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Fourier Transforms
Wavelets
Theory and Applications

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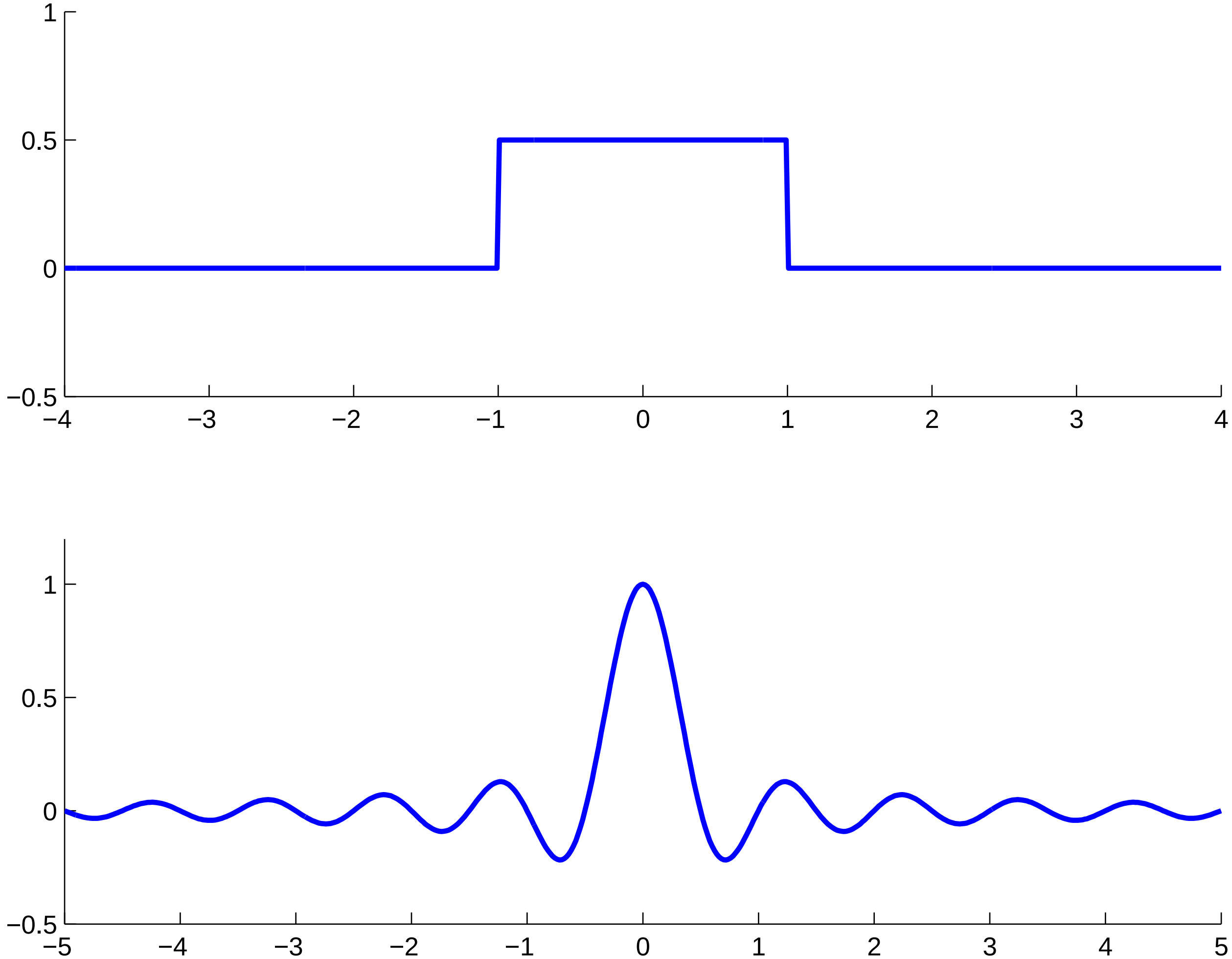
Fourier Integrals



Program

- Heuristic
- Fourier transform for L^1 functions
- Derivatives
- Fourier transform for L^2 functions
- Extensions
- Duality observations

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$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_1 \equiv \int |f(t)|\,\mathrm{d}t = \int_{-\infty}^{+\infty} |f(t)|\,\mathrm{d}t < \infty.$

To ease notation, we often drop the integration bounds, when the bounds are clear from the context:

$$\int f \equiv \int f(t)\,\mathrm{d}t \equiv \int_{-\infty}^{+\infty} f(t)\,\mathrm{d}t.$$

$L^1(\mathbb{R})$ is the space of
all functions $f : \mathbb{R} \to \mathbb{C}$ for which $\|f\|_1 < \infty$.

Similarly, $L^p(\mathbb{R}) \equiv \{f : \mathbb{R} \to \mathbb{C} \mid \|f\|_p < \infty\}$
(with [in this lecture] integration form $-\infty$ to $+\infty$).

$f:\mathbb{R}\to\mathbb{C}$ st $\|f\|_1 \equiv \int |f(t)|\,\mathrm{d}t = \int_{-\infty}^{+\infty} |f(t)|\,\mathrm{d}t < \infty.$

With $$\gamma_k^T \equiv \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-2\pi i t\frac{k}{T}}\,\mathrm{d}t \qquad (k\in\mathbb{Z}),$$

and $f\in C^{(1)}(\mathbb{R})$, $$f(t) = \sum_{k\in\mathbb{Z}} \gamma_k^T e^{2\pi i t\frac{k}{T}} \qquad (|t|<T/2)$$

(restrict f to $[-T/2, T/2]$, extend T-periodic, use Th. 2.4.b)

What happens if $T\to\infty$?

$f\colon\mathbb{R}\to\mathbb{C}$ st $\|f\|_1\equiv\int|f(t)|\,\mathrm{d}t=\int_{-\infty}^{+\infty}|f(t)|\,\mathrm{d}t<\infty.$

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and $f\in C^{(1)}(\mathbb{R})$, $$f(t)=\sum_{k\in\mathbb{Z}}\gamma_k^T e^{2\pi i t\frac{k}{T}}\quad(|t|<T/2)$$

(restrict f to $[-T/2,T/2]$, extend T-periodic, use Th. 2.4.b)

With $$\widehat{f}(\omega)\equiv\int f(t)e^{-2\pi i t\omega}\,\mathrm{d}t$$

we have that $T\gamma_k^T\approx\widehat{f}(\frac{k}{T})$. Hence, (Riemann sum)

$$f(t)\approx\sum_{k\in\mathbb{Z}}\frac{1}{T}\widehat{f}(\frac{k}{T})e^{2\pi i t\frac{k}{T}}\approx\int\widehat{f}(\omega)e^{2\pi i t\omega}\,\mathrm{d}\omega$$

Conjecture. $f(t)=\widehat{\widehat{f}}(-t)$.

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$$\boxed{\widehat{f}(\omega) \equiv \int f(t) e^{-2\pi i t \omega}\, \mathsf{d}t \qquad (\omega \in \mathbb{R})}$$

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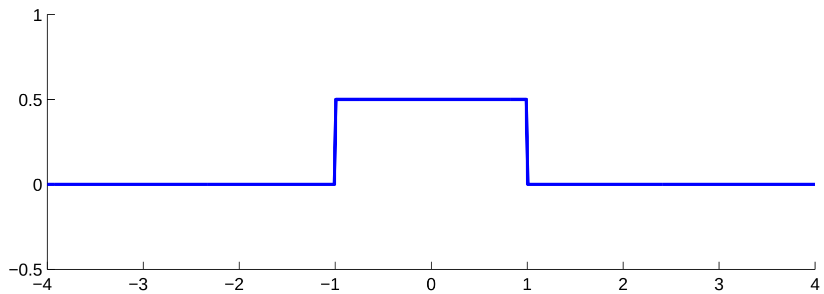
Example. For $T > 0$, $f(t) \equiv \Pi_T(t) \equiv \begin{cases} 1 \text{ if } |t| \leq T, \\ 0 \text{ if } |t| > T. \end{cases}$

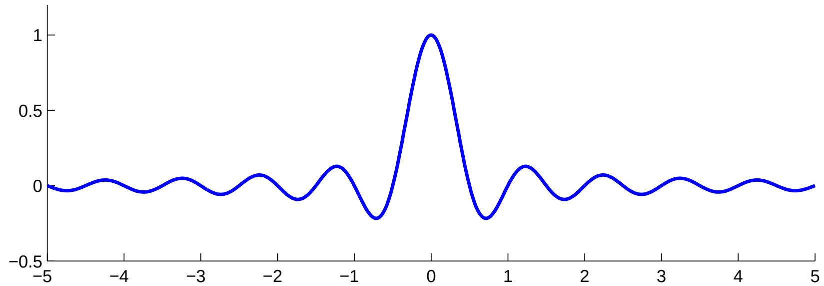
Then $\widehat{\Pi}_T(\omega) = 2T\mathrm{sinc}(2T\omega)$, where $\mathrm{sinc}(t) \equiv \frac{\sin(\pi t)}{\pi t}$.

Fourier transform

of a **tophat**



is a **sinc**.



$$f : \mathbb{R} \to \mathbb{C} \quad \text{st} \quad \|f\|_1 \equiv \int |f(t)|\,\mathrm{d}t = \int_{-\infty}^{+\infty} |f(t)|\,\mathrm{d}t < \infty$$

$$\boxed{\widehat{f}(\omega) \equiv \int f(t) e^{-2\pi i t\omega}\,\mathrm{d}t \qquad (\omega \in \mathbb{R})}$$

Example. f is the **Gaussian** $f(t) = e^{-\pi t^2}$. Then

$$\widehat{f}(\omega) = e^{-\pi\omega^2} \qquad (\omega \in \mathbb{R}).$$

Proof.

$$\widehat{f}(\omega) = \int e^{-\pi(t^2+2it\omega)}\,\mathrm{d}t = e^{-\pi\omega^2} \int e^{-\pi(t+i\omega)^2}\,\mathrm{d}t.$$

Complex function theory:

$\int_\Gamma e^{-\pi\zeta^2}\,\mathrm{d}\zeta = 0$ for each closed curved Γ in $\mathbb{C}$.

Take Γ the boundary curve of $[-T, T] \times [0, i\omega]$. Then $T \to \infty$ implies

$$\int e^{-\pi(t+i\omega)^2}\,\mathrm{d}t = \int e^{-\pi t^2}\,\mathrm{d}t = 1.$$

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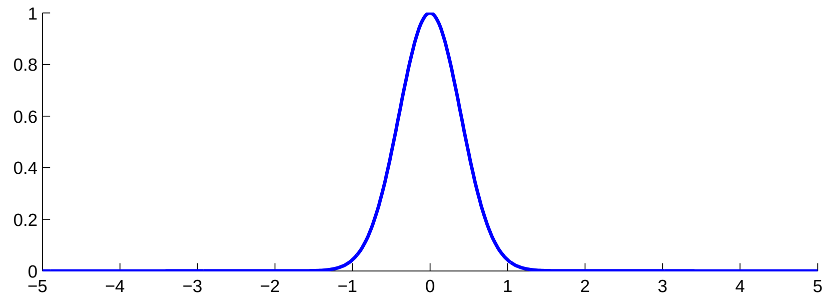
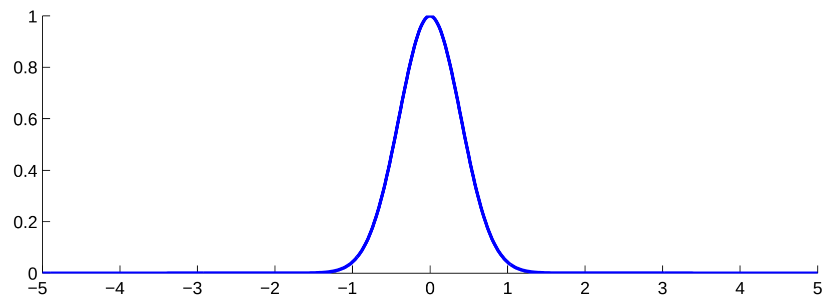
Fourier transform of a **Gaussian**



is a **Gaussian**.



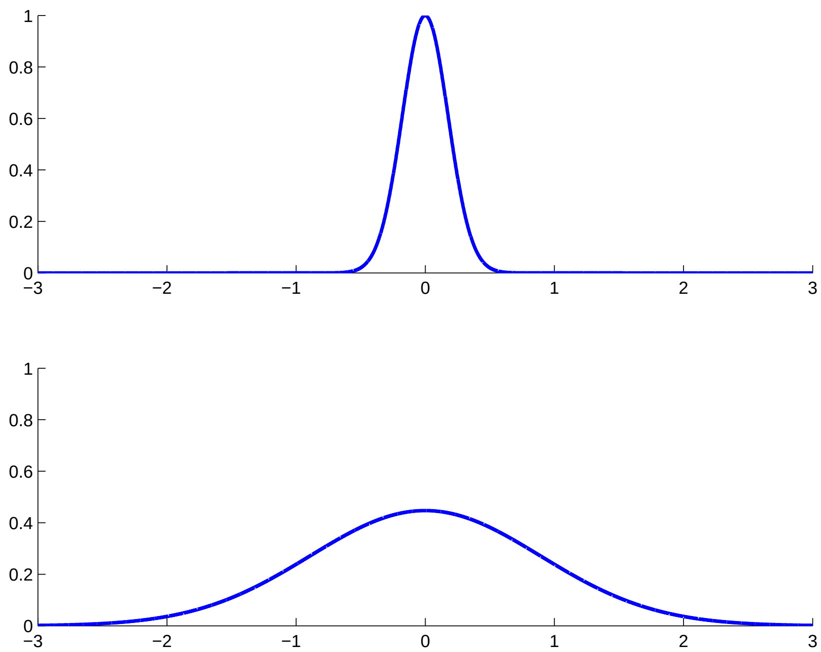
$$f:\mathbb{R}\to\mathbb{C}\ \text{ st }\ \|f\|_1\equiv\int|f(t)|\,\mathrm{d}t=\int_{-\infty}^{+\infty}|f(t)|\,\mathrm{d}t<\infty$$

$$\boxed{\widehat{f}(\omega)\equiv\int f(t)e^{-2\pi i t\omega}\,\mathrm{d}t\qquad(\omega\in\mathbb{R})}$$

Example. f is the **Gaussian** $f(t)=e^{-\alpha\pi t^2}$. Then

$$\widehat{f}(\omega)=\sqrt{\frac{1}{\alpha}}e^{-\frac{1}{\alpha}\pi\omega^2}\qquad(\omega\in\mathbb{R}).$$

Fourier transform of a **Gaussian**



is a **Gaussian**.

$$f : \mathbb{R} \to \mathbb{C} \quad \text{st} \quad \|f\|_1 \equiv \int |f(t)|\,\mathrm{d}t = \int_{-\infty}^{+\infty} |f(t)|\,\mathrm{d}t < \infty$$

$$\boxed{\widehat{f}(\omega) \equiv \int f(t) e^{-2\pi i t \omega}\,\mathrm{d}t \qquad (\omega \in \mathbb{R})}$$

Theorem. $\|f\|_1 < \infty$

- $\widehat{f}$ is **bounded**: $\|\widehat{f}\|_\infty \leq \|f\|_1$.

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Proof.

$$|\widehat{f}(\omega)| = |\int f(t)\exp(-2\pi i t\omega)\,\mathrm{d}t| \leq \int |f(t)|\,|\exp(-2\pi i t\omega)|\,\mathrm{d}t = \int |f(t)|\,\mathrm{d}t$$

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- $\widehat{f}$ is **uniformly continuous**:

$$\sup_\omega |\widehat{f}(\omega + \delta) - \widehat{f}(\omega)| \to 0 \text{ if } \delta \to 0.$$

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Proof.

$$\begin{aligned}
\widehat{f}(\omega+\delta)-\widehat{f}(\omega)| &= |\textstyle\int f(t)[e^{-2\pi it(\omega+\delta)}-e^{-2\pi it\omega}]\,\mathrm{d}t| \\
&= |\textstyle\int f(t)e^{-\pi it(2\omega+\delta)}[e^{-\pi it\delta}-e^{\pi it\delta}]\,\mathrm{d}t \\
&\leq \textstyle\int|f(t)|\,|2\sin(\pi t\delta)|\,\mathrm{d}t \\
&\leq \textstyle\int_{-T}^{T}|f(t)|\,|2\sin(\pi t\delta)|\,\mathrm{d}t+\frac{1}{2}\varepsilon\leq\varepsilon
\end{aligned}$$

Here $T>0$ is selected st $\int_{|t|>T}|f(t)|\,\mathrm{d}t<\frac{1}{2}\varepsilon$, and subsequently, $\delta>0$ is selected st $2|\sin(\pi t\delta)|\leq\varepsilon/(2\|f\|_1)$ all $t\in[-T,T]$.

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- $\widehat{f}$ **vanishes at** ∞: $\widehat{f}(\omega) \to 0$ if $|\omega| \to \infty$.

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- $\widehat{f}$ **vanishes at ∞**: $\widehat{f}(\omega) \to 0$ if $|\omega| \to \infty$.

Proof. Select an $\varepsilon > 0$. Is there an $\Omega > 0$ st $|\widehat{f}(\omega)| \leq \varepsilon$ all $|\omega| \geq \Omega$?

$$|\widehat{f}(\omega)| \leq |\widehat{(f-g)}(\omega)| + |\widehat{g}(\omega)| \leq \|f-g\|_1 + |\widehat{g}(\omega)| \quad (g \in L^1(\mathbb{R})).$$

Select $g \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ st $\|f-g\|_1 \leq \frac{1}{2}\varepsilon$ and $g(t) = 0$ if $|t| \geq T$.

If the claim is correct for g, then $\exists\Omega > 0$ st $|\widehat{g}(\omega)| \leq \frac{1}{2}\varepsilon$ if $|\omega| \geq \Omega$ and

$|\widehat{f}(\omega)| \leq \varepsilon$ if $|\omega| \geq \Omega$, which completes the proof.

Therefore, to prove claim, assume $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$, $f(t) = 0$ if $|t| \geq T$.

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Proof. $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ st $f(t) = 0$ if $|t| \geq T$.

$$\widehat{f}(\omega) = -\int f(t + \tfrac{1}{2\omega}) e^{-2\pi i t \omega}\,\mathrm{d}t \qquad (t \to t + \tfrac{1}{2\omega}, \quad e^{-\pi i} = -1)$$

$$\widehat{f}(\omega) = \tfrac{1}{2}[\widehat{f}(\omega) + \widehat{f}(\omega)] = \int \tfrac{1}{2}\left[f(t) - f(t + \tfrac{1}{2\omega})\right] e^{-2\pi i t \omega}\,\mathrm{d}t$$

$$|\widehat{f}(\omega)| \leq \tfrac{1}{2} \int \left|f(t) - f(t + \tfrac{1}{2\omega})\right|\,\mathrm{d}t$$

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Proof. $f \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ st $f(t) = 0$ if $|t| \geq T$. For $|\omega| > 1$,

$$|\widehat{f}(\omega)| \leq \tfrac{1}{2}\int \left|f(t) - f(t+\tfrac{1}{2\omega})\right|\,\mathrm{d}t = \tfrac{1}{2}\int_{-T-1}^{T+1} \left|f(t) - f(t+\tfrac{1}{2\omega})\right|\,\mathrm{d}t$$

Since f is uniformly continuous, $\exists\Omega > 0$ st $\forall|\omega| \geq \Omega$,

$$\sup_{|t|\leq T+1} \left|f(t) - f(t+\tfrac{1}{2\omega})\right| \leq \tfrac{\varepsilon}{2T+2}.$$

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$$L^1(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{C} \mid \|f\|_1 < \infty\}, \text{ norm } \|\cdot\|_1$$

$$C_\infty(\mathbb{R}) = \{g \in C(\mathbb{R}) \mid g \text{ vanishes at } \infty\}, \text{ norm } \|\cdot\|_\infty.$$

$$f \in L^1(\mathbb{R}) \;\Rightarrow\; \widehat{f} \in C_\infty(\mathbb{R}) \qquad \text{and} \qquad \|\widehat{f}\|_\infty \le \|f\|_1$$

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Theorem. If $f, f' \in L^1(\mathbb{R})$ then

$$\widehat{f'}(\omega) = 2\pi i\omega \widehat{f}(\omega) \qquad (\omega \in \mathbb{R})$$

Proof. Integrate by parts.

Theorem. If $f, f' \in L^1(\mathbb{R})$ then

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Theorem. If $f, tf \in L^1(\mathbb{R})$ then $\widehat{f} \in C^{(1)}(\mathbb{R})$ and

$$\widehat{f}^{(1)}(\omega) = -2\pi i(\widehat{tf})(\omega) \qquad (\omega \in \mathbb{R})$$

Proof. If $tf \in L^1(\mathbb{R})$ then

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\widehat{f}(\omega) = \frac{\mathrm{d}}{\mathrm{d}\omega}\int f(t)\, e^{-2\pi i t\omega}\,\mathrm{d}t = \int f(t)\,\frac{\partial}{\partial\omega}e^{-2\pi i t\omega}\,\mathrm{d}t.$$

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support f is **bounded** by T if $f(t) = 0$ all $|t| > T$.

Corollary. $f \in L^1(\mathbb{R})$ with support bounded by T, then

$$\widehat{f} \in C^{(\infty)}(\mathbb{R}), \quad \|\widehat{f}^{(n)}\|_\infty \leq (2\pi T)^n \|f\|_1 \quad (n \in \mathbb{N}_0)$$

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Proof. $t^n f \in L^1(\mathbb{R})$. Apply the last theorem inductively.

Corollary. $f \in L^1(\mathbb{R})$ with support bounded by T, then $\widehat{f}$ is **analytic** on $\mathbb{R}$, i.e., $\widehat{f} \in C^{(\infty)}(\mathbb{R})$ and

$$\widehat{f}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{k!} \widehat{f}^{(k)}(0) \quad (\omega \in \mathbb{R}).$$

To be precise,
with (Taylor's theorem on Taylor series)

$$\widehat{f}(\omega) = \sum_{k=0}^{n-1} \frac{\omega^k}{k!} \widehat{f}^{(k)}(0) + \frac{\omega^n}{n!} \widehat{f}^{(n)}(\xi)$$

for some ξ in between 0 and ω,
we have that

$$|\frac{\omega^n}{n!} \widehat{f}^{(n)}(\xi)| \leq \frac{(2\pi T\omega)^n}{n!} \|f\|_1 \to 0 \quad \text{if } n \to \infty.$$

Applications

- **Differential equations.**
- **Insight** Smoothness f relates to decrease $\widehat{f}$ at ∞
- **New concept of derivative.**

Differential equations.

See exercises.

Insight

First note that

$$f, tf, t^2 f, \ldots, t^n f \in L^1(\mathbb{R}) \quad \Leftrightarrow \quad (1+|t|)^n f \in L^1(\mathbb{R}).$$

Therefore,

$$(1+|t|)^n f \in L^1(\mathbb{R}), \;\text{ then }\; \widehat{f} \in C^{(k)}(\mathbb{R}) \text{ for } k = 0, \ldots, n.$$

$$f, f', \ldots, f^{(n)} \in L^1(\mathbb{R}), \;\text{ then }\; (1+|\omega|)^n \widehat{f} \text{ bounded.}$$

- *'Size' of f at ∞ determines smoothness of $\widehat{f}$.*
- *Smoothness of f determines 'size' of $\widehat{f}$ at ∞.*

$\widehat{\cdot}$ *identifies $L^2(\mathbb{R})$ with $L^2(\mathbb{R})$ (see later):*
'size' of f at ∞ corresponds to smoothness of $\widehat{f}$.

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$\widehat{\cdot}$ *identifies $L^2(\mathbb{R})$ with $L^2(\mathbb{R})$ (see later):*
'size' of $\widehat{f}$ at ∞ corresponds to smoothness of f.

New concept of derivative.

For the moment (see later), assume that

$$\widehat{\cdot} \text{ identifies } L^2(\mathbb{R}) \text{ with } L^2(\mathbb{R}).$$

If $(1+|\omega|)^n \widehat{f} \in L^2(\mathbb{R})$ then, $\forall\, k = 0, \ldots, n$, $\omega^k \widehat{f} \in L^2(\mathbb{R})$ and $\exists\, g \in L^2(\mathbb{R})$ st $\widehat{g} = (2\pi i \omega)^k \widehat{f}$. Denote $f^{(k)} \equiv g$.

Consistent. If $f, \ldots, f^{(k)} \in L^1(\mathbb{R})$, then $g = f^{(k)}$.

Let $\gamma > 0$. Suppose $(1+|\omega|)^\gamma \widehat{f} \in L^2(\mathbb{R})$.
Then, $\exists g \in L^2(\mathbb{R})$ st $\widehat{g} = (2\pi i \omega)^\gamma \widehat{f}$. Denote $f^{(\gamma)} \equiv g$.
$f^{(\gamma)}$ is a **pseudo (or fractional) derivative** of f.

$$H^{(\gamma)} \equiv \{ f \mid (1+|\omega|)^\gamma \widehat{f} \in L^2(\mathbb{R}) \}$$

is the **Sobolev space** of order γ.

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$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \,\mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

Note that $f_n \equiv f \Pi_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \qquad (n \in \mathbb{N})$.

Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2$.

$f:\mathbb{R}\to\mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2\,\mathrm{d}t} < \infty$: $f\in L^2(\mathbb{R})$.

Note that $f_n \equiv f\Pi_n \in L^2(\mathbb{R})\cap L^1(\mathbb{R}) \qquad (n\in\mathbb{N})$.

Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2$.

Proof. For $L>2n$, consider the restriction of f_n to $[-L/2, L/2]$, and its L-periodic extension.

$$L\gamma_k(f_n) = \int_{-L/2}^{L/2} f_n(t)e^{-2\pi i t\frac{k}{L}}\,\mathrm{d}t = \int f_n(t)e^{-2\pi i t\frac{k}{L}}\,\mathrm{d}t = \widehat{f}_n(\frac{k}{L}).$$

Apply Parceval to see that

$$\|f_n\|_2^2 = L\,\frac{1}{L}\int_{-L/2}^{L/2}|f_n(t)|^2\,\mathrm{d}t = L\sum_{k=-\infty}^{\infty}\frac{1}{L^2}\left|\widehat{f}_n(\frac{k}{L})\right|^2.$$

The limit for $L\to\infty$ exists and equals $\|f_n\|_2^2$.

Since $\widehat{f}_n$ is uniformly continuous, we also have that

$$\sum_{k=-\infty}^{\infty}\frac{1}{L}\left|\widehat{f}_n(\frac{k}{L})\right|^2 \to \int |\widehat{f}_n(\omega)|^2\,\mathrm{d}\omega = \|\widehat{f}_n\|_2^2 \quad (L\to\infty).$$

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \,\mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

Note that $f_n \equiv f \Pi_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \qquad (n \in \mathbb{N})$.

Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2$.

Along the same line (using Th.2.4.a) we have

$$f_n(t) = \int \widehat{f_n}(\omega)\, e^{+2\pi i t \omega} \,\mathrm{d}\omega$$

in some L^2-sense.

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

Note that $f_n \equiv f \Pi_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \qquad (n \in \mathbb{N})$.

Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2 \qquad \& \qquad (\widehat{f}_n)$ is Cauchy in $L^2(\mathbb{R})$.

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

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Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2 \quad \& \quad (\widehat{f}_n)$ is Cauchy in $L^2(\mathbb{R})$.

Proof.

$$\|\widehat{f}_n - \widehat{f_m}\|_2 = \|f_n - f_m\|_2 \to 0 \quad (n > m \to \infty).$$

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

Note that $f_n \equiv f \Pi_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \qquad (n \in \mathbb{N})$.

Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2$ & $(\widehat{f}_n)$ is Cauchy in $L^2(\mathbb{R})$.

$$\exists g \in L^2(\mathbb{R}) \text{ st } \|\widehat{f_n} - g\|_2 \to 0 \quad (n \to \infty).$$

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

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Proposition. $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \quad \Rightarrow \quad g = \hat{f}$.

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

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Proof. $\|\widehat{f_n} - \widehat{f}\|_\infty \le \|f_n - f\|_1 \to 0$ if $n \to \infty$ implies $\|g - \widehat{f}\|_2 = 0$.

$f:\mathbb{R}\to\mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2\,\mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

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Proposition. $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \quad \Rightarrow \quad g = \widehat{f}$.

Proof. $\|\widehat{f}_n - \widehat{f}\|_\infty \le \|f_n - f\|_1 \to 0$ if $n \to \infty$ implies $\|g - \widehat{f}\|_2 = 0$:

to be more precise, with $\|h\|_{2,T} \equiv \sqrt{\int_{-T}^{T} |h(\omega)|^2\,\mathrm{d}\omega}$,

$$\begin{aligned}
\|g - \widehat{f}\|_{2,T} &\le \|g - \widehat{f}_n\|_{2,T} + \|\widehat{f}_n - \widehat{f}\|_{2,T} \\
&\le \|g - \widehat{f}_n\|_2 + \sqrt{2T}\|\widehat{f_n - f}\|_\infty \\
&\le \|g - \widehat{f}_n\|_2 + \sqrt{2T}\|f_n - f\|_1.
\end{aligned}$$

For $n \to \infty$, this shows that $\|g - \widehat{f}\|_{2,T} = 0$ for all $T > 0$.
Therefore,

$$\|g - \widehat{f}\|_2 = \lim_{T\to\infty} \|g - \widehat{f}\|_{2,T} = 0.$$

$f:\mathbb{R}\to\mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2\,\mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

Note that $f_n \equiv f\Pi_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \qquad (n \in \mathbb{N})$.

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$$\exists g \in L^2(\mathbb{R}) \text{ st } \|\widehat{f_n} - g\|_2 \to 0 \quad (n \to \infty).$$

Proposition. $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \quad \Rightarrow \quad g = \widehat{f}$.

Definition. $\widehat{f} \equiv g$. **Plancherel.** $\|\widehat{f}\|_2 = \|f\|_2$.

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

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Proof. $\|\widehat{f}\|_2 = \|g\|_2 = \lim \|\widehat{f_n}\|_2 = \lim \|f_n\|_2 = \|f\|_2$.

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

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Definition. $\widehat{f} \equiv g$. **Plancherel.** $\|\widehat{f}\|_2 = \|f\|_2$.

Corollary. $(f, g) = (\widehat{f}, \widehat{g}) \quad (f, g \in L^2(\mathbb{R}))$.

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

Note that $f_n \equiv f \Pi_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \quad (n \in \mathbb{N})$.

Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2 \quad \& \quad (\widehat{f_n})$ is Cauchy in $L^2(\mathbb{R})$.

$$\exists g \in L^2(\mathbb{R}) \text{ st } \|\widehat{f_n} - g\|_2 \to 0 \quad (n \to \infty).$$

Proposition. $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \;\Rightarrow\; g = \widehat{f}$.

Definition. $\widehat{f} \equiv g$. **Plancherel.** $\|\widehat{f}\|_2 = \|f\|_2$.

Note. If (f_n) in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ converges to f in $L^2(\mathbb{R})$, then $(\widehat{f_n})$ converges to g in $L^2(\mathbb{R})$: in other words, definition $\widehat{f}$ independent of selected approx. in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \, \mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

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Definition. $\widehat{f} \equiv g$. **Plancherel.** $\|\widehat{f}\|_2 = \|f\|_2$.

Note. Usually

$$\widehat{f}(\omega) = \lim_{\Omega \to \infty} \int_{-\Omega}^{\Omega} f(t) e^{-2\pi i t \omega} \, \mathrm{d}t \qquad \text{for almost all } \omega \in \mathbb{R}.$$

Therefore,

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \,\mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

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Proposition. $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \ \Rightarrow \ g = \widehat{f}$.

Definition. $\widehat{f} \equiv g$. **Plancherel.** $\|\widehat{f}\|_2 = \|f\|_2$.

For $f \in L^2(\mathbb{R})$, we also put $\widehat{f}(\omega) = \int f(t) e^{-2\pi i t\omega} \,\mathrm{d}\omega$
(for ease of notation).

$f:\mathbb{R}\to\mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2\,\mathrm{d}t} < \infty$: $f\in L^2(\mathbb{R})$.

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Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2$ & $(\widehat{f_n})$ is Cauchy in $L^2(\mathbb{R})$.

$$\exists g\in L^2(\mathbb{R}) \text{ st } \|\widehat{f_n}-g\|_2 \to 0 \quad (n\to\infty).$$

Proposition. $f\in L^2(\mathbb{R})\cap L^1(\mathbb{R}) \;\Rightarrow\; g=\widehat{f}$.

Definition. $\widehat{f}\equiv g$. **Plancherel.** $\|\widehat{f}\|_2 = \|f\|_2$.

For $f\in L^2(\mathbb{R})$, we also put $\widehat{f}(\omega) = \int f(t)e^{-2\pi i t\omega}\,\mathrm{d}\omega$.

Theorem. $f\in L^2(\mathbb{R})$ then $\|\widehat{f}\|_2 = \|f\|_2$, and

$$\widehat{f}(\omega) = \int f(t)e^{-2\pi i t\omega}\,\mathrm{d}\omega, \qquad f(t) = \int \widehat{f}(\omega)\,e^{+2\pi i t\omega}\,\mathrm{d}\omega.$$

Interpretation. $f(t) = \int \widehat{f}(\omega)\, e^{2\pi i t \omega}\, \mathrm{d}\omega$:

f is a superposition of harmonic oscilations:

with $$\widehat{f}(\omega) = |\widehat{f}(\omega)|\, e^{2\pi i \phi(\omega)},$$

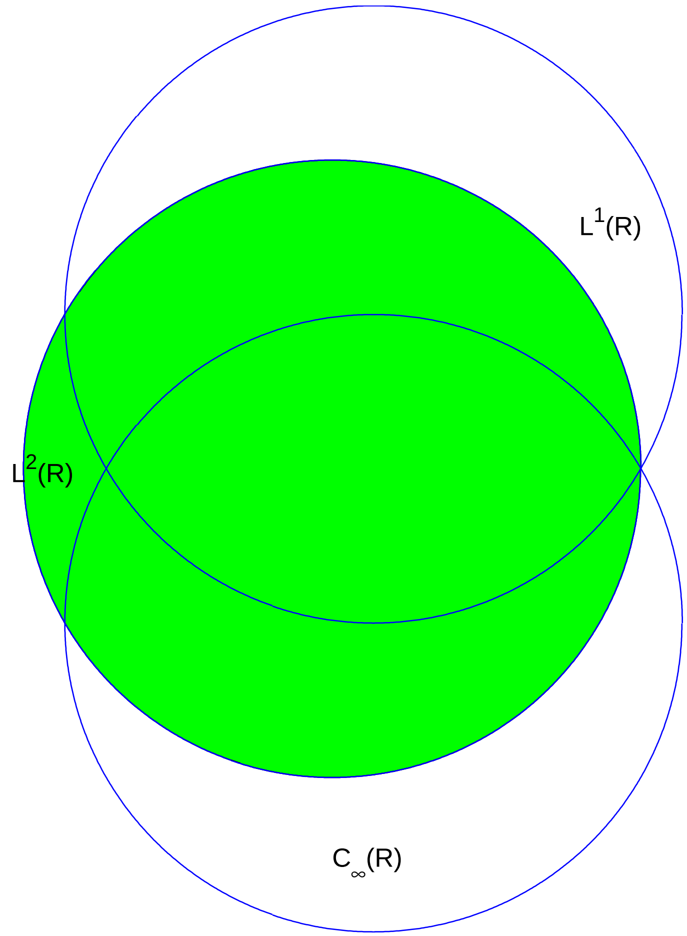
$|\widehat{f}(\omega)|$ is the **amplitude** of the oscilation
with **frequency** ω,
$\phi(\omega)$ is the **phase**.

De Fourier transform $\widehat{f}$ is also denoted by $\mathcal{F}(f)$:

(Plancherel:) $\mathcal{F}$ is a linear operator and a norm preserving bijection from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$.

$$L^1(\mathbb{R}) \xrightarrow{\widehat{\ }} C_\infty(\mathbb{R}), \quad \|\widehat{f}\|_\infty \le \|f\|_1, \quad \text{not surjective}$$

$$L^2(\mathbb{R}) \xrightarrow{\widehat{\ }} L^2(\mathbb{R}), \quad \|\widehat{f}\|_2 = \|f\|_2, \quad \text{inversion exists.}$$



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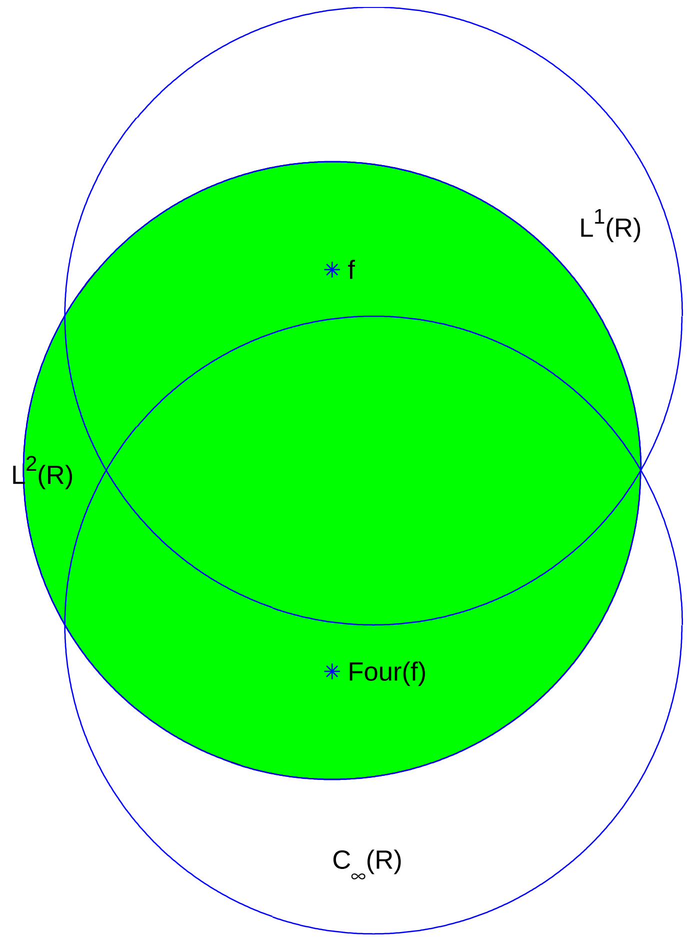

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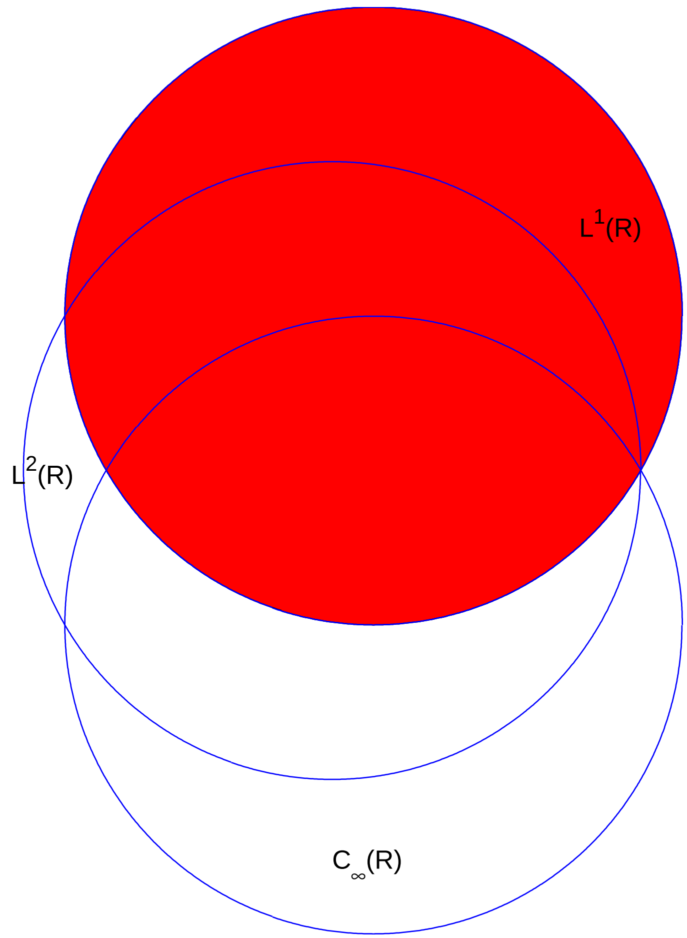

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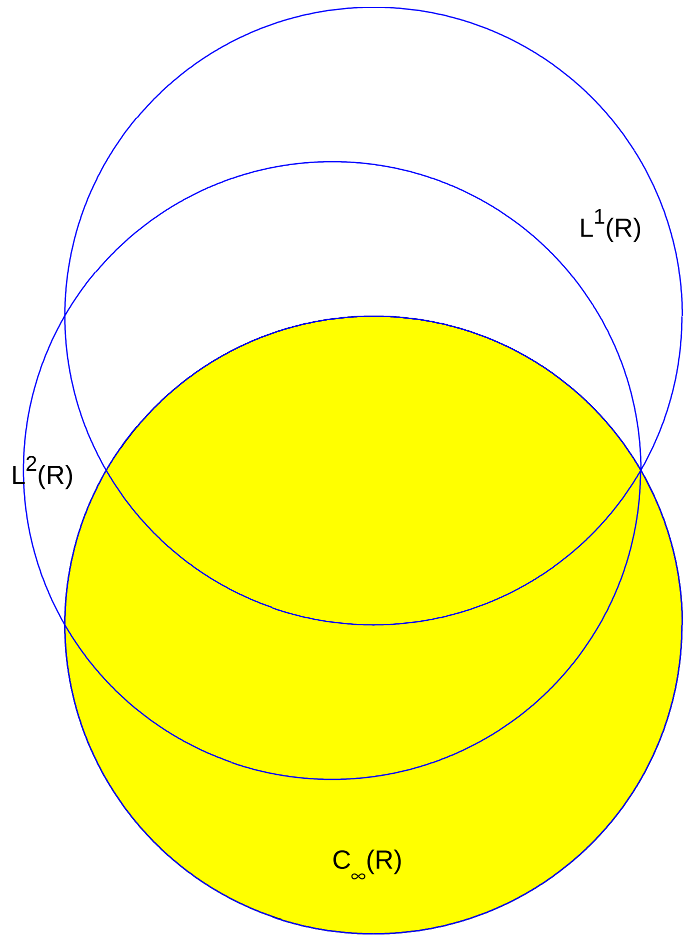

$$L^1(\mathbb{R}) \xrightarrow{\widehat{}} C_\infty(\mathbb{R}), \quad \|\widehat{f}\|_\infty \le \|f\|_1, \quad \text{not surjective}$$

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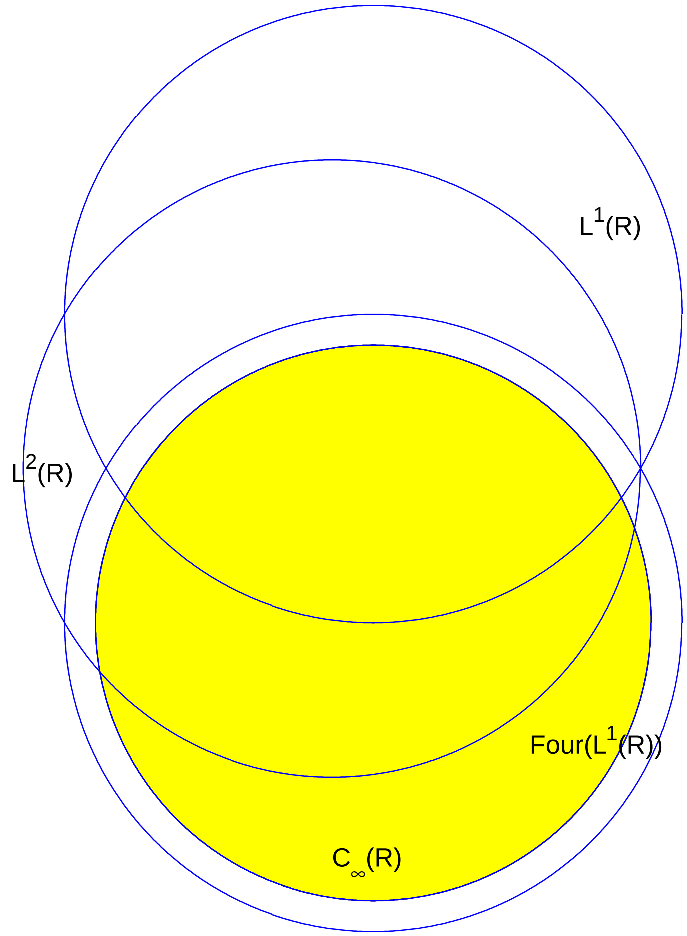

Program

- Heuristic
- Fourier transform for L^1 functions
- Derivatives
- Fourier transform for L^2 functions
- Extensions
- Duality observations

Interpretation. $f(t)=\int \widehat{f}(\omega)\,e^{2\pi i t\omega}\,\mathrm{d}\omega$:

f is a superposition of harmonic oscilations:

with $$\widehat{f}(\omega)=|\widehat{f}(\omega)|\,e^{2\pi i t\phi(\omega)},$$

> $|\widehat{f}(\omega)|$ is the **amplitude** of the oscilation
> with **frequency** ω,
> $\phi(\omega)$ is the **phase**.

Let $\nu\in\mathbb{R}$ be a frequency. Can the function ϕ_ν, with

$$\phi_\nu(t)\equiv e^{2\pi i t\nu}\qquad (t\in\mathbb{R})$$

be viewed as a superposition of harmonic oscilations?

The Dirac δ function

$$e^{2\pi i t \nu} = \int \delta_\nu(\omega)\, e^{2\pi i t \omega}\, \mathrm{d}\omega \qquad (t \in \mathbb{R})$$

Here δ_ν is the **Dirac δ function** or **point measure** at ν defined by the following two properties:

$$\delta_\nu(\omega) = 0 \ \text{ for all } \omega \neq \nu \quad \text{and}$$

$$\int \delta_\nu(\omega)\, g(\omega)\, \mathrm{d}\omega = g(\nu) \quad (g \in C(\mathbb{R})).$$

δ_ν can be view as some **weak limit** of, e.g., $\frac{1}{2\varepsilon}\Pi_\varepsilon$ for $\varepsilon \to 0$.

In some sense $\widehat{\phi_\nu} = \delta_\nu$ and $\phi_\nu(t) = \widehat{\delta_\nu}(-t)$.

Application of the Dirac δ-function.

Suppose f is $C^{(1)}$ on both $(-\infty,\tau)$ and (τ,∞) and $f(\tau+)$ and $f(\tau-)$ exists. Then, with $\alpha\equiv f(\tau+)-f(\tau-)$,

$$f(t)=f(0)+\int_0^t\Big(f'(s)+\alpha\delta_\tau(s)\Big)\,\mathrm{d}s\qquad (t\in\mathbb{R}).$$

The function $f'+\alpha\delta_\tau$ can be viewed as the derivative of f.

Exercise. Consider the approximate derivatives $\partial_{\Delta t}f$:

$$\partial_{\Delta t}f(t)\equiv\frac{f(t+\Delta t)-f(t-\Delta t)}{2\Delta t}.$$

Show that the behaviour for $(\partial_{\Delta t}f)$ for $\Delta t\to 0$ is consistent with the point of view that $f'+\alpha\delta_\tau$ is the derivative of f and the definition of δ_τ. Pay special attention to t's for which $\tau\in(t-\Delta t,t+\Delta t)$

Application of the Dirac δ-function.

Exercise. For $\lambda \in \mathbb{C}$, Re$(\lambda) \neq 0$,

consider the differential equation

$$f'(t) = \lambda f(t) \ (t \in \mathbb{R}, t \neq 0), \quad f(0-) = 0, \ f(0+) = 1$$

- Solve this eq. for an $f \in L^2(\mathbb{R})$ (if exist).
- Is the eq. equivalent to

$$f \in L^2(\mathbb{R}) \quad \text{st} \quad f' = \lambda f + \delta_0$$

- Use Fourier transform to show that

$$\widehat{f}(\omega) = \frac{1}{2\pi i\omega - \lambda} \quad (\omega \in \mathbb{R})$$

- Discuss the situation for Re$(\lambda) < 0$ and Re$(\lambda) > 0$.

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$f \in L^2(\mathbb{R})$.

Energy:

$$E \equiv \int |f(t)|^2 \, \mathrm{d}t = \int |\widehat{f}(\omega)|^2 \, \mathrm{d}\omega.$$

Energy center:

$$t_0 \equiv \frac{1}{E} \int t |f(t)|^2 \, \mathrm{d}t, \qquad \omega_0 \equiv \frac{1}{E} \int \omega |\widehat{f}(\omega)|^2 \, \mathrm{d}\omega.$$

Spread:

$$\sigma_t^2 \equiv \frac{1}{E} \int (t - t_0)^2 |f(t)|^2 \, \mathrm{d}t, \quad \sigma_\omega^2 \equiv \frac{1}{E} \int (\omega - \omega_0)^2 |\widehat{f}(\omega)|^2 \, \mathrm{d}\omega.$$

Heisenberg uncertainty principle.

$$\sigma_t \sigma_\omega \geq \frac{1}{4\pi}.$$

$$\sigma_t \sigma_\omega = \frac{1}{4\pi} \;\Leftrightarrow\; f(t) = c\, e^{\gamma (t - t_0)^2} \; (t \in \mathbb{R})$$

Duality

f	$\Rightarrow$	$\widehat{f}$
real		even
even		real
smooth		rapid decrease at ∞
rapid decrease at ∞		smooth
localized		spread out