(x, y)$ is the gray-value of the picture at position (x, y).

In practice, $[a, b] \times [c, d]$ is discretized into pixels. With $\Delta x = (b-a)/n$, $\Delta y = (d - c)/m$, $I_{i,j} = [a + i\Delta x, c + j\Delta y]$ is the (i, j)th **pixel**. f has a constant color value at each pixel: so, actually f is a step function (piece-wise constant). The pixels have size $\Delta x \times \Delta y$. Smaller pixels (higher n and m) imply *higher resolution*. The function values are also discretized. They may take integer values betwee 0 (black) and 255 (white). For mathematical analysis, it is often more convenient to assume function values in the whole of $\mathbb{R}$ and to assume some smoothness.

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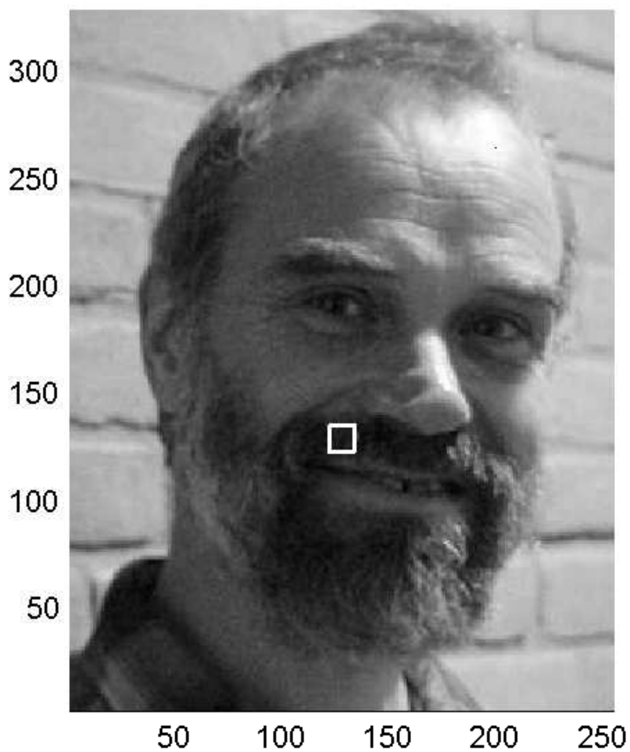

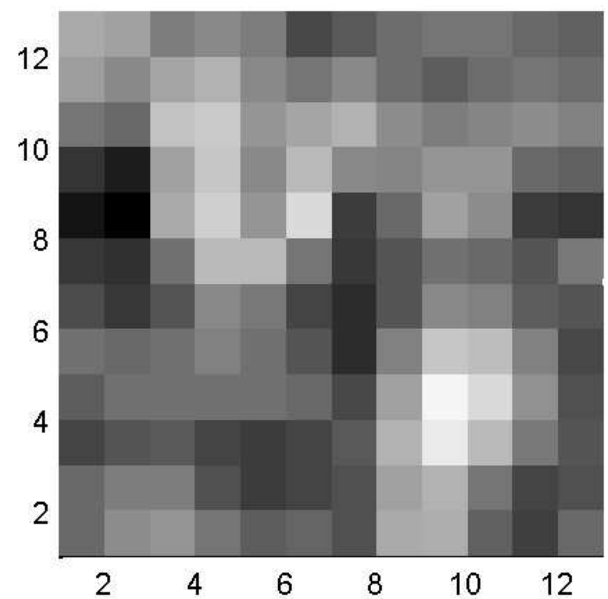

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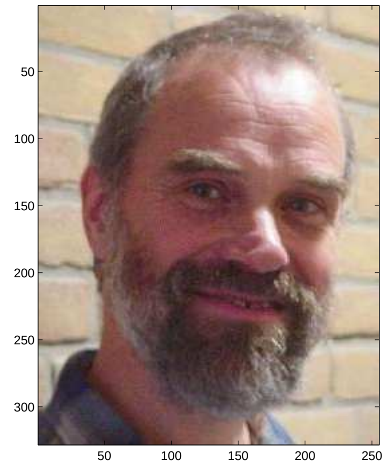
Example. Color pictures:



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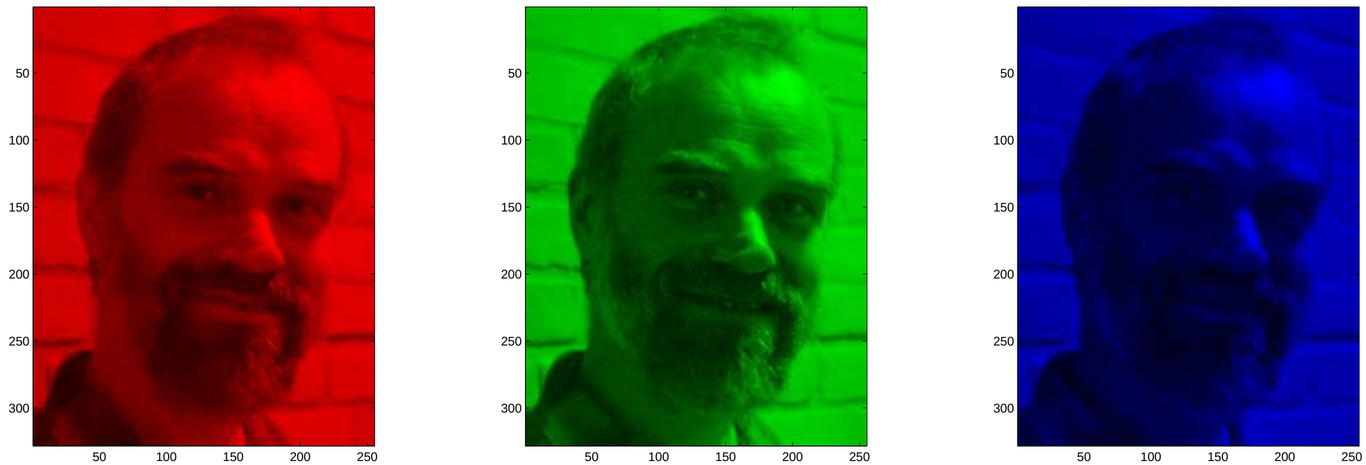

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Example. Color pictures:

Colors are a combination

of monochromatic colors **red**, **green** and **blue** (RGB).

$f(x, y) = f_R(x, b)$ is the red-value of the picture at position (x, y).

The picure can be described by

$$(x, y) \rightsquigarrow \vec{f}(x, y) = (f_R(x, y), f_G(x, y), f_B(x, y))^\top$$

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f can be defined on $\mathbb{R}^d$ or on a (nice) subset I of $\mathbb{R}^d$.

Example. $d = 3$, Movies.

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Example. $d = 3$, Computerized Tomography (PET scan, MRI). Voxels

Remark. If $I \subset \mathbb{R}^d$ and $f : I \to \mathbb{C}^\ell$ then

$$f = (f_1, \ldots, f_\ell)^\top$$

and **we can study the functions $f_i : I \to \mathbb{C}$ separately**.

Remark. If $I \subset \mathbb{R}^d$ and $f : I \to \mathbb{C}^\ell$ then

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However, there is no convenient way to restrict the analysis further, to functions defined on (a subset of) $\mathbb{R}$:
e.g., $x \rightsquigarrow f_1(x, x_2, \ldots, x_d)$ depends on $(x_2, \ldots, x_d)$!

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$$f : \mathbb{R}^2 \to \mathbb{C}.$$

Remark. If f is defined on a subset I of $\mathbb{R}^d$, then f can be extended to a function defined on $\mathbb{R}^d$, for instance, by defining $f(x) = 0$ for $x \notin I$ (or by periodicity).

Purpose

We want to analyse functions, reveal hidden structures.

Applications.

- De-noising, de-blurring
- Compression

Ex. For some $k \in \mathbb{Z}$ and $T > 0$, $f(t) = \sin(2\pi kt/T)$ for $t \in [0, 10]$.

Store $f(j\Delta t)$ for $j = 0, 1, \ldots, 10^5$ with $\Delta t = 10^{-4}$ (as on a CD).

Alternative, store k and T.

Compression also important to facilitate analysis.

- . . .

Strategy

Find a suitable basis to represent the class of functions that are of interest.

(ϕ_k) (infinite set of) 'basisfunctions'.
Then $f = \sum_k \gamma_k \phi_k$ in some sense.

Find (ϕ_k) such that

1) $f \approx \sum_{k \in E} \gamma_k \phi_k$, with E finite (small) subset of indices k.
2) E is 'small' and can 'easily' be detected.
3) $\sum_{k \in E} \gamma_k \phi_k(t)$ can efficiently be computed.

1) Approximation, 2) Extraction, 3) Computation

Example. $f \in C([-1,1])$, $\phi_k(t) = t^k \quad (k \in \mathbb{N}_0, |t| \leq 1)$

Approximation. Weierstrass. $\forall\, \varepsilon > 0$

$\exists$ a polynomial p st $\forall\, t \in [-1,1]$, $|f(t) - p(t)| \leq \varepsilon$.

Extraction. Taylor. If f is sufficiently smooth:

$$p(t) = \sum_{j<k} \frac{t^j}{j!} f^{(j)}(0), \qquad f(t) - p(t) = \frac{t^k}{k!} f^{(k)}(\xi).$$

Evaluation. Horner. If $p(t) = \gamma_0 + \gamma_1 t + \ldots + \gamma_k t^k$ then

$$p(t) = \gamma_0 + (\ldots(\gamma_{k-2} + (\gamma_{k-1} + \gamma_k t)t)t \ldots)t :$$

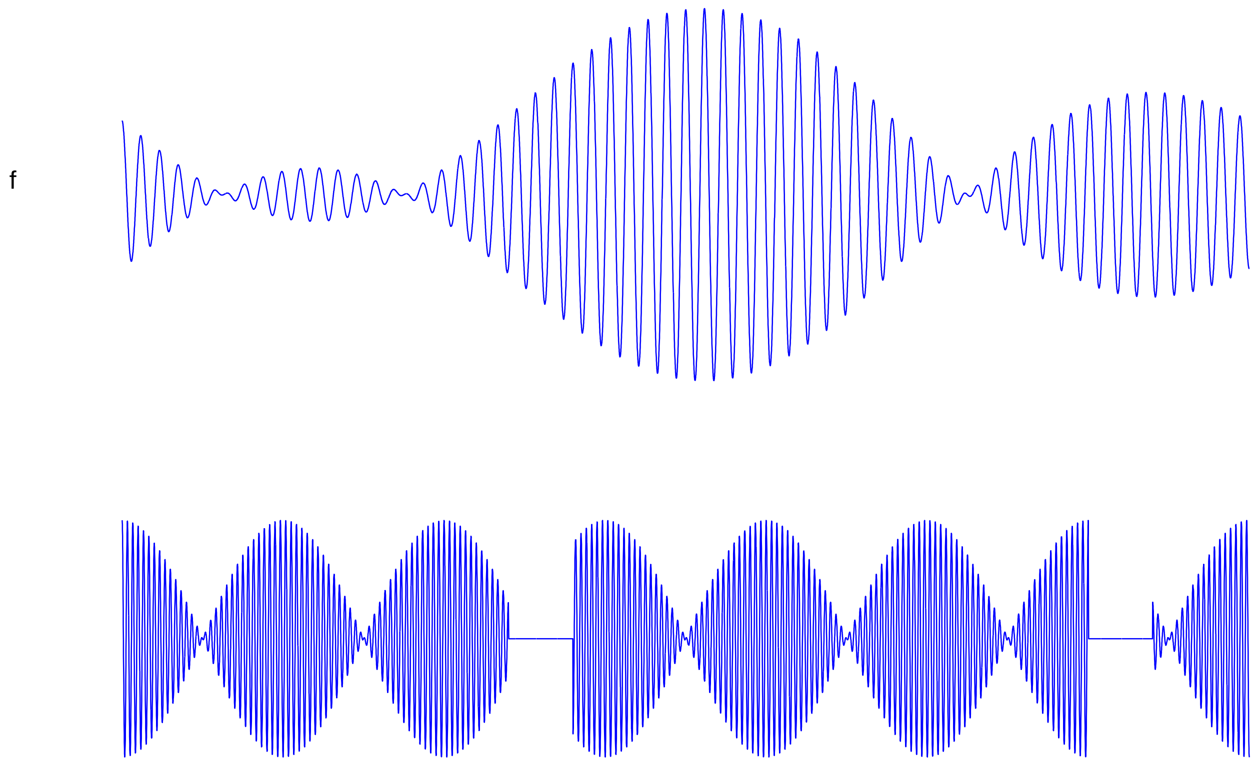
$s_0 = \gamma_k$, $s_j = \gamma_{k-j} + s_{j-1} t$ for $j = 1, \ldots, k$. Then $p(t) = s_k$.

Polynomials well suited for computing (but not t^k),
less suitable for analysis.

Example. $f \in C([0,1])$, $\phi_k(t) \equiv \cos(2\pi k t)$.

Reveals periodic structures in f:

test against ϕ_k ($k \in \mathbb{N}_0$), i.e., compute $\int f(t)\phi_k(t)\,\mathrm{d}t$



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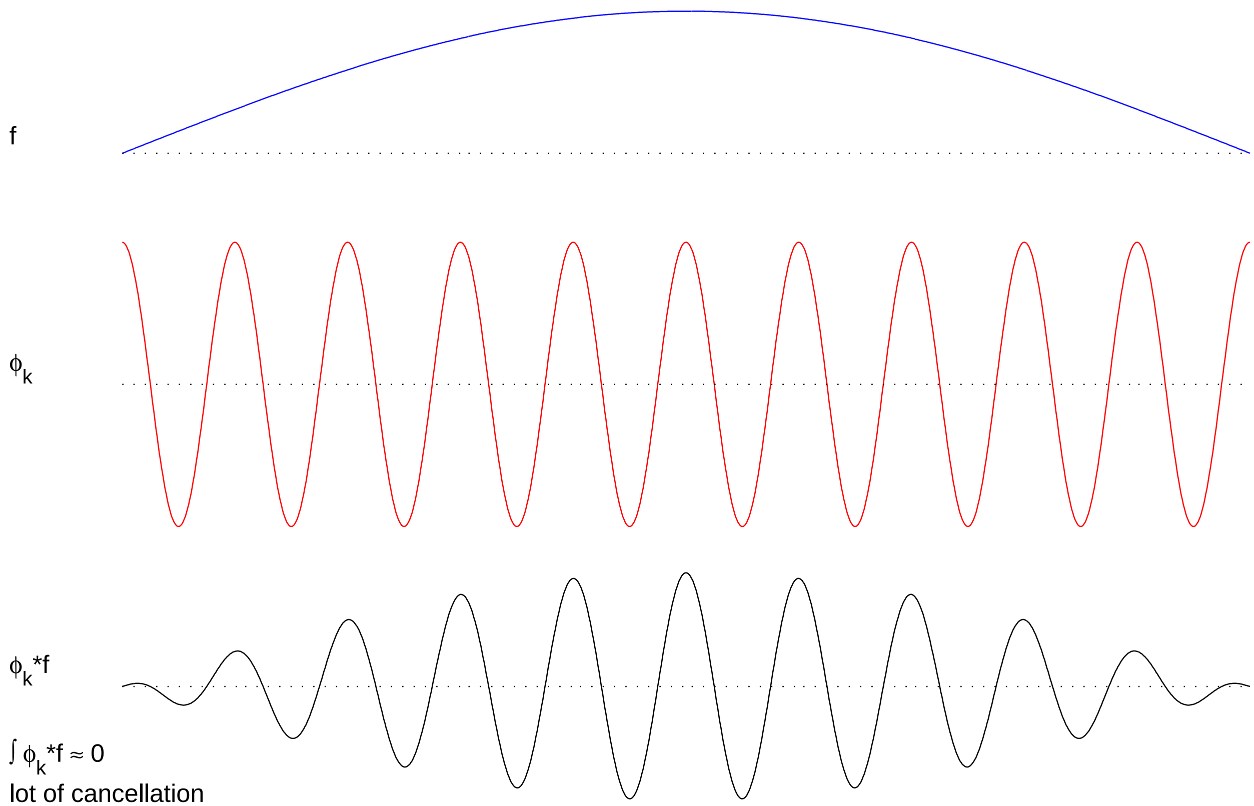

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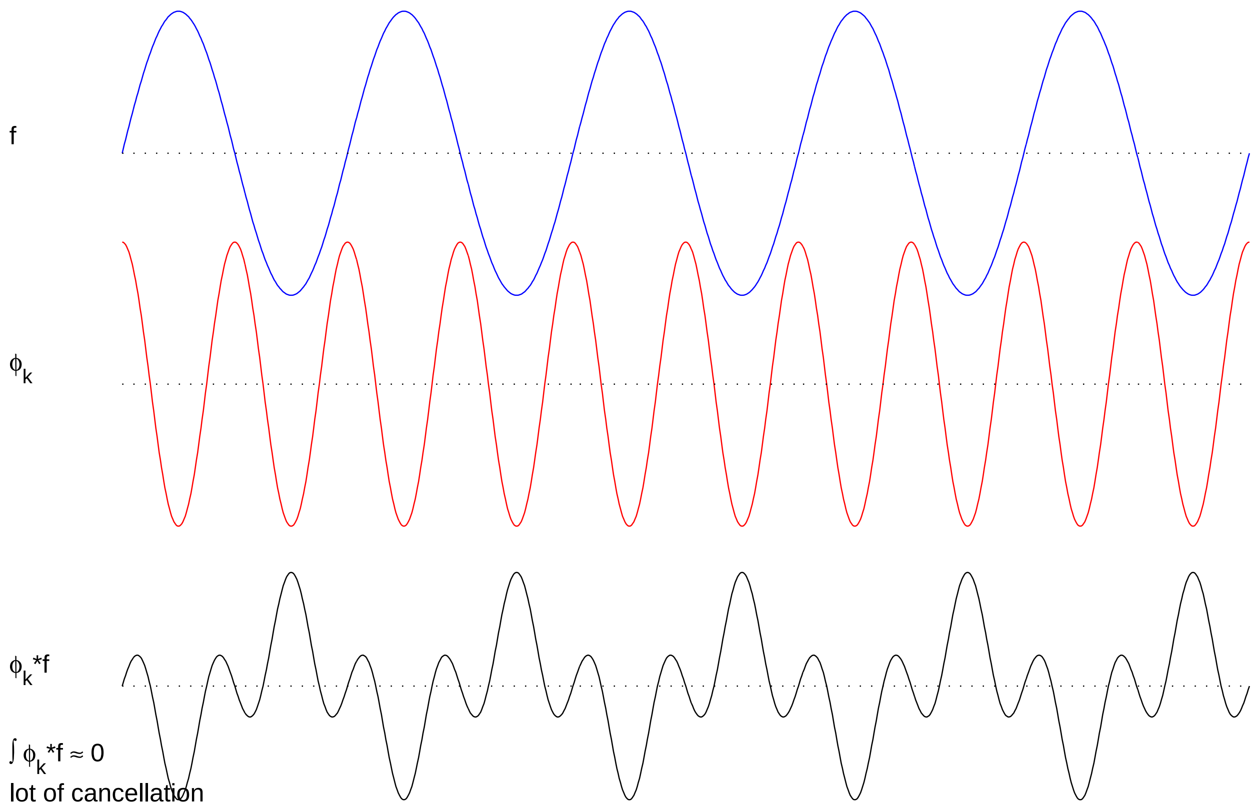

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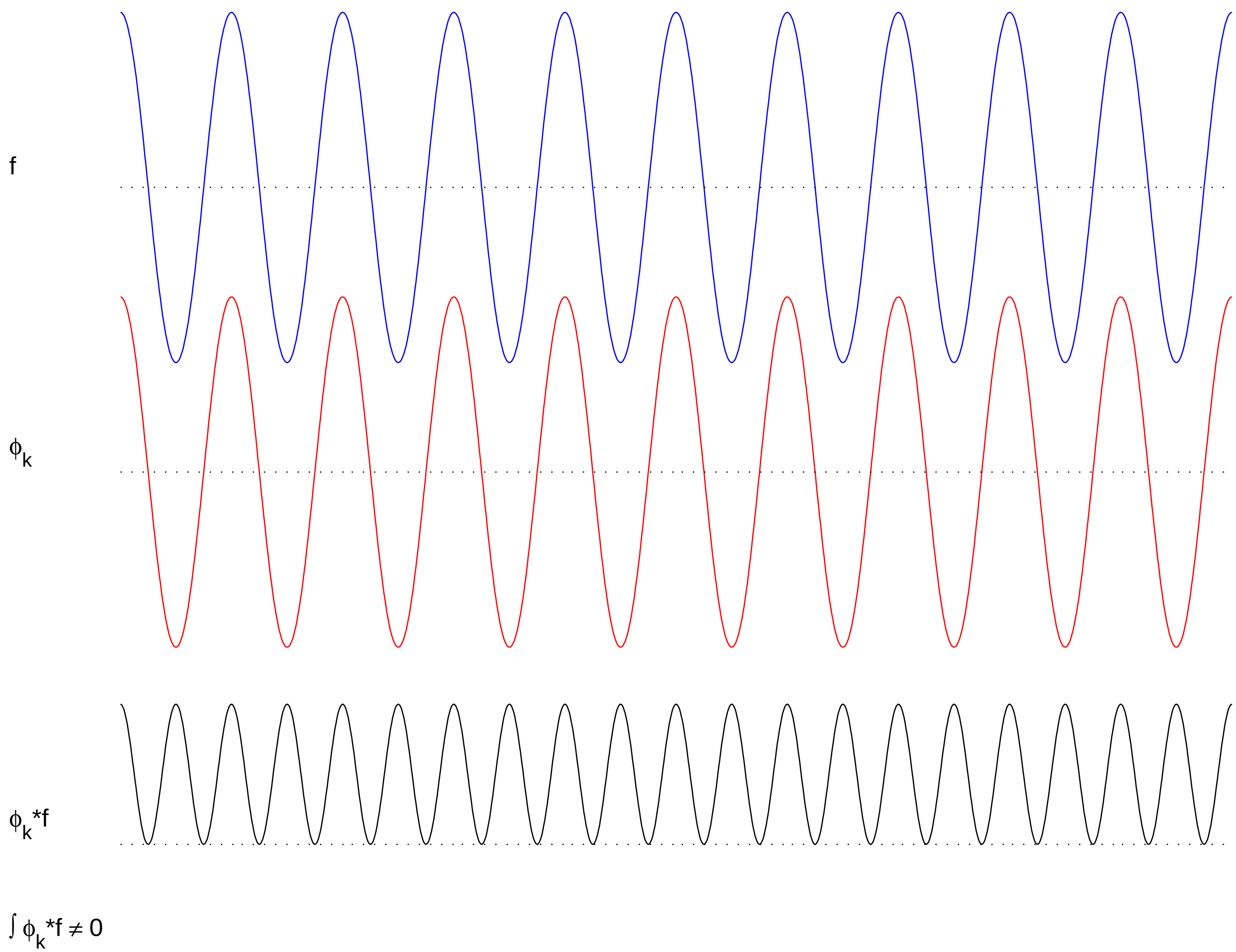

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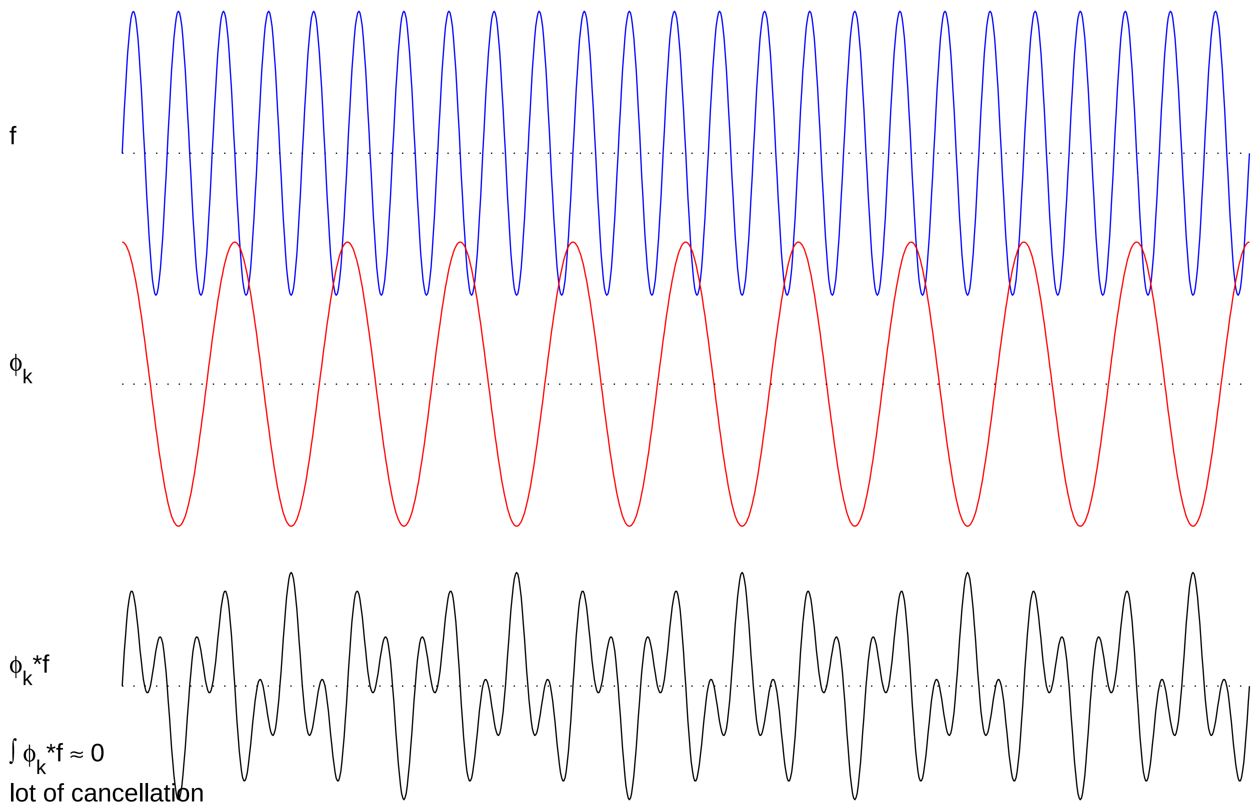

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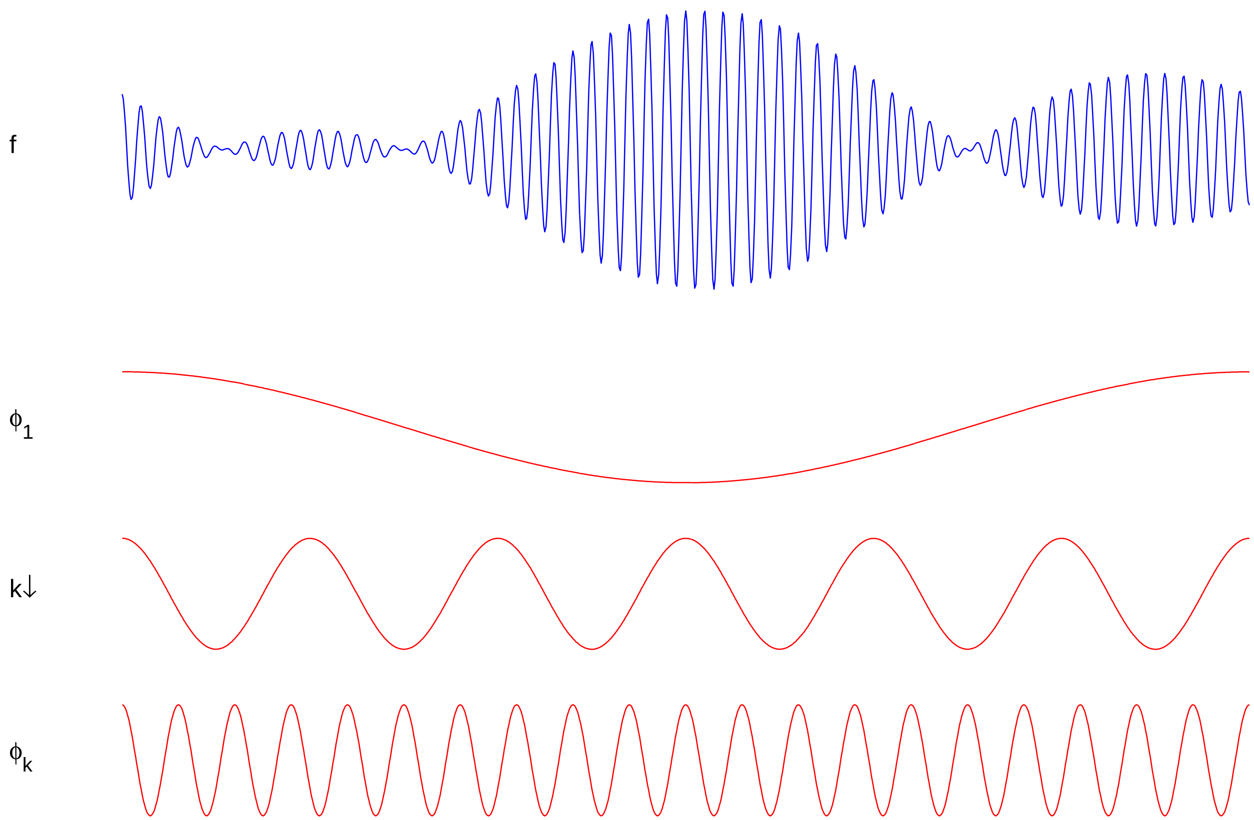


Applications Fourier analysis.

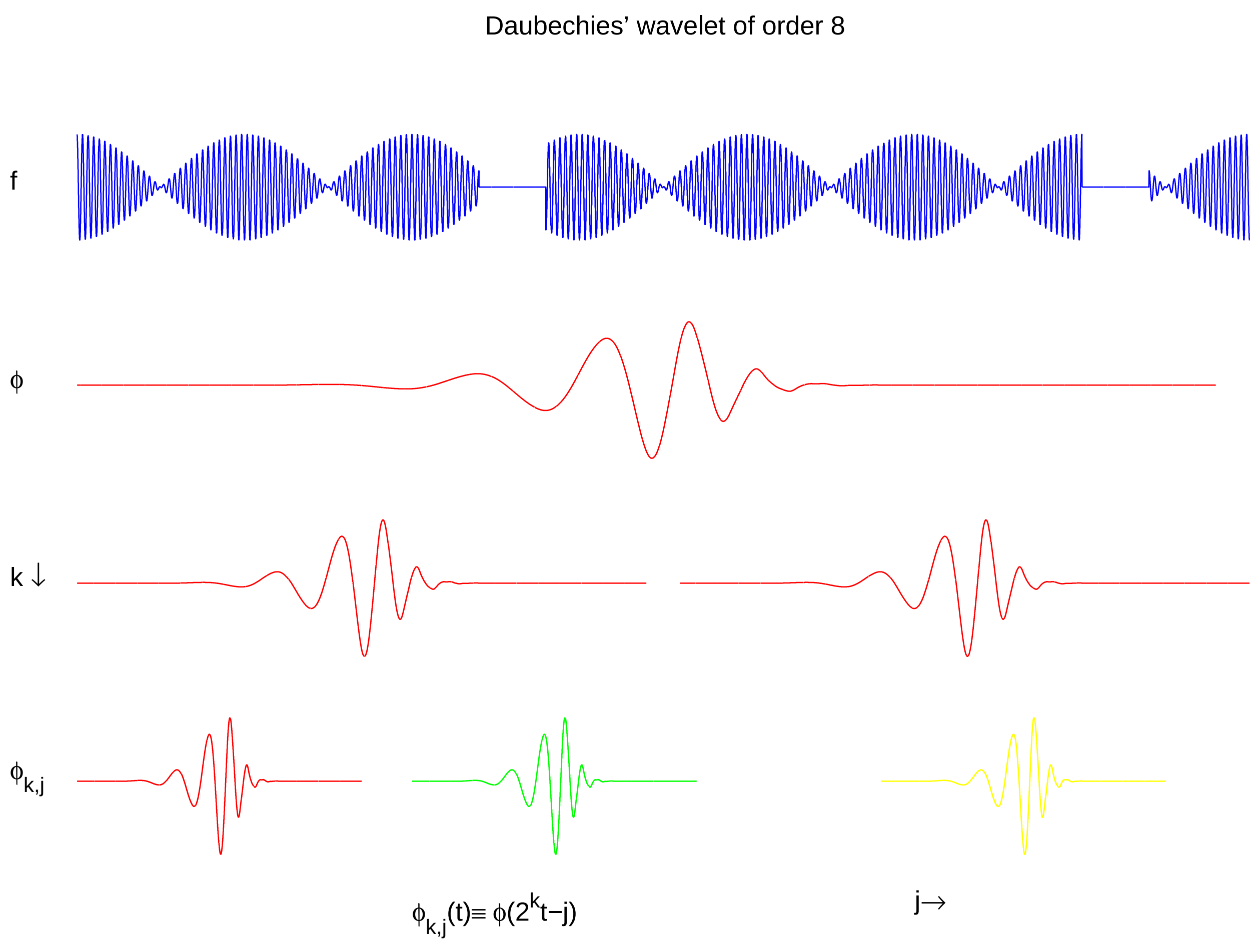
- Audio technique (equalizers, amplyfiers, tuner, CDs)
- MP3 and other audio compression techniques
- biology, ear, eye, . . .
- radar, echo location, CT, MRI, . . .
- Cristallography, Geophysics, . . .
- denoising, deblurring of images, JPEG compression, MJPEG
- Theory (partial) differential equations

⋮

Example. $f \in C([0,1])$, $\phi_{k,j}(t) = \psi(2^k t - j)$.

Reveals periodic structures in f **and localized changes**:

compute $\int f(t)\phi_{k,j}(t)\,\mathrm{d}t$ for $k, j \in E \subset \mathbb{Z}$



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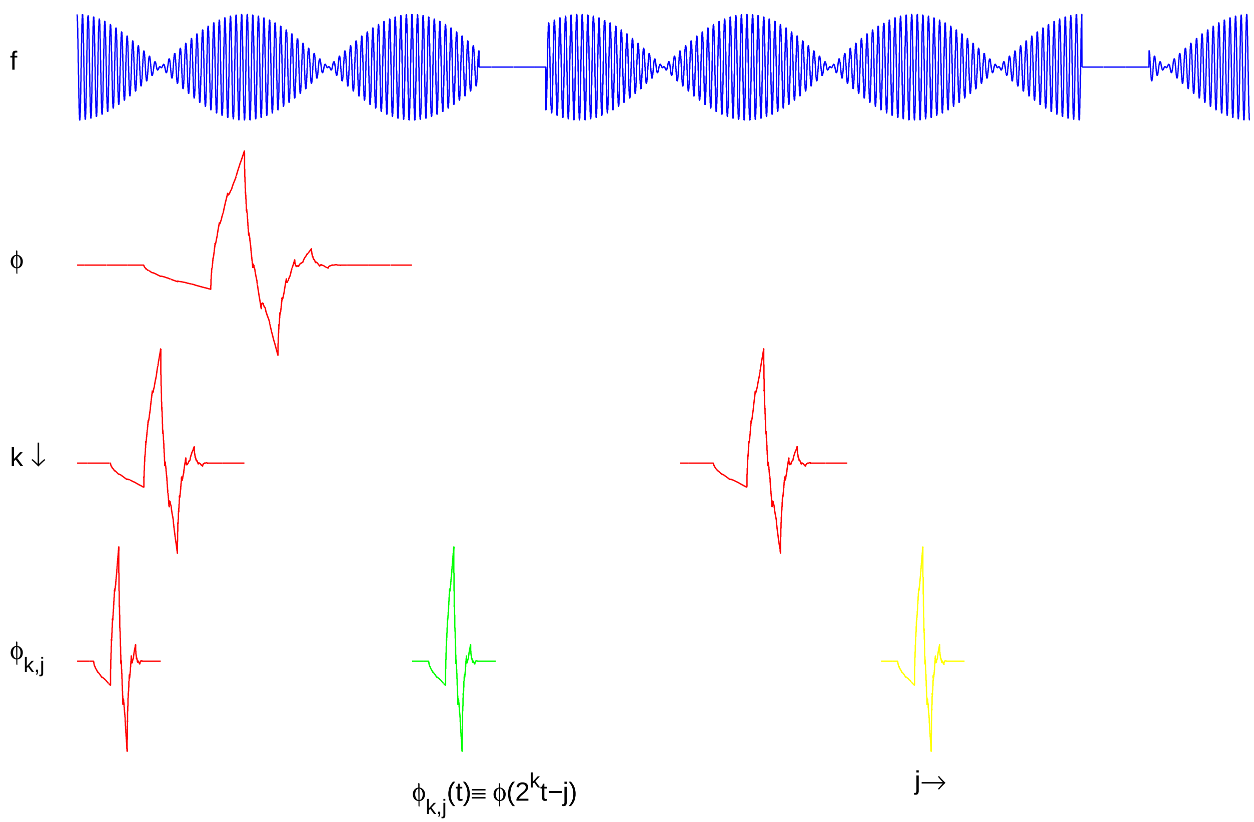
Daubechies' wavelet of order 2



Application wavelet analysis.

As Fourier, tends to be more practical

∘ Storing and detection of fingerprints (to help police investigations)

∘ Computational techniques for partial differential equations

⋮

Example. $\phi_k(t) = t^k$ polynomials.

Example. $\phi_k(t) \equiv \cos(2\pi kt)$
Harmonic oscillations, Fourier modes

Example. **Wavelets**

Example. Bessel functions, . . .

Example. Splines (smooth, piece-wise polynomials)

Example. Finite element basis functions

⋮

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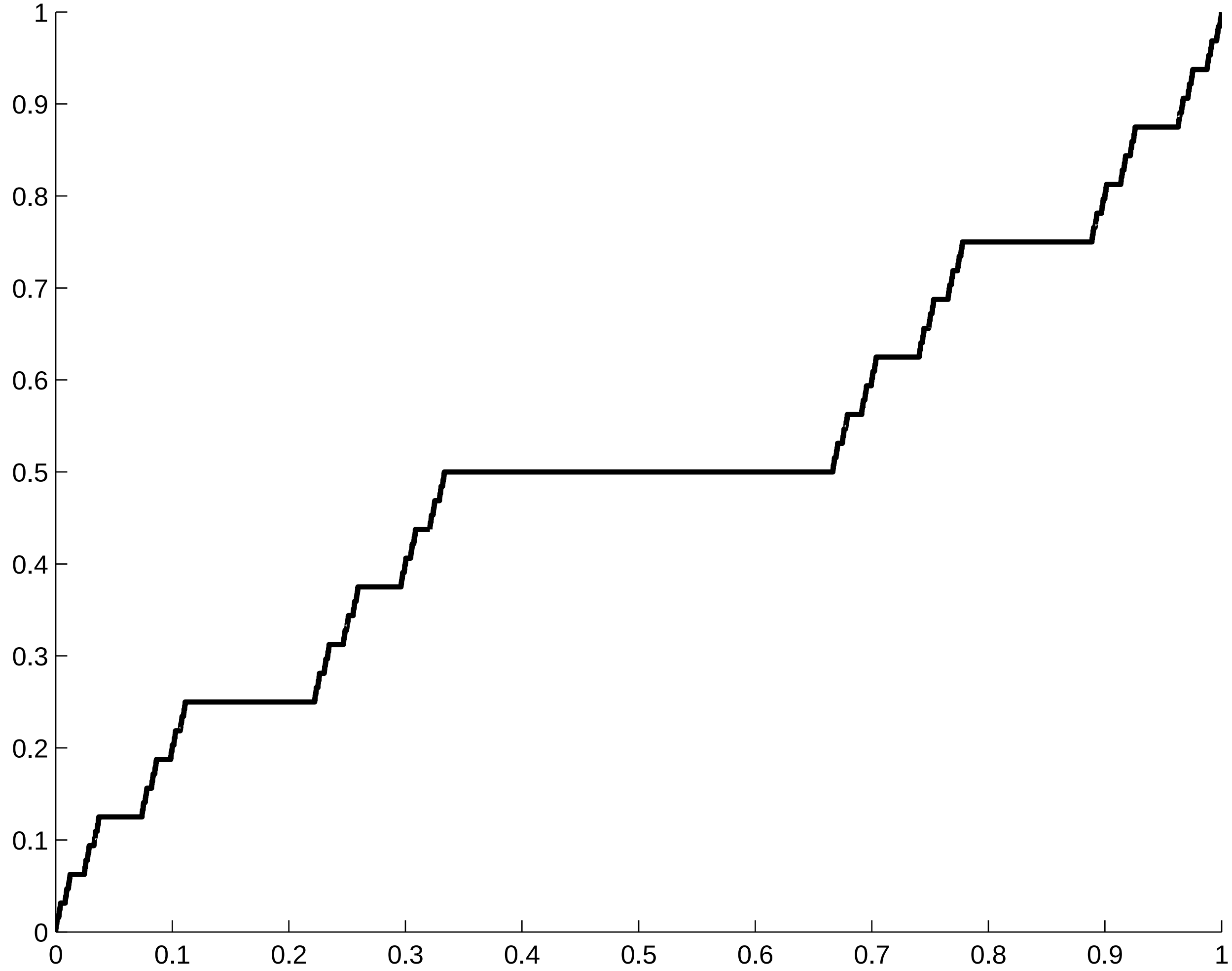
Preliminaries



Program

- Norms and inner products
- Convergence
- Almost everywhere
- Function spaces
- Point-wise convergence
- Function values
- Derivatives

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Norms

Let $\mathcal{V}$ be a (real or) complex vector space.

A map $\|\cdot\| : \mathcal{V} \to [0, \infty)$ is a **norm** if

1) $\|f\| = 0$ iff $f = 0$ $\qquad (f \in \mathcal{V})$

2) $\|\lambda f\| = |\lambda|\,\|f\|$ $\qquad (f \in \mathcal{V}, \lambda \in \mathbb{C})$

3) $\|f + g\| \leq \|f\| + \|g\|$ $\qquad (f, g \in \mathcal{V}, \lambda \in \mathbb{C})$

Examples. $\mathcal{V} = C([a, b])$

$$\|f\|_\infty = \max\{|f(t)| \mid t \in [a, b]\}$$

$$\|f\|_1 = \int_a^b |f(t)|\, \mathrm{d}t$$

$$\|f\|_2 = \sqrt{\int_a^b |f(t)|^2\, \mathrm{d}t}$$

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Exercise.

Inner products

Let $\mathcal{V}$ be a (real or) complex vector space.

A map $(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ is an **inner product** if

1) $(f, f) \geq 0$, $\quad (f, f) = 0$ iff $f = 0 \quad (f \in \mathcal{V})$

2) $(f, g) = \overline{(g, f)} \quad (f, g \in \mathcal{V})$

3) $f \rightsquigarrow (f, g)$ is linear $\quad (g \in \mathcal{V})$

Example. $\mathcal{V} = C([a, b])$

$(f, g) = \int_a^b f(t)\,\overline{g(t)}\,\mathrm{d}t$

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Theorem. If $(\cdot,\cdot)$ is an inner product on $\mathcal{V}$, then $f \rightsquigarrow \sqrt{(f, f)}$ defines a norm on $\mathcal{V}$.

Example. $\|f\|_2 = \sqrt{(f, f)}$ on $\mathcal{V} = C([a, b])$.

$\mathcal{V}$ is an inner product space with associated norm $\|\cdot\|_2$.

Pythagoras. If $f, g \in \mathcal{V}$ such that $f \perp g$, i.e. $(f, g) = 0$, then
$$\|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2.$$

Proof.

$$\begin{aligned}\|f+g\|_2^2 = (f+g, f+g) &= (f,f) + (f,g) + (g,f) + (g,g) \\ &= \|f\|_2^2 + (f,g) + \overline{(f,g)} + \|g\|_2^2 \\ &= \|f\|_2^2 + 2\text{Re}(f,g) + \|g\|_2^2\end{aligned}$$

If $(f, g) = 0$ the claim follows.

$\mathcal{V}$ is an inner product space with associated norm $\|\cdot\|_2$.

Pythagoras. If $f, g \in \mathcal{V}$ such that $f \perp g$, i.e. $(f, g) = 0$, then
$$\|f + g\|_2^2 = \|f\|_2^2 + \|g\|_2^2.$$

Cauchy–Schwartz. $(f, g) \leq \|f\|_2 \, \|g\|_2 \qquad (f, g \in \mathcal{V})$.
$(f, g) = \|f\|_2 \, \|g\|_2$ iff f is a scalar multiple of g.

Proof. Assume $\|g\|_2 = 1$. Note $f - (f, g)g \perp g$.

Hence, (Pythagoras) $\|f\|_2^2 = \|f - (f, g)g\|_2^2 + \|(f, g)g\|_2^2 \geq |(f, g)|^2$.

Equality only if $\|f - (f, g)g\|_2 = 0$.

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$(f, g) = \|f\|_2 \, \|g\|_2$ iff f is a scalar multiple of g.

Example. $\mathcal{V} = C([a, b])$

$$\|f\|_1 \leq \sqrt{b - a} \, \|f\|_2 \leq (b - a) \, \|f\|_\infty \qquad (f \in C([a, b]))$$

$\mathcal{V}$ is an inner product space with associated norm $\|\cdot\|_2$.

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Example. $\mathcal{V} = C([a, b])$

$$\|f\|_1 \leq \sqrt{b-a}\, \|f\|_2 \leq (b-a)\, \|f\|_\infty \qquad (f \in C([a, b]))$$

Exercise. $\mathcal{V} = C([0, 1])$

Is there a $\kappa > 0$ such that $\|f\|_\infty \leq \kappa \|f\|_2$ for all $f \in C([0, 1]]$?

Is there a $\kappa > 0$ such that $\|f\|_2 \leq \kappa \|f\|_1$ for all $f \in C([0, 1]]$?

$\mathcal{V}$ is an inner product space with associated norm $\|\cdot\|_2$.

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Example. $\mathcal{V} = C([a, b])$

$$\|f\|_1 \leq \sqrt{b-a} \, \|f\|_2 \leq (b-a) \, \|f\|_\infty \qquad (f \in C([a, b]))$$

Example.

$$\|f\|_\infty \leq |f(a)| + \sqrt{b-a} \, \|f'\|_2 \qquad (f \in C^{(1)}([a, b]))$$

Program

- Norms and inner products
- Convergence
- Almost everywhere
- Function spaces
- Point-wise convergence
- Function values
- Derivatives

$\mathcal{V}$ is a space with norm $\|\cdot\|$.

A sequence (f_n) in $\mathcal{V}$ **converges** to an $f \in \mathcal{V}$ if

$$\lim_{n\to\infty} \|f_n - f\| = 0$$

Exercise. $\mathcal{V} = C([0,1])$, $f_n(t) = t^n \qquad (n \in \mathbb{N}, t \in [0,1])$.

Does (f_n) converge with respect to $\|\cdot\|_1$?

Does (f_n) converge with respect to $\|\cdot\|_\infty$?

Exercise. $\mathcal{V} = C([0,2])$, $f_n(t) = \min(t^n, 1)$.

Does (f_n) converge with respect to $\|\cdot\|_1$?

(f_n) is a **Cauchy sequence** with respect to a norm $\|\cdot\|$

if $$\|f_n - f_m\| \to 0 \quad \text{if } n > m,\ m \to \infty$$

Exercise. $\mathcal{V} = C([0,2])$, $f_n(t) = \min(t^n, 1)$.

Is (f_n) a Cauchy sequence wrt $\|\cdot\|_1$?

Is (f_n) a Cauchy sequence wrt $\|\cdot\|_2$?

Is (f_n) a Cauchy sequence wrt $\|\cdot\|_\infty$?

(f_n) is a **Cauchy sequence** with respect to a norm $\| \cdot \|$

if $$\|f_n - f_m\| \to 0 \quad \text{if } n > m,\, m \to \infty$$

A space $\mathcal{V}$ with norm $\| \cdot \|$ is **complete** if each Cauchy sequence (f_n) in $\mathcal{V}$ converges to an $f \in \mathcal{V}$.

(f_n) is a **Cauchy sequence** with respect to a norm $\|\cdot\|$

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A space $\mathcal{V}$ with norm $\|\cdot\|$ is **complete** if each Cauchy sequence (f_n) in $\mathcal{V}$ converges to an $f \in \mathcal{V}$.

Exercise. $\mathcal{V} = C([0,2])$.

Is $\mathcal{V}$ complete wrt $\|\cdot\|_1$?

Is $\mathcal{V}$ complete wrt $\|\cdot\|_2$?

Is $\mathcal{V}$ complete wrt $\|\cdot\|_\infty$?

(f_n) is a **Cauchy sequence** with respect to a norm $\| \cdot \|$ if

$$\|f_n - f_m\| \to 0 \quad \text{if } n > m,\ m \to \infty$$

A space $\mathcal{V}$ with norm $\| \cdot \|$ is **complete** if each Cauchy sequence (f_n) in $\mathcal{V}$ converges to an $f \in \mathcal{V}$.

Exercise. $\mathcal{V} = C([0, 2])$.

Is $\mathcal{V}$ complete wrt $\| \cdot \|_1$?

Is $\mathcal{V}$ complete wrt $\| \cdot \|_2$?

Is $\mathcal{V}$ complete wrt $\| \cdot \|_\infty$?

Can we complete $C([0, 2])$ wrt the $\| \cdot \|_2$?
What kind of objects are contained in the completion?

Program

- Norms and inner products
- Convergence
- Almost everywhere
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Consider two functions f and g on $[a, b]$.

f and g coincide **almost everywhere** ($f = g$ a.e.)

if the set $\mathcal{N} \equiv \{t \in [a, b] \mid f(t) \neq g(t)\}$ on which they differ is negligible, i.e., has measure zero, i.e., $\int_a^b \chi_{\mathcal{N}}(t)\,\mathrm{d}t = 0$, where

$$\chi_{\mathcal{N}}(t) = \begin{cases} 1 & \text{if } t \in \mathcal{N} \\ 0 & \text{if } t \notin \mathcal{N} \end{cases}$$

Consider two functions f and g on $[a, b]$.

f and g coincide **almost everywhere** ($f = g$ a.e.)

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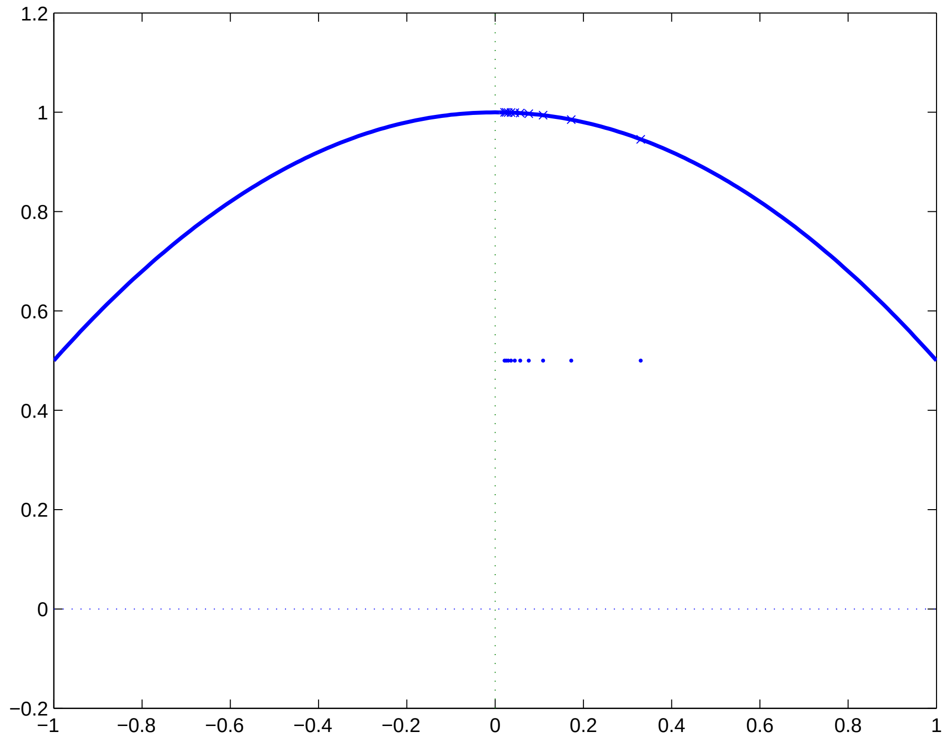
Example. Let $f(t) = 1$ for $t > 0$ and $f(t) = 0$ elsewhere, and let $\widetilde{f}(t) = 1$ for $t \geq 0$ and $\widetilde{f}(t) = 0$ elsewhere.
Then $f = \widetilde{f}$ a.e..



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and let $\widetilde{f}(t) = 1$ for $t \geq 0$ and $\widetilde{f}(t) = 0$ elsewhere.
Then $f = \widetilde{f}$ a.e..

Unless stated otherwise,

we will identify functions that coincide a.e.

Program

- Norms and inner products
- Convergence
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For functions $f : [a, b] \to \mathbb{C}$

$$\|f\|_1 \equiv \int_a^b |f(t)|\, \mathrm{d}t, \qquad \|f\|_2 \equiv \sqrt{\int_a^b |f(t)|^2\, \mathrm{d}t}$$

We implicitly assume that
for all functions that we consider
integration is possible,
but we allow integrals to have value ∞.

For functions $f : [a, b] \to \mathbb{C}$

$$\|f\|_1 \equiv \int_a^b |f(t)|\,\mathrm{d}t, \qquad \|f\|_2 \equiv \sqrt{\int_a^b |f(t)|^2\,\mathrm{d}t}$$

Note that $\|f - g\|_1 = \|f - g\|_2 = 0$ if $f = g$ a.e.

How to define $\|f\|_\infty$?

For functions $f : [a,b] \to \mathbb{C}$

$$\|f\|_1 \equiv \int_a^b |f(t)| \, \mathrm{d}t, \qquad \|f\|_2 \equiv \sqrt{\int_a^b |f(t)|^2 \, \mathrm{d}t}$$

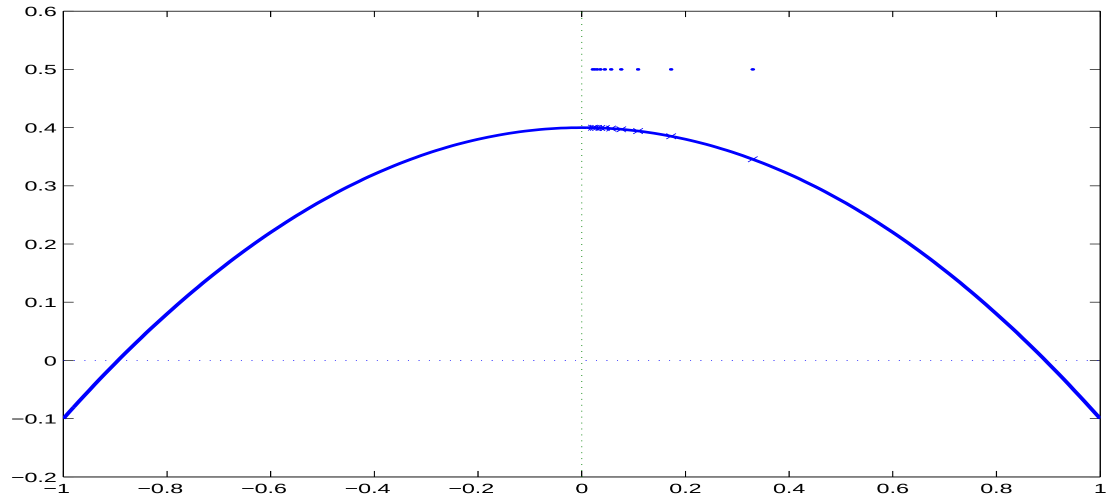
$$\|f\|_\infty \equiv \text{ess-sup}\{|f(t)| \mid t \in [a,b]\}$$

Here **ess-sup** is the **essential supremum**, i.e., essentially we discart negligible sets. More formally,

$$\|f\|_\infty \equiv \inf\{\|g\|_\infty \mid g = f \text{ a.e.}\},$$

where $\|g\|_\infty = \sup\{|g(t)| \mid t \in [a,b]\}$ as before.

Example.



For functions $f : [a,b] \to \mathbb{C}$

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$$\|f\|_\infty \equiv \text{ess-sup}\{|f(t)| \mid t \in [a,b]\}$$

Theorem. $\|f\|_1 \le \sqrt{b-a}\,\|f\|_2 \le (b-a)\,\|f\|_\infty$

$L^1([a,b])$, $L^2([a,b])$, $L^\infty([a,b])$ is the space of all functions $f : [a,b] \to \mathbb{C}$ for which $\|f\|_1 < \infty$, $\|f\|_2 < \infty$, $\|f\|_\infty < \infty$, respectively, and we identify functions that coincide a.e..

$L^2([a,b])$ is an inner product space: $(f,g) \equiv \int_a^b f(t)\,\overline{g(t)}\,\mathrm{d}t$.

Theorem. $C([a,b]) \subset L^\infty([a,b]) \subset L^2([a,b]) \subset L^1([a,b])$

Exercise. Show that all inclusions are strict.

(f_n) is a **Cauchy sequence** wrt a norm $\|\cdot\|$

if $$\|f_n - f_m\| \to 0 \quad \text{if } n > m,\ m \to \infty$$

Completeness Theorem.

The spaces $L^p([a,b])$, for $p = 1, 2, \infty$, are **complete** that is, if (f_n) is a **Cauchy sequence** in $L^p([a,b])$ then there is an $f \in L^p([a,b])$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

(f_n) is a **Cauchy sequence** wrt a norm $\|\cdot\|$

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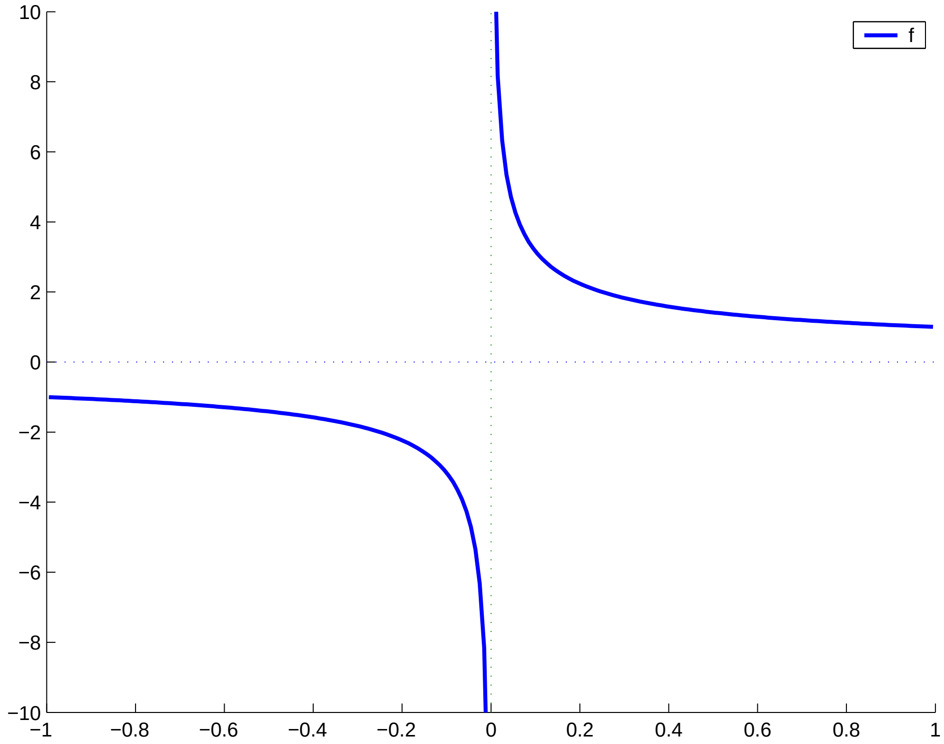
Density Theorem. $C([a,b])$ is **dense** in $L^p([a,b])$ for $p = 1$ as well as for $p = 2$, i.e., for each $f \in L^p([a,b])$ and each $\varepsilon > 0$ there is a $g \in C([a,b])$ such that $\|f - g\|_p < \varepsilon$.

(f_n) is a **Cauchy sequence** wrt a norm $\|\cdot\|$

if $\|f_n - f_m\| \to 0$ if $n > m$, $m \to \infty$

Completeness Theorem.

The spaces $L^p([a,b])$, for $p = 1, 2, \infty$, are **complete** that is, if (f_n) is a **Cauchy sequence** in $L^p([a,b])$ then there is an $f \in L^p([a,b])$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.



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Completeness Theorem.
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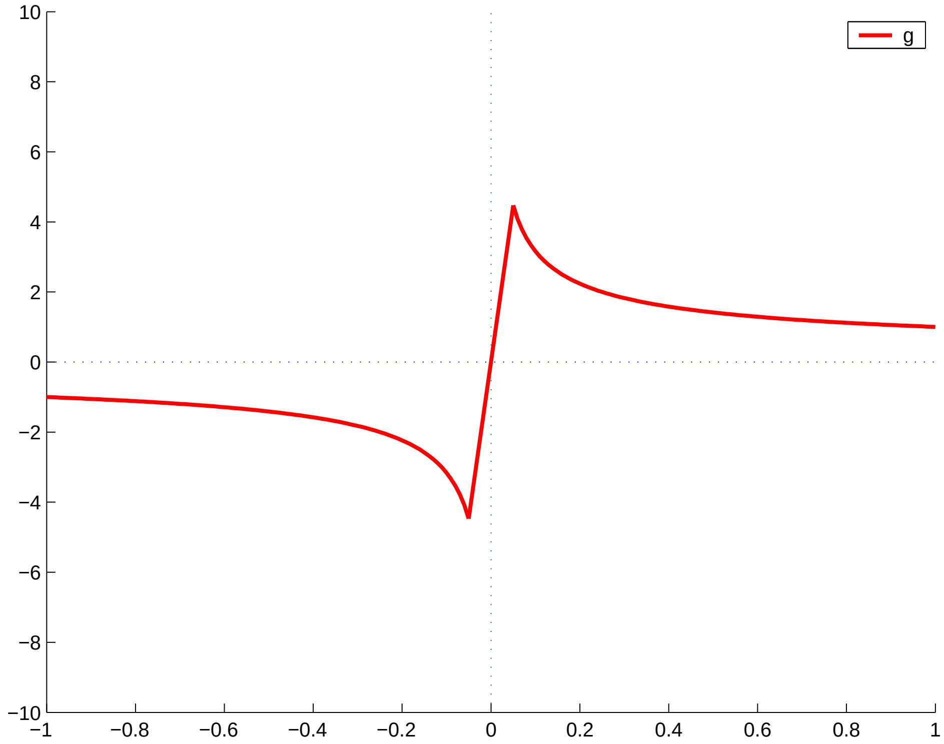

(f_n) is a **Cauchy sequence** wrt a norm $\|\cdot\|$

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Completeness Theorem.
The spaces $L^p([a,b])$, for $p = 1, 2, \infty$, are **complete** that is, if (f_n) is a **Cauchy sequence** in $L^p([a,b])$ then there is an $f \in L^p([a,b])$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

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Exercise. $C([a,b])$ is **not** dense in $L^\infty([a,b])$ (with $f(t) = 1$ for $t > 0$ and $f(t) = -1$ for $t \le 0$ ($|t| \le 1$) show that $\|f - g\|_\infty \ge 1$ for all $g \in C([-1,+1])$.)

For sequences $(\gamma_k)_{k\in\mathbb{Z}}$ in $\mathbb{C}$. With $\gamma(k)=\gamma_k$, $\gamma:\mathbb{Z}\to\mathbb{C}$.

$$\|\gamma\|_1 \equiv \sum_{k=-\infty}^{\infty} |\gamma_k|, \quad \|\gamma\|_2 \equiv \sqrt{\sum_{k=-\infty}^{\infty} |\gamma_k|^2}, \quad \|\gamma\|_\infty \equiv \sup_{k\in\mathbb{Z}} |\gamma_k|$$

$\ell^1(\mathbb{Z})$, $\ell^2(\mathbb{Z})$, $\ell^\infty(\mathbb{Z})$ is the space of all sequences γ in $\mathbb{C}$ for which $\|\gamma\|_1<\infty$, $\|\gamma\|_2<\infty$, $\|\gamma\|_\infty<\infty$, resp.

$\ell^2(\mathbb{Z})$ is an inner product space: $<\gamma,\mu>\equiv\sum\gamma_k\,\overline{\mu_k}$.

Theorem. $\quad \|\gamma\|_\infty \leq \|\gamma\|_2 \leq \|\gamma\|_1 \quad (\gamma:\mathbb{Z}\to\mathbb{C})$

$$\ell^1(\mathbb{Z})\subset\ell^2(\mathbb{Z})\subset\ell^\infty(\mathbb{Z})$$

For functions $f : \mathbb{R} \to \mathbb{C}$

$$\|f\|_1 \equiv \int_{-\infty}^{\infty} |f(t)|\, \mathrm{d}t, \qquad \|f\|_2 \equiv \sqrt{\int_{-\infty}^{\infty} |f(t)|^2\, \mathrm{d}t}$$

$$\|f\|_\infty \equiv \text{ess-sup}\{|f(t)| \mid t \in \mathbb{R}\}$$

$L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $L^\infty(\mathbb{R})$ is the space of all functions $f : \mathbb{R} \to \mathbb{C}$ for which $\|f\|_1 < \infty$, $\|f\|_2 < \infty$, $\|f\|_\infty < \infty$, respectively, and we identify functions that coincide a.e..

$L^2(\mathbb{R})$ is an inner product space: $(f, g) \equiv \int_{-\infty}^{\infty} f(t)\, \overline{g(t)}\, \mathrm{d}t$.

Exercise. Discuss the inclusions

$$C(\mathbb{R}) \subset L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \subset L^1(\mathbb{R})$$

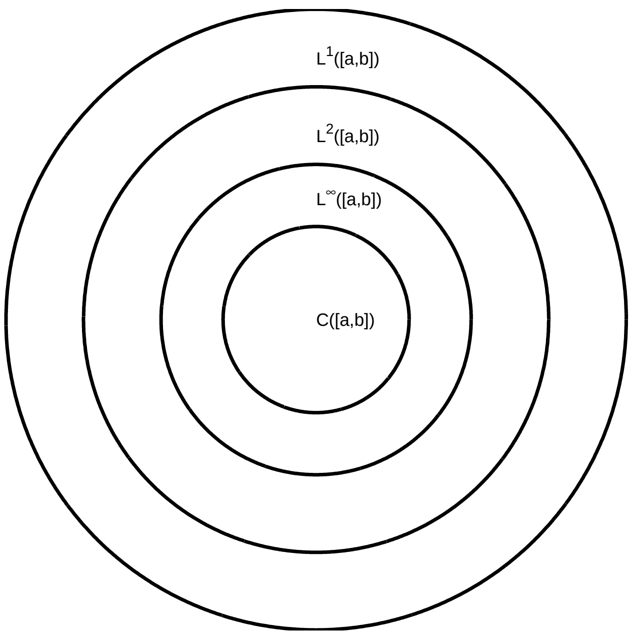
On $[a,b]$: $C([a,b]) \subset L^\infty([a,b]) \subset L^2([a,b]) \subset L^1([a,b])$

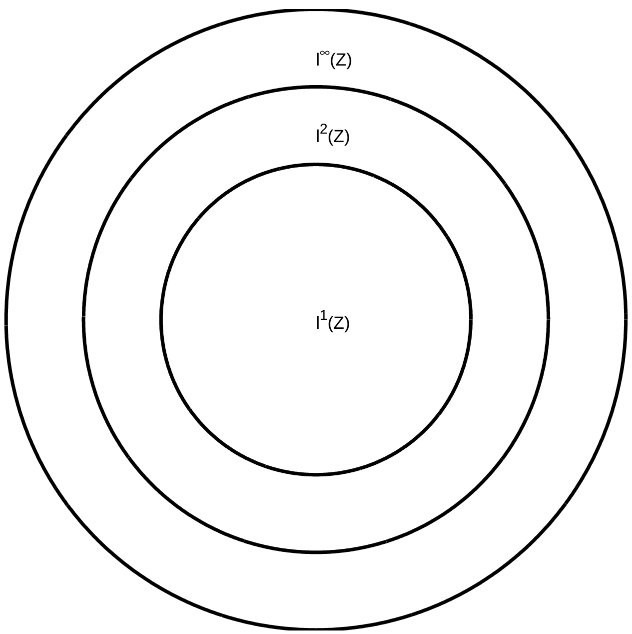
On $\mathbb{Z}$: $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$

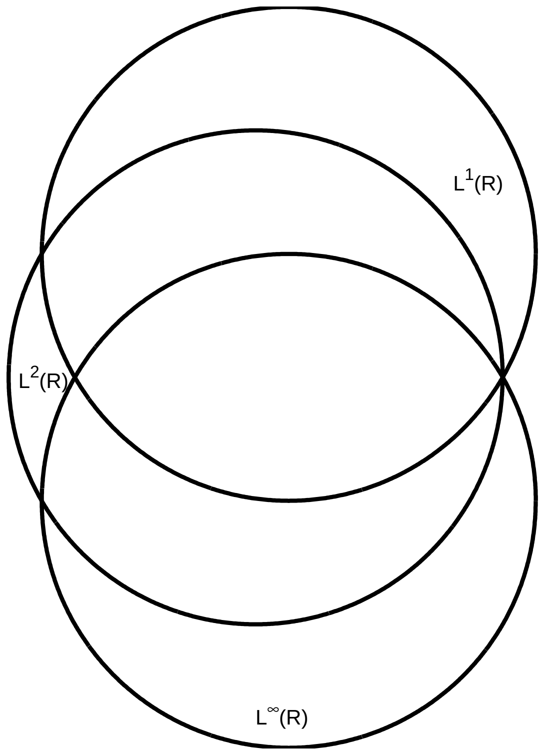
On $\mathbb{R}$: $C(\mathbb{R}) \,??\, L^\infty(\mathbb{R}) \,??\, L^2(\mathbb{R}) \,??\, L^1(\mathbb{R})$

Explanation: $\|f\|_1 = \sum_{k\in\mathbb{Z}} \|f|_{[k,k+1]}\|_1$ for $f : \mathbb{R} \to \mathbb{C}$:

mixure of 'on $[a,b]$' and 'on $\mathbb{Z}$.







On $[a, b]$: $C([a, b]) \subset L^\infty([a, b]) \subset L^2([a, b]) \subset L^1([a, b])$

On $\mathbb{Z}$: $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$

On $\mathbb{R}$: $C(\mathbb{R}) \,??\, L^\infty(\mathbb{R}) \,??\, L^2(\mathbb{R}) \,??\, L^1(\mathbb{R})$

Explanation: $\|f\|_1 = \sum_{k\in\mathbb{Z}} \|f|_{[k,k+1]}\|_1$ for $f : \mathbb{R} \to \mathbb{C}$:

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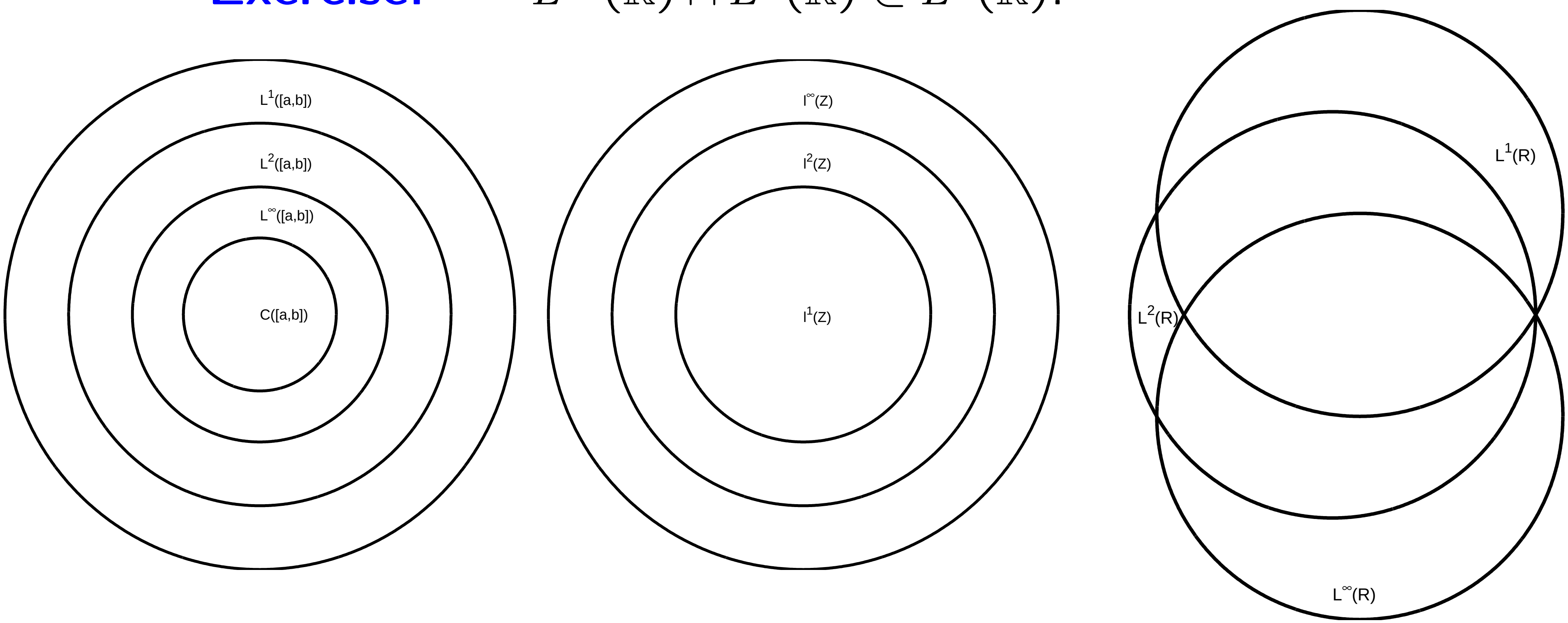
Exercise. $L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^2(\mathbb{R})$.



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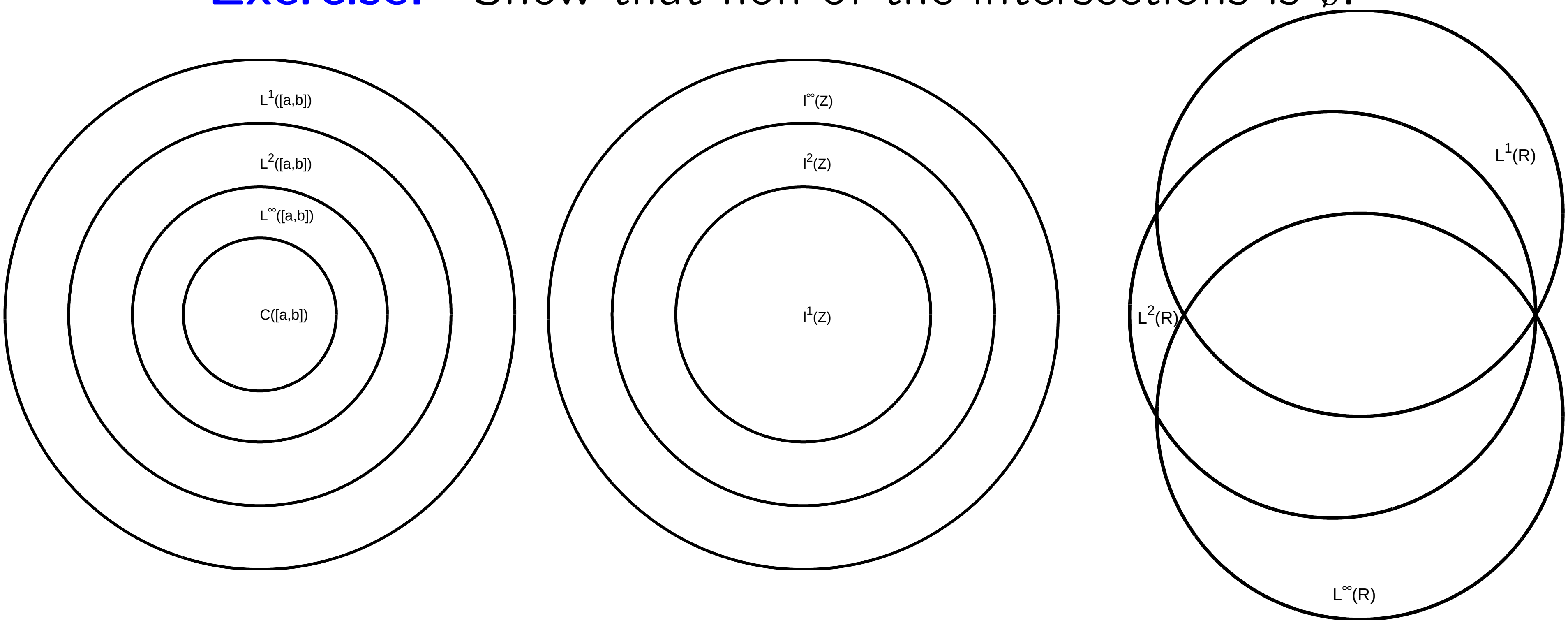
On $\mathbb{R}$: $C(\mathbb{R}) \,??\, L^\infty(\mathbb{R}) \,??\, L^2(\mathbb{R}) \,??\, L^1(\mathbb{R})$

Explanation: $\|f\|_1 = \sum_{k\in\mathbb{Z}} \|f|_{[k,k+1]}\|_1$ for $f : \mathbb{R} \to \mathbb{C}$:

mixure of 'on $[a,b]$' and 'on $\mathbb{Z}$.

Exercise. Show that non of the intersections is $\emptyset$.



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On $\mathbb{Z}$: $\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$

On $\mathbb{R}$: $C(\mathbb{R}) \,??\, L^\infty(\mathbb{R}) \,??\, L^2(\mathbb{R}) \,??\, L^1(\mathbb{R})$

Explanation: $\|f\|_1 = \sum_{k\in\mathbb{Z}} \|f|_{[k,k+1]}\|_1$ for $f : \mathbb{R} \to \mathbb{C}$:

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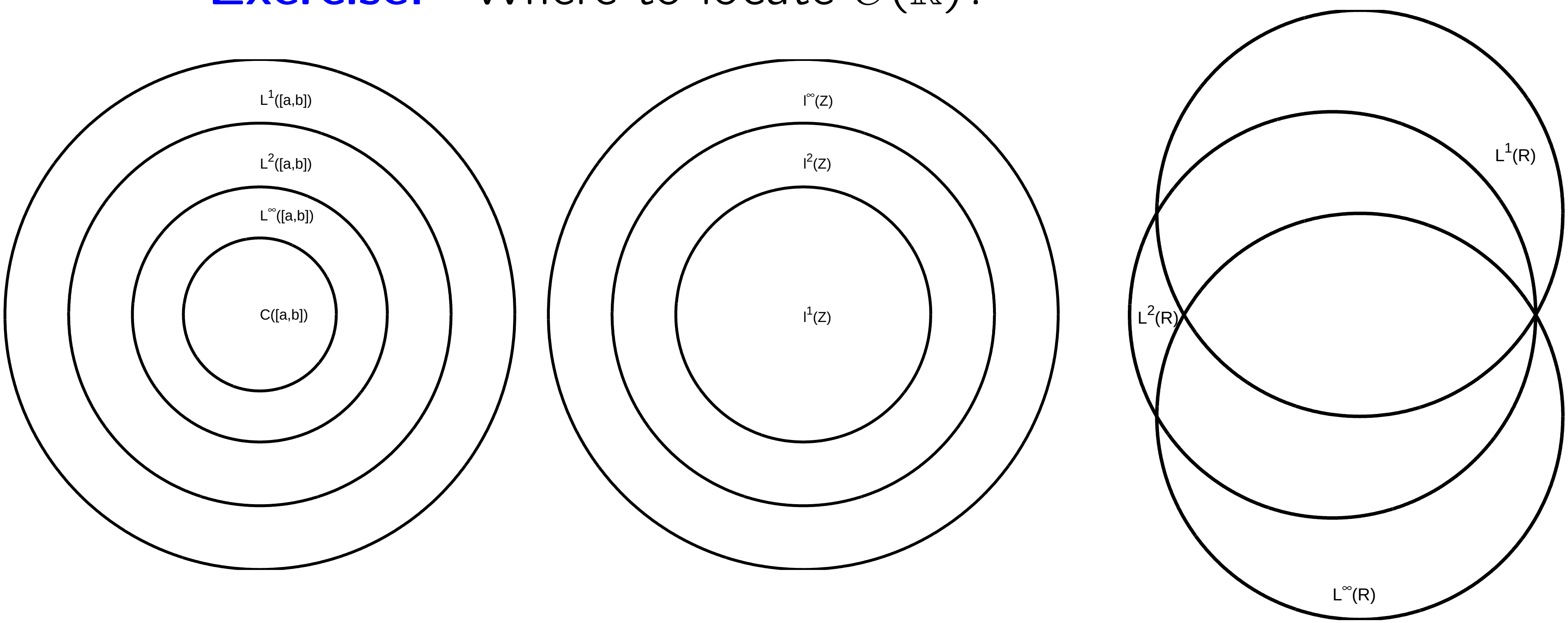
Exercise. Where to locate $C(\mathbb{R})$?



Program

- Norms and inner products
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For $I = [a, b]$ or $I = \mathbb{R}$,

consider a sequence (f_n) in $L^1(\mathbb{R})$ and an $f \in L^1(\mathbb{R})$ st

$$\lim_{n \to \infty} f_n(t) = f(t) \qquad (t \in I).$$

The sequence **converges point-wise**.

Exercise.

Does point-wise convergence imply $\|\cdot\|_1$ convergence?

For $I = [a, b]$ or $I = \mathbb{R}$,

consider a sequence (f_n) in $L^1(\mathbb{R})$ and an $f \in L^1(\mathbb{R})$ st

$$\lim_{n\to\infty} f_n(t) = f(t) \qquad (t \in I).$$

The sequence **converges point-wise**.

Fatou's lemma. If there is a g st

$$g \in L^1(I) \quad \text{and} \quad |f_n(t)| \leq |g(t)| \ \ (t \in I, n \in \mathbb{N}),$$

then $\quad \lim_{n\to\infty} f_n(t) = f(t) \ \ (t \in I) \ \Rightarrow \ \lim_{n\to\infty} \|f_n - f\|_1 = 0$

For $I = [a, b]$ or $I = \mathbb{R}$,

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then $\displaystyle \lim_{n\to\infty} f_n(t) = f(t) \;\; (t \in I) \;\; \Rightarrow \;\; \lim_{n\to\infty} \|f_n - f\|_1 = 0$

Exercise. Suppose $f, tf \in L^1(I)$. Consider g defined by

$$g(\omega) \equiv \int_I f(t) \sin(2\pi t\omega)\, \mathrm{d}t \quad (\omega \in \mathbb{R}).$$

Show that

$$g'(\omega) = 2\pi \int_I t f(t) \cos(2\pi t\omega)\, \mathrm{d}t \quad (\omega \in \mathbb{R}).$$

For $I = [a, b]$ or $I = \mathbb{R}$,

consider a sequence (f_n) in $L^1(\mathbb{R})$ and an $f \in L^1(\mathbb{R})$ st

$$\lim_{n\to\infty} f_n(t) = f(t) \qquad (t \in I).$$

The sequence **converges point-wise**.

Fatou's lemma. If there is a g st

$$g \in L^1(I) \quad \text{and} \quad |f_n(t)| \leq |g(t)| \;\; (t \in I, n \in \mathbb{N}),$$

then $\displaystyle \lim_{n\to\infty} f_n(t) = f(t) \;\; (t \in I) \;\Rightarrow\; \lim_{n\to\infty} \|f_n - f\|_1 = 0$

Exercise. Does Fatou's lemma hold for

- L^2-functions and $\|\cdot\|_2$-convergence?
- L^∞ functions and $\|\cdot\|_\infty$ convergence?

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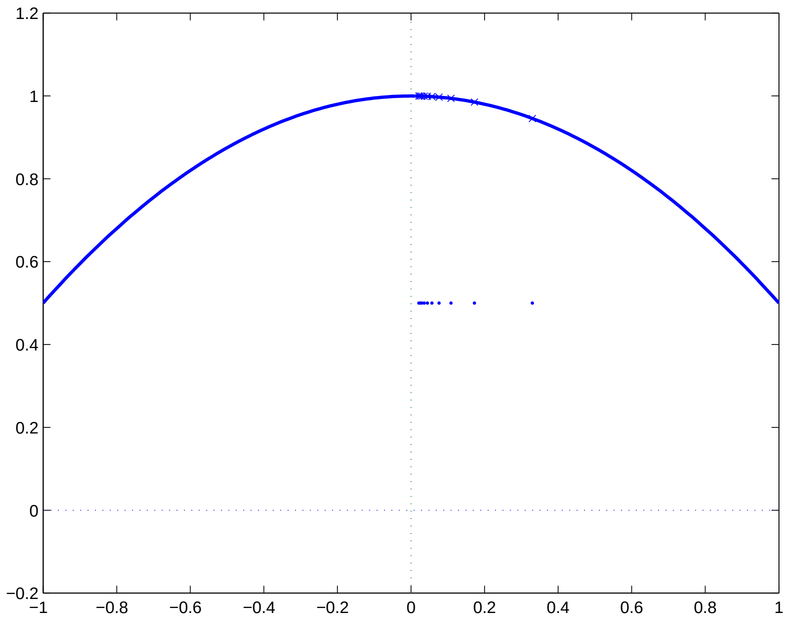
Function values

Note. Formally, $f(t)$ does not have a meaning.

However, if $f = g$ a.e. and g is continuous at t, then $g(t)$ is well-defined and

Convention. With $f(t)$ **we will denote this value** $g(t)$.



Function values

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In particular $f(t)$ has a well-defined value if f is continuous.

Function values

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More generally, we put $f(t+)$,
if $f = g$ a.e. for a function g that is left continuous at t ($\lim_{\varepsilon>0,\varepsilon\to 0} g(t+\varepsilon) = g(t)$). Then $f(t+)$ has the value $g(t)$.

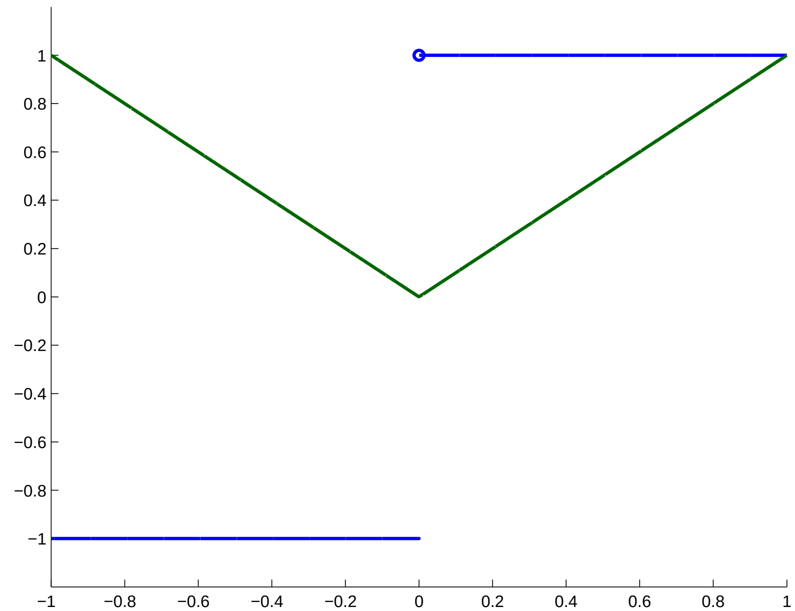
Similarly,
$f(t-) = g(t)$ if $f = g$, a.e., and $\lim_{\varepsilon>0,\varepsilon\to 0} g(t-\varepsilon) = g(t)$

Program

- Norms and inner products
- Convergence
- Almost everywhere
- Function spaces
- Point-wise convergence
- Function values
- Derivatives

Weak Derivatives

Example. The function $f(t) \equiv |t|$ is a.e. differentiable with derivative g given by $g(t) = 1$ if $t > 0$ and $g(t) = -1$ else.



More generally,

Weak Derivatives

Consider a function f on $[a,b]$. We will put f' if there is a function g on $[a,b]$ and a $c \in [a,b]$ such that

$$f(t) = f(c) + \int_c^t g(s)\,\mathrm{d}s \qquad (t \in [a,b]).$$

Then, f' will denote the function g.

g is unique if we identify functions that coincide a.e..

Exercise. Does f' exists for

(a) $f(t) \equiv |t| \qquad (|t| \leq 1)$

(b) $f(t) = 1$ if $t > 0$ and $f(t) = -1$ elsewhere $\qquad (|t| \leq 1)$

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Theorem. If $f' \in L^1([a, b])$ then $f \in C([a, b])$.

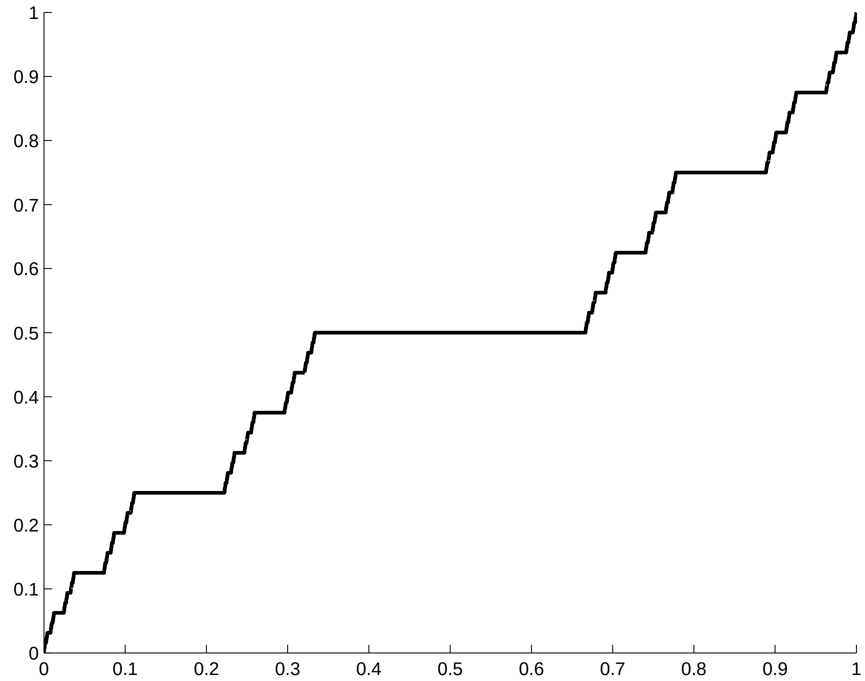
f is said to be **absolutely continuous** if $f' \in L^1([a, b])$.

Weak Derivatives

There is a **continuous non-decreasing** function f on $[0,1]$ with $f(0)=0$, $f(1)=1$ such that

$$f'(t)=0 \text{ for almost all } t\in[0,1]:$$

Allthough most values $f'(t)$ exists, f' **does not exists!**



Integration by parts

If $f', g' \in L^1([a,b])$ then

$$\int_a^b f'(t)g(t)\,\mathrm{d}t = f(b)g(b) - f(a)g(a) - \int_a^b f(t)g'(t)\,\mathrm{d}t$$

It is essential that both f and g are continuous on $[a,b]$, the functions f' and g' need not be continuous.