Scientific Computing, Utrecht, February 17, 2014

Fourier Transforms
Wavelets
Theory and Applications

Gerard Sleijpen

Universiteit Utrecht
Department of Mathematics

http://www.staff.science.uu.nl/~sleij101/

Summary Lecture 1

Find appropriate set of basis functions to approximate functions from interesting class.

$f : [a, b] \to \mathbb{C}$, $\quad \|f\|_1 \le \sqrt{b-a}\,\|f\|_2 \le (b-a)\,\|f\|_\infty$

Identify f and g if $f = g$ a.e..

$\|\cdot\|_2$ attractive in theory because, $\|f\|_2^2 = (f, f)$,

Pythagoras, Cauchy-Schwartz, orthogonality.

$\|\cdot\|_p$ with $p \in [1, \infty)$ attractive in theory because

$C([a, b])$ dense in $L^p([a, b])$

Fatou's lemma (Lebesgue's dominated convergence):

$f_n(t) \to f(t)$ all t, $|f_n(t)| < g(t)$ all n, t, $\|g\|_p < \infty$

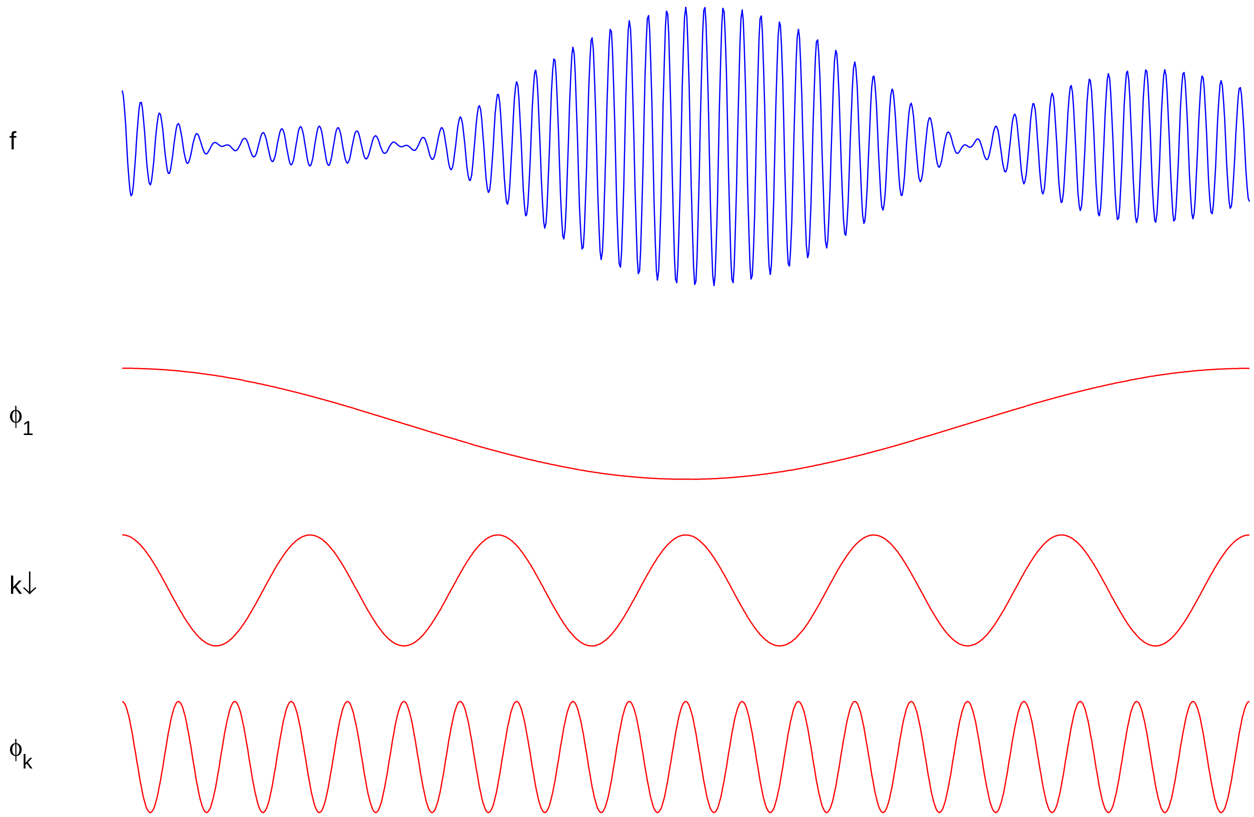
$\Rightarrow \quad \|f_n - f\|_p \to 0$

Also correct with $[a, \infty)$ or $(-\infty, \infty)$ instead of $[a, b]$.

$\|\cdot\|_\infty$ attractive in practice.

$\gamma : \mathbb{Z} \to \mathbb{C}$, $\quad \big|\gamma\big|_\infty \le \big|\gamma\big|_2 \le \big|\gamma\big|_1$

Fourier Series



Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

$T > 0$

T-periodic functions

$f : \mathbb{R} \to \mathbb{C}$ is **T-periodic** if $f(t+T) = f(t) \quad \forall t \in \mathbb{R}$

Example. For each $k \in \mathbb{Z}$,
$t \rightsquigarrow \cos(2\pi t \frac{k}{T})$ and $t \rightsquigarrow \sin(2\pi t \frac{k}{T})$ are T-periodic.

Fourier: these are essentially all T-periodic functions: each T-periodic function is in some sense a linear combinations of these sines and cosines

T is the *length of the period*.

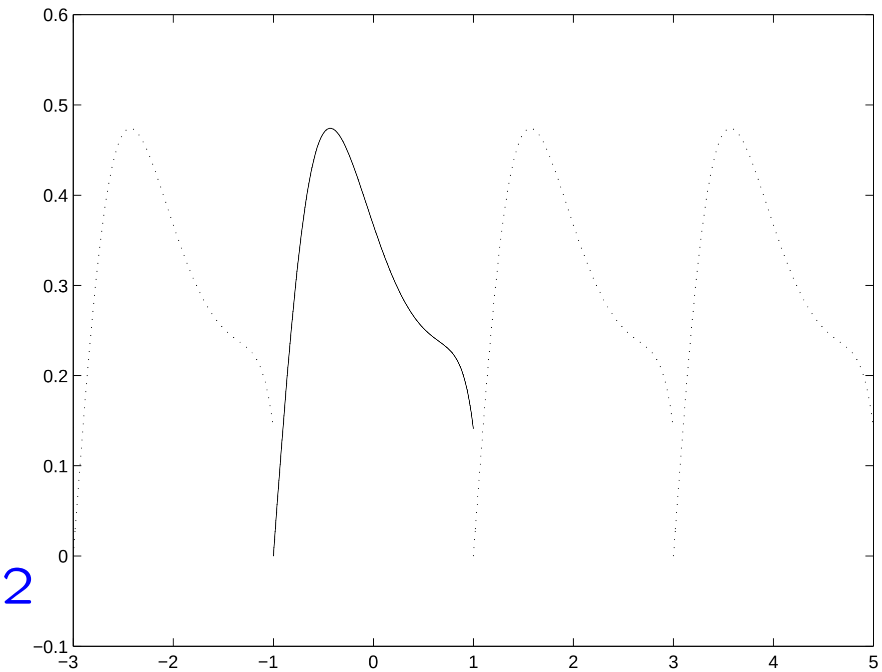
Note that $\exp(2\pi i t \frac{k}{T}) = \cos(2\pi t \frac{k}{T}) + i \sin(2\pi t \frac{k}{T})$

Functions on $[a, a+T]$ can be identified with T-periodic functions:

If g is defined on $[a, a+T]$, then

$$f(t) \equiv g(t+kT) \quad (t \in \mathbb{R}) \quad k \in \mathbb{Z} \text{ s.t. } t+kT \in [a, a+T)$$

defines a T-periodic function and $f = g$ on $[a, a+T]$.



— graph g
$a = -1$, $T = 2$
... graph f,
f is 2-periodic

Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

If f is T-periodic, then

$$\int_0^T f(t)\,\mathrm{d}t = \int_\tau^{\tau+T} f(t)\,\mathrm{d}t \qquad (\tau \in \mathbb{R})$$

For T-periodic, integrable functions f on $\mathbb{R}$ define

$$\|f\|_1 \equiv \frac{1}{T}\int_0^T |f(t)|\,\mathrm{d}t$$

The space of all complex-valued T-periodic functions for which $\|f\|_1 < \infty$ is denoted by $L^1_T(\mathbb{R})$.

$$\|f\|_2 \equiv \sqrt{\frac{1}{T}\int_0^T |f(t)|^2\,\mathrm{d}t}$$

The space of all complex-valued T-periodic functions for which $\|f\|_2 < \infty$ is denoted by $L^2_T(\mathbb{R})$.

Note. **We identify functions that coincide a.e.**.

$$\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty \equiv \text{ess-sup}\{|f(x)| \mid x \in \mathbb{R}\}$$

$$C_T(\mathbb{R}) \equiv \{f \in C(\mathbb{R}) \mid f \text{ is } T\text{-periodic}\} \subset L^2_T(\mathbb{R}) \subset L^1_T(\mathbb{R})$$

$L^2_T(\mathbb{R})$ is an inner product space w.r.t.

$$(f, g) \equiv \frac{1}{T} \int_0^T f(t)\, \overline{g(t)}\, \mathrm{d}t \qquad (f, g \in L^2_T(\mathbb{R}))$$

For each $k \in \mathbb{Z}$, put $\phi_k(t) \equiv \exp(2\pi i t \frac{k}{T}) \qquad (t \in \mathbb{R})$.

Theorem. The ϕ_k form an orthonormal system in $L^2_T(\mathbb{R})$:

$$(\phi_k, \phi_j) = 0 \quad \text{if } k \neq j \quad \text{and} \quad \|\phi_k\|_2 = 1 \quad (j, k \in \mathbb{Z})$$

Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

Fourier series

For $f \in L^1_T(\mathbb{R})$, put

$$\gamma_k(f) \equiv \frac{1}{T}\int_0^T f(t)e^{-2\pi i t\frac{k}{T}}\,\mathrm{d}t \qquad (k \in \mathbb{Z})$$

$\gamma_k(f)$ is the **kth Fourier coefficient**. For $n \in \mathbb{N}$,

$$S_n(f)(t) \equiv \sum_{k=-n}^{n} \gamma_k\, e^{2\pi i t\frac{k}{T}} \qquad (t \in \mathbb{R})$$

$S_n(f)$ is the **nth partial Fourier series**.
The formal infinite sum is the **Fourier series of f**:

$$f \sim \sum \gamma_k\, e^{2\pi i t\frac{k}{T}}.$$

Note. This is not statement on convergence!

Use $\exp(2\pi i t\frac{k}{T}) = \cos(2\pi t\frac{k}{T}) + i\sin(2\pi t\frac{k}{T})$ for a formulation in sines and cosines.

Fourier series

For $f \in L^2_T(\mathbb{R})$,

$$\gamma_k(f) \equiv \frac{1}{T}\int_0^T f(t)e^{-2\pi i t\frac{k}{T}}\,\mathrm{d}t = (f, \phi_k)$$

$\gamma_k(f)$ is the **kth Fourier coefficient**. For $n \in \mathbb{N}$,

$$S_n(f)(t) \equiv \sum_{k=-n}^{n} \gamma_k\, e^{2\pi i t\frac{k}{T}} = \sum_{|k|\leq n} (f, \phi_k)\,\phi_k$$

Note that $S_n(f) \in C^{(\infty)}(\mathbb{R})$.

Exercise. Consider f and g defined by

$$f(t) \equiv |t|, \quad g(t) = \begin{cases} +1 & \text{for } t \in (0,1] \\ -1 & \text{for } t \in (-1,0] \end{cases}$$

(a) Extend f and g to 2-periodic functions ($T = 2$).

(b) Are these functions in $C^{(1)}(\mathbb{R})$, $C_T(\mathbb{R})$, $L_T^2(\mathbb{R})$, $L_T^1(\mathbb{R})$?

(c) Compute the Fourier series of g.

Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

L^2-Convergence

Theorem. $\|S_n(f) - f\|_2 \to 0 \quad (n \to \infty)$ if $f \in L^2_T(\mathbb{R})$.

Therefore, for $f \in L^2_T(\mathbb{R})$, in the L^2_T-sense, we have that

$$f = \sum_{k=-\infty}^{\infty} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

L^2-Convergence

Theorem. $\|S_n(f) - f\|_2 \to 0 \quad (n \to \infty)$ if $f \in L^2_T(\mathbb{R})$.

Therefore, for $f \in L^2_T(\mathbb{R})$, in the L^2_T-sense, we have that

$$f = \sum_{k=-\infty}^{\infty} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

What about $\|S_n(f) - f\|_\infty$ or $|S_n(f)(t) - f(t)|$ for $n \to \infty$?

L^2-Convergence

Theorem. $\|S_n(f) - f\|_2 \to 0 \quad (n \to \infty)$ if $f \in L^2_T(\mathbb{R})$.

Therefore, for $f \in L^2_T(\mathbb{R})$, in the L^2_T-sense, we have that

$$f = \sum_{k=-\infty}^{\infty} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

What about $\|S_n(f) - f\|_\infty$ or $|S_n(f)(t) - f(t)|$ for $n \to \infty$?

$\exists f \in C_T(\mathbb{R})$ **and a** $t \in \mathbb{R}$ **for which** $(S_n(f)(t))$ **diverges**:

additional smoothness is required for stronger convergence.

Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T^{(1)}(\mathbb{R})$.

Theorem. $S_n(f)(t) \to f(t) \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and $f \in C^{(1)}([t-\delta, t+\delta])$ for some $\delta > 0$.

This last theorem is quite remarkable, since the Fourier series requires f over a whole period, while the convergence results needs some additional smoothness of f only close to the point t of interest.

Results can be relaxed.

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** f is of bounded variation.

A function f on $\mathbb{R}$ is of **bounded variation** (BV) if it is a finite linear combination of non-decreasing functions.

Example. $f(t) \equiv |t|$ on $[-1, +1]$.

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** f is of bounded variation.

A function f on $\mathbb{R}$ is of **bounded variation** (BV) if it is a finite linear combination of non-decreasing functions.

Example. $f(t) \equiv |t|$ on $[-1, +1]$.

Example. If $f \in C^{(1)}(\mathbb{R})$ then f is of BV.

Proof. $f(t) = f(0) + \int_0^t f'(s)\,\mathrm{d}s = f(0) + f_+(t) - f_-(t)$ with

$$f_+(t) \equiv \int_0^t \max(f'(s), 0)\,\mathrm{d}s \quad \text{and} \quad f_-(t) \equiv \int_0^t \max(-f'(s), 0)\,\mathrm{d}s$$

Uniform Convergence

Theorem. $\|S_n(f)-f\|_\infty \to 0 \quad (n\to\infty)$ if $f\in C_T(\mathbb{R})$ **and** f is of bounded variation.

A function f on $\mathbb{R}$ is of **bounded variation** (BV) if it is a finite linear combination of non-decreasing functions.

Example. $f(t)\equiv |t|$ on $[-1,+1]$.

Example. If $f(t)=f(0)+\int_0^t g(s)\,\mathrm{d}s$ with $g\in L^1_T(\mathbb{R})$,
f is **absolutely continuous** (AC), then f is of BV: .

Proof. $f(t)=f(0)+\int_0^t g(s)\,\mathrm{d}s=f(0)+f_+(t)-f_-(t)$ with

$$f_+(t)\equiv\int_0^t \max(g(s),0)\,\mathrm{d}s \quad \text{and} \quad f_-(t)\equiv\int_0^t \max(-g(s),0)\,\mathrm{d}s$$

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of bounded variation.

A function f on $\mathbb{R}$ is of **bounded variation** (BV) if it is a finite linear combination of non-decreasing functions.

Example. $f(t) \equiv |t|$ on $[-1, +1]$.

Example. With

$$g(t) = \begin{cases} +1 & \text{if } t \in [0, 1] \\ -1 & \text{if } t \in [-1, 0) \end{cases}$$

we have that $|t| = \int_0^t g(s)\,\mathrm{d}s$.

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of bounded variation.

A function f on $\mathbb{R}$ is of **bounded variation** (BV) if it is a finite linear combination of non-decreasing functions.

Example. $f(t) \equiv |t|$ on $[-1, +1]$.

Example. With

$$g(t) = \begin{cases} +1 & \text{if } t \in [0, 1] \\ -1 & \text{if } t \in [-1, 0) \end{cases}$$

we have that $|t| = \int_0^t g(s)\, \mathrm{d}s$.

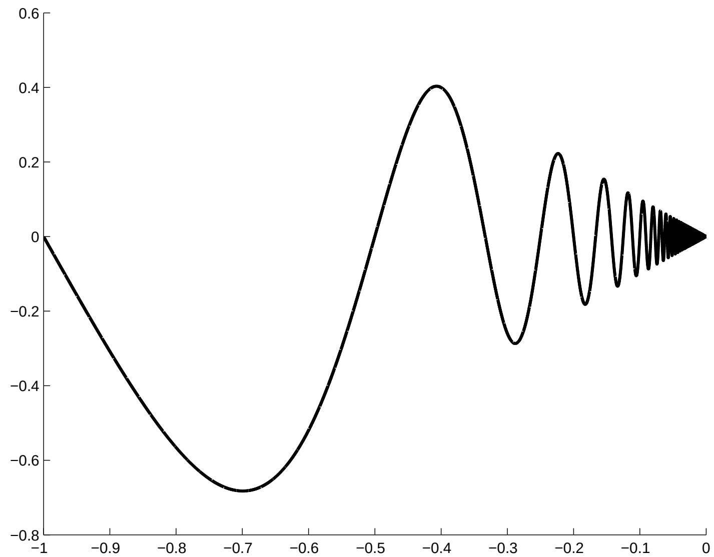
$|t|$ is AC, g is **not** AC, g is of BV.

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** f is of bounded variation.

A function f on $\mathbb{R}$ is of **bounded variation** (BV) if it is a finite linear combination of non-decreasing functions.

Example. f 2-periodic s.t. $f(t) = t\sin(2\pi/t) \qquad (|t| \leq 1)$
f is continuous, but **not** of BV.



Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of bounded variation.

A function f on $\mathbb{R}$ is of **bounded variation** (BV) if it is a finite linear combination of non-decreasing functions.

Totaal variation norm. f BV iff $\mathrm{TV}(f) < \infty$, where

$$\mathrm{TV}(f) \equiv \int_{-\infty}^{\infty} \ell(c)\,\mathrm{d}c, \quad \text{where } \ell(c) \equiv \#\{t \mid f(t) = c\}.$$

If $f(t) = f(0) + \int_0^t g(s)\,\mathrm{d}s$ then $\mathrm{TV}(f) = \|g\|_1$.

If f defined on subset $\mathbb{R}^2$ then $\ell(c)$ is length $\{t \mid f(t) = c\}$.

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** f is of bounded variation.

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** f is of bounded variation.

Theorem. $\|\sigma_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ iff $f \in C_T(\mathbb{R})$

Here, $$\sigma_n(f) \equiv \frac{1}{n} \sum_{j=0}^{n-1} S_n(f)$$ **Césaro sum**

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** f is of bounded variation.

Theorem. $\|\sigma_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ iff $f \in C_T(\mathbb{R})$

Here, $$\sigma_n(f) \equiv \frac{1}{n} \sum_{j=0}^{n-1} S_n(f)$$ **Césaro sum**

Example. $f(t) = \cos(2\pi t \frac{1}{T})$. Then,

$$S_n(f) = \frac{1}{2}(e^{-2\pi i t/T} + e^{2\pi i t/T}) = f \quad (n \geq 1),$$

whereas $\sigma_n(f) = \frac{n-1}{n} f \qquad (n \in \mathbb{N}).$

Point-wise Convergence

Theorem. $S_n(f)(t) \to f(t) \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** $f \in C^{(1)}([t-\delta, t+\delta])$ for some $\delta > 0$.

Point-wise Convergence

Theorem. $S_n(f)(t) \to f(t) \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of BV on $[t-\delta, t+\delta]$ for some $\delta > 0$.

Point-wise Convergence

Theorem. $S_n(f)(t) \to f(t) \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of BV on $[t-\delta, t+\delta]$ for some $\delta > 0$.

Even continuity is not needed

Point-wise Convergence

Theorem. $S_n(f)(t) \to f(t) \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of BV on $[t-\delta, t+\delta]$ for some $\delta > 0$.

Theorem.

$S_n(f)(t) \to \frac{1}{2}[f(t+) + f(t-)] \quad (n \to \infty)$ if $f \in L^1_T(\mathbb{R})$
and f is of BV on $[t-\delta, t+\delta]$ for some $\delta > 0$.

Here,
$$f(t+) \equiv \lim_{\varepsilon > 0,\, \varepsilon \to 0} f(t+\varepsilon)$$
$$f(t-) \equiv \lim_{\varepsilon > 0,\, \varepsilon \to 0} f(t-\varepsilon)$$

and we assume that the (essential) limits exist.

Note that, for $f \in L^1_T(\mathbb{R})$, $f(t)$ is not well-defined.

Point-wise Convergence

Theorem. $S_n(f)(t) \to f(t) \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of BV on $[t-\delta, t+\delta]$ for some $\delta > 0$.

Theorem.

$S_n(f)(t) \to \frac{1}{2}[f(t+) + f(t-)] \quad (n \to \infty)$ if $f \in L^1_T(\mathbb{R})$
and f is of BV on $[t-\delta, t+\delta]$ for some $\delta > 0$.

Example. g 2-periodic with

$$g(t) = +1 \text{ if } t \in (0,1] \text{ and } g(t) = -1 \text{ if } t \in (-1,0]$$

is in $L^1_T(\mathbb{R})$ and of BV.

Theorem. $\|S_n(f) - f\|_2 \to 0 \quad (n \to \infty)$ if $f \in L^2_T(\mathbb{R})$.

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** f is of bounded variation.

Theorem. $\|\sigma_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ iff $f \in C_T(\mathbb{R})$

Theorem.

$S_n(f)(t) \to \frac{1}{2}[f(t+) + f(t-)] \quad (n \to \infty)$ if $f \in L^1_T(\mathbb{R})$

and f is of BV on $[t-\delta, t+\delta]$ for some $\delta > 0$.

There is an $f \in C_T(\mathbb{R})$ and a $t \in \mathbb{R}$ for which $(S_n(f)(t))$ diverges.

Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

Error estimates

Theorem. If $f \in C_T(\mathbb{R})$ and $f' \in L^1_T(\mathbb{R})$ then

$$\boxed{\gamma_k(f) = \frac{T}{2\pi i k}\,\gamma_k(f') \quad (k \in \mathbb{Z}, k \neq 0)}$$

Proof. Integrate by parts.

Error estimates

Theorem. If $f \in C_T(\mathbb{R})$ and $f' \in L^1_T(\mathbb{R})$ then

$$\boxed{\gamma_k(f) = \tfrac{T}{2\pi i k}\,\gamma_k(f') \quad (k \in \mathbb{Z}, k \neq 0)}$$

Exercise. If $f \in C_T^{(1)}(\mathbb{R})$ then $f' \in C_T(\mathbb{R})$ and

$$\gamma_0(f') = 0, \qquad \gamma_k(f) = \frac{T\gamma_k(f')}{2\pi i k} \quad (k \neq 0).$$

Exercise. f is 2-periodic given by $f(t) = |t|$ $(|t| \leq 1)$. Compute $\gamma_k(f)$. (Hint. Consider f').

Error estimates

Theorem. If $f \in C_T(\mathbb{R})$ and $f' \in L^1_T(\mathbb{R})$ then

$$\boxed{\gamma_k(f) = \frac{T}{2\pi i k}\,\gamma_k(f') \qquad (k \in \mathbb{Z}, k \neq 0)}$$

Theorem. $f \in L^1_T(\mathbb{R})$.

$$|\gamma_k(f)| \leq \|f\|_1 \leq \|f\|_\infty$$

$$\gamma_k(f) \to 0 \quad \text{if} \quad |k| \to \infty. \qquad \textbf{(Riemann–Lebesgue)}$$

$$|\gamma_k(f)| \leq \frac{1}{|k|^\ell}\,(\tfrac{T}{2\pi})^\ell\,\|f^{(\ell)}\|_1 \quad \text{if} \quad f \in C_T^{(\ell)}(\mathbb{R})$$

Error estimates

Theorem. If $f \in C_T(\mathbb{R})$ and $f' \in L^1_T(\mathbb{R})$ then

$$\boxed{\gamma_k(f) = \tfrac{T}{2\pi i k}\,\gamma_k(f') \quad (k \in \mathbb{Z}, k \neq 0)}$$

Theorem. $f \in L^1_T(\mathbb{R})$.

$$|\gamma_k(f)| \leq \|f\|_1 \leq \|f\|_\infty$$

$\gamma_k(f) \to 0$ if $|k| \to \infty$. (**Riemann–Lebesgue**)

$$|\gamma_k(f)| \leq \tfrac{1}{|k|^\ell}\,(\tfrac{T}{2\pi})^\ell\,\|f^{(\ell)}\|_1 \quad \text{if } f \in C_T^{(\ell)}(\mathbb{R})$$

Theorem. $f \in C_T^{(\ell)}(\mathbb{R})$

$$\|S_n(f) - f\|_\infty \leq \tfrac{1}{n^{\ell-1}}\,(\tfrac{T}{2\pi})^\ell\,\|f^{(\ell)}\|_1$$

Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

Differential equations

Turn differential equations into algebraic equations.

With $f \in C_T(\mathbb{R})$, $a, b, c \in \mathbb{C}$, find a T-periodic u s.t.

$$a\, u'' + b\, u' + c\, u = f$$

Solution.
$$\begin{aligned} \gamma_k(f) &= a\, \gamma_k(u'') + b\, \gamma_k(u') + c\, \gamma_k(u) \\ &= [a(\tfrac{2\pi i k}{T})^2 + b\, \tfrac{2\pi i k}{T} + c]\, \gamma_k(u) \end{aligned}$$

What about boundary conditions?

Applications. Electric circuits.

Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

Parseval. $$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |\gamma_k(f)|^2 \qquad (f \in L_T^2(\mathbb{R}))$$

Proof. $\|f\|_2^2 = \|f - S_n(f)\|_2^2 + \|S_n(f)\|_2^2$ (Pythagoras)

$\|S_n(f)\|_2^2 = \sum_{k=-n}^{n} |\gamma_k(f)|^2$ (Pythagoras)

$\lim_{n\to\infty} \|f - S_n(f)\|_2^2 = 0$ (Theorem)

Parseval. $$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |\gamma_k(f)|^2 \qquad (f \in L_T^2(\mathbb{R}))$$

Consider $\ell^2(\mathbb{Z}) \equiv \{(\gamma_k)_{k\in\mathbb{Z}} \mid \gamma_k \in \mathbb{C}, \|(\gamma_k)\|_2 \equiv \sum |\gamma_k|^2 < \infty\}$

with inner product $\quad <(\gamma_k),(\mu_k)> \equiv \sum \gamma_k \overline{\mu_k}$.

Riesz–Fischer. The Fourier transform $f \rightsquigarrow (\gamma_k(f))_{k\in\mathbb{Z}}$

identifies the inner product spaces $L_T^2(\mathbb{R})$ and $\ell^2(\mathbb{Z})$.

In particular, $\quad (f,g) = <(\gamma_k(f)),(\gamma_k(g))> \quad (f,g \in L_T^2(\mathbb{R}))$.

Proof. $(\gamma_k) \in \ell^2(\mathbb{Z})$ then $(\sum_{|k|<n} \gamma_k\,\phi_k)_n$ Cauchy sequence in $L_T^2(\mathbb{R})$.

$\|f+\zeta g\|_2^2 = \|f\|_2^2 + 2\mathrm{Re}(\zeta(f,g)) + \|g\|_2^2$ for all $f,g \in L_T^2(\mathbb{R})$, $\zeta \in \mathbb{C}$.

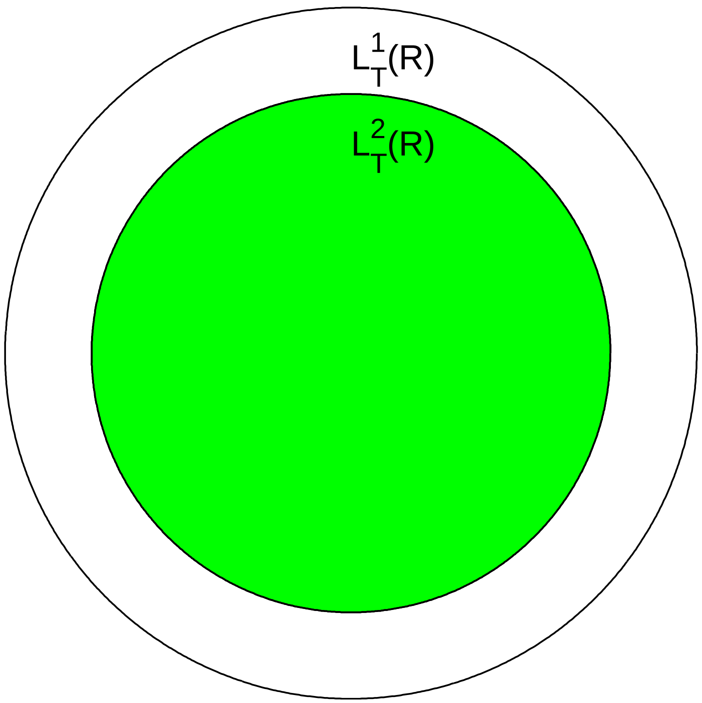
$\|(\gamma_k)+\zeta(\mu_k)\|_2^2 = \|(\gamma_k)\|_2^2 + 2\mathrm{Re}(\zeta<(\gamma_k),(\mu_k)>) + \|(\mu_k)\|_2^2$ for all . . . .

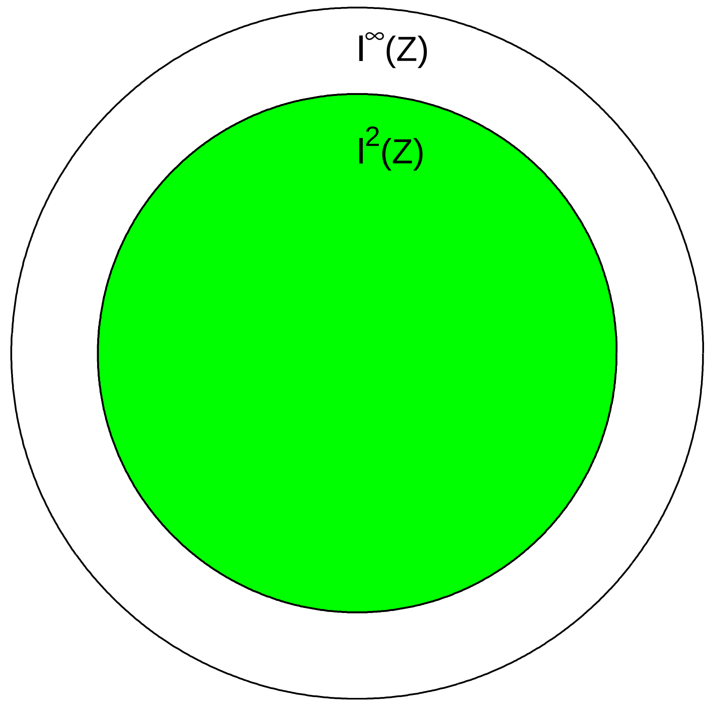
Now, apply Parseval and take $\zeta = 1$ and $\zeta = i$.

$$L^1_T(\mathbb{R}) \stackrel{\gamma(\cdot)}{\longrightarrow} \ell^\infty(\mathbb{Z}), \quad \big\|\gamma(f)\big\|_\infty \le \|f\|_1, \quad \text{not surjective}$$

$$L^2(\mathbb{R}) \stackrel{\gamma(\cdot)}{\longrightarrow} \ell^2(\mathbb{Z}), \qquad \big\|\gamma(f)\big\|_2 = \|f\|_2, \qquad \text{inversion exists.}$$

Here, $\ell^\infty(\mathbb{Z}) \equiv \{(\gamma_k) \mid \big\|(\gamma_k)\big\|_\infty < \infty\}$ and $\gamma(f) \equiv (\gamma_k(f))$.





$$L^1_T(\mathbb{R}) \overset{\gamma(\cdot)}{\longrightarrow} \ell^\infty_0(\mathbb{Z}), \quad \big|\gamma(f)\big|_\infty \leq \|f\|_1, \quad \text{not surjective}$$

$$L^2(\mathbb{R}) \overset{\gamma(\cdot)}{\longrightarrow} \ell^2(\mathbb{Z}), \qquad \big|\gamma(f)\big|_2 = \|f\|_2, \qquad \text{inversion exists.}$$

Here, $\ell^\infty_0(\mathbb{Z}) \equiv \{(\gamma_k) \in \ell^\infty(\mathbb{Z}) \,|\, \gamma_k \to 0\,(|k| \to \infty)\}$ and $\gamma(f) \equiv (\gamma_k(f))$.