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Fourier Transforms
Wavelets
Theory and Applications

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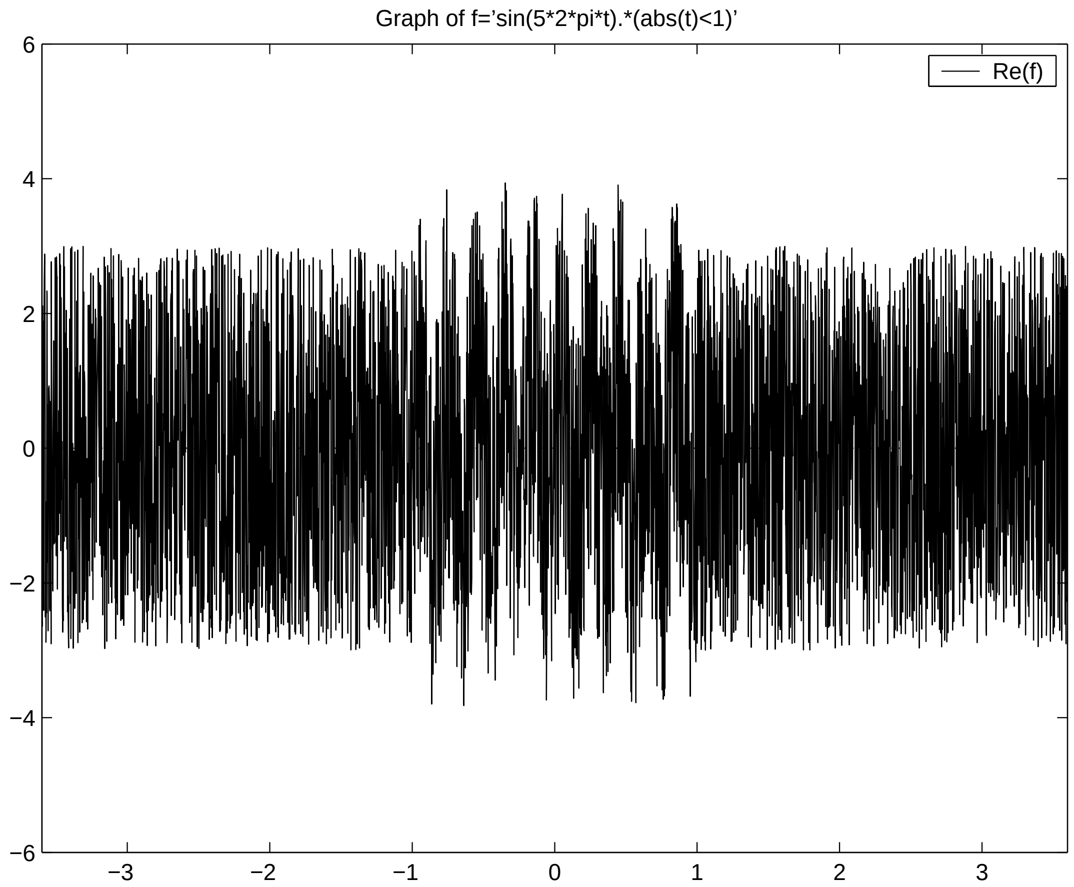
Convolution products



$\widetilde{f} \equiv f + n/\delta$. In picture: $s = 0$,

reflected signal is scaled.

Program

- Convolution products
- Correlation
- Radar

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Well-defined if $f \in L^1(\mathbb{R}), h \in L^\infty(\mathbb{R})$

$f \in L^2(\mathbb{R}), h \in L^2(\mathbb{R})$ (apply Cauchy-Schwartz)

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Intermezzo.

$f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})$, where $p, q \in [1, \infty]$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$

Here,

$$\|f\|_p \equiv \left(\int |f(t)|^p \,\mathrm{d}t \right)^{\frac{1}{p}}$$

Then

$$|(f, g)| \leq \|f\|_q \, \|g\|_q,$$

$$\|f\|_p = \sup\{|(f, g)| \mid q \text{ s.t } \|g\|_q \leq 1\}$$

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Well-defined if $f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})$ $(p, q \in [1, \infty],\ \frac{1}{p} + \frac{1}{q} = 1)$

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Then $\|f * h\|_\infty \leq \|f\|_p \, \|h\|_q$ and $f * h$ is uniformly continuous.

Proof. Suppose $p < \infty$.
Consider (f_n) in $L^p(\mathbb{R})$

each f_n is continuous, bounded and of bounded support

such that $\|f_n - f\|_p \to 0$ if $n \to \infty$.

Then $f_n * h$ is continuous and $\|f_n * h - f * h\|_\infty \to 0$ if $n \to \infty$.

$$f * h(t) = \int_{-\infty}^{+\infty} f(t-s)h(s)\,\mathrm{d}s$$

Well-defined if $f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})$ $(p, q \in [1, \infty],\ \frac{1}{p} + \frac{1}{q} = 1)$

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Convolution products tend to make functions smoother.
This is exploited in many applications.

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Application. Narrow Gaussians,

$$h(x, y) = \frac{1}{\delta}\, e^{-\pi \frac{1}{\delta^2}(x^2+y^2)},$$

are used to smooth ('**denoise**') pictures:

$$f_{\text{denoised}} = (f + n) * h$$

(using the 2d-variant of the convolution product).

The idea here is the $n * h \approx 0$ if n is noise.

$$f * h(t) = \int_{-\infty}^{+\infty} f(t-s)h(s)\,\mathsf{d}s$$

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Application. The solution of the heat equation on a bounded spatial domain can be expressed as the convolution of a Gaussian and the initial conditions.
In 1-d, on $\mathbb{R}$ with constant diffusion coefficient γ: **[Ex.6.14]**

$$u(x,t) = \phi * h_t,$$

where $u(\cdot, 0) = \phi \in L^2(\mathbb{R})$ and

$$h_t(x) \equiv \frac{1}{2\sqrt{\gamma t}}\, e^{-\pi(\frac{x}{2\sqrt{\gamma t}})}$$

$$f * h(t) = \int_{-\infty}^{+\infty} f(t-s)h(s)\,\mathrm{d}s$$

Well-defined if $f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R}) \quad (p, q \in [1, \infty], \frac{1}{p} + \frac{1}{q} = 1)$

Then $\|f * h\|_\infty \leq \|f\|_p \, \|h\|_q$ and $f * h$ is uniformly continuous.

Discussion. The smoothing ('melting') effect of spreading heat (modelled by the heat equation) is exploited in techniques to denoise pictures. The denoised picture is obtained by solving the heat equation at time t_0:

$$f_{\text{denoised}} = u(\cdot, t_0) \quad \text{with} \quad u(\cdot, 0) = \phi = f + n$$

To avoid '**blurring**' (i.e., to maintain sharp lines), the head equation is extended with advection type of terms.

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Then $\|f * h\|_\infty \leq \|f\|_p \, \|h\|_q$ and $f * h$ is uniformly continuous.

Application. With $B_0(x) \equiv 1$ if $x \in [0, 1]$, $B_0(x) \equiv 0$ elsewhere, define

$$B_k \equiv B_0 * B_{k-1} \qquad (k \in \mathbb{N})$$

Theorem. For all $k \in \mathbb{N}$: [Ex.6.5]

- $B_k \in C^{(k-1)}(\mathbb{R})$
- On $[j, j+1]$, B_k is a polynomial of degree k $(j \in \mathbb{Z})$.
- $B_k(x) > 0$ for all $x \in (0, k+1)$
- $B_k(x) = 0$ for all $x \notin (0, k+1)$.

The B_k are **basis splines** or **Box splines** of degree k.

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Shifted version $B_k(\cdot - j)$ $(j \in \mathbb{Z})$ form a basis of the space of all **splines** of degree $\leq k$, i.e., functions $f \in C^{(k-1)}(\mathbb{R})$ that are polynomials on each interval $[j, j+1]$.

$$f * h(t) = \int_{-\infty}^{+\infty} f(t-s)h(s)\,\mathrm{d}s$$

Well-defined if $f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})$ $(p, q \in [1, \infty], \frac{1}{p} + \frac{1}{q} = 1)$

Then $\|f * h\|_\infty \leq \|f\|_p \|h\|_q$ and $f * h$ is uniformly continuous.

Notation. $f_s(t) \equiv f(t-s)$; s is a **delay**. $f^\top(t) = \overline{f(-t)}$.

Prop. $f * h(t) = (h, f_t^\top)$, $\|f\|_p = \|\overline{f}\|_p = \|f_t\|_p = \|f^\top\|_p$.

$$(f * h, g) = (h, f^\top * g).$$

$$f * h(t) = \int_{-\infty}^{+\infty} f(t-s)h(s)\,\mathrm{d}s$$

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$$(f * h, g) = (h, f^\top * g).$$

$f \in L^p, h \in L^1$. Taking the sup over all $g \in L^q$, $\|g\|_q \le 1$,

$$\begin{aligned}
\|f * h\|_p &= \sup |(f * h, g)| = \sup |(h, f^\top * g)| \\
&\le \sup \|h\|_1\,\|f^\top * g\|_\infty \\
&\le \sup \|h\|_1\,\|f^\top\|_p\,\|g\|_q = \|h\|_1\,\|f\|_p
\end{aligned}$$

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Prop. $f * h(t) = (h, f_t^\mathsf{T})$, $\quad \|f\|_p = \|\overline{f}\|_p = \|f_t\|_p = \|f^\mathsf{T}\|_p$.

$$(f * h, g) = (h, f^\mathsf{T} * g).$$

$f \in L^p, h \in L^1$. Then $\|f * h\|_p \leq \|h\|_1 \, \|f\|_p$.

$$f * h(t) = \int_{-\infty}^{+\infty} f(t-s)h(s)\,\mathrm{d}s$$

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$$(f * h, g) = (h, f^\top * g).$$

$f \in L^p, h \in L^1$. Then $\|f * h\|_p \le \|h\|_1 \, \|f\|_p$.

Proof. Correct if $f \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

If $f \in L^p$, define $f_n(t) \equiv f(t)$ if $|f(t)| < n$ and $|t| < n$ and

$f_n(t) \equiv 0$ elsewhere.

Then, $\|f_n - f\|_p \to 0$, $f_n \in L^p(\mathbb{R}) \cap L^\infty(\mathbb{R})$, and $\|f_n * h - f_m * h\|_p \to 0$.

Hence, limit exist in $L^p(\mathbb{R})$: $f * h \equiv \lim_{n\to\infty} f_n * h$.

Theorem. $f, h \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$. Then

$$\widehat{(f * h)} = \widehat{f} \cdot \widehat{h}$$

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Proof. (sketch)

$$\begin{aligned}
\widehat{f * h}(\omega) &= \int f * h(t)\, e^{-2\pi i \omega t}\, \mathrm{d}t \\
&= \int \int f(t-s)\, h(s)\, e^{-2\pi i t \omega}\, \mathrm{d}t\, \mathrm{d}s \\
&= \int \int f(t-s)\, h(s)\, e^{-2\pi i (t-s)\omega}\, e^{-2\pi i s \omega}\, \mathrm{d}t\, \mathrm{d}s \\
&= \int \left(\int f(t-s)\, e^{-2\pi i (t-s)\omega}\, \mathrm{d}t \right) h(s)\, e^{-2\pi i s \omega}\, \mathrm{d}s \\
&= \int \widehat{f}(\omega)\, h(s)\, e^{-2\pi i s \omega}\, \mathrm{d}s \\
&= \widehat{f}(\omega)\, \widehat{h}(\omega)
\end{aligned}$$

Theorem. $f, h \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$. Then

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Application. In some applications a known (learning) signal f is transmitted, the signal $f*h$ with unknown 'blurring' function h is received.
In other applications, the blurring function h is known and $f * h$ is received with f some unknown signal.
In both examples, a signal, say f has to be reconstructed from a received (known) signal $f * h$ with h known.

Theorem. $f, h \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$. Then

$$\widehat{(f * h)} = \widehat{f} \cdot \widehat{h}$$

Application. h and $f * h$ are known. Construct f.

Solution. In principle

$$\widehat{f} = \frac{\widehat{f * h}}{\widehat{h}}.$$

Discussion. The received signal $f * h$ (and h?) will be affected by noise: received $f * h + n$. On average, the noise n will have (equally large) components in all frequencies, while h will be concentrated in a frequency interval J around a certain frequency ω_0: $J \equiv \{\omega \mid \big| |\omega| - |\omega_0| \big| < \delta\}$. If $\omega \notin J$, then $\widehat{h}(\omega) = 0$. Therefore, the above approach is unstable.

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Remedy. Use a filter (see next lecture) to remove frequencies in $f * h + n$ outside J

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Remedy. **(Tikhonov) Regularise**: for some appropriate **regularisation parameter** τ (which one?)

$$f^{\mathsf{r}} \equiv \mathrm{armin}_g \left(\|g * h - [f * h + n]\,\|_2^2 + \tau \|g\|_2^2 \right)$$

(and combine with filtering).

Program

- Convolution products
- Correlation
- Radar

Application.

$$(h, f_t) = \int \overline{f(s-t)}\, h(s)\, \mathrm{d}s = f^{\top} * h(t)$$

The map $f \odot h(t) \equiv (h, f_t)$
is called the **correlation product** of f and h:

it tests how much h is correlated to a shifted variant of f.

Note that the correlation product is the *adjoint* of the convolution product:

$$(f * g, h) = (g, f^{\top} * h).$$

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$$(h, f_t) = \int \overline{f(s-t)}\, h(s)\, \mathrm{d}s = f^{\mathsf{T}} * h(t)$$

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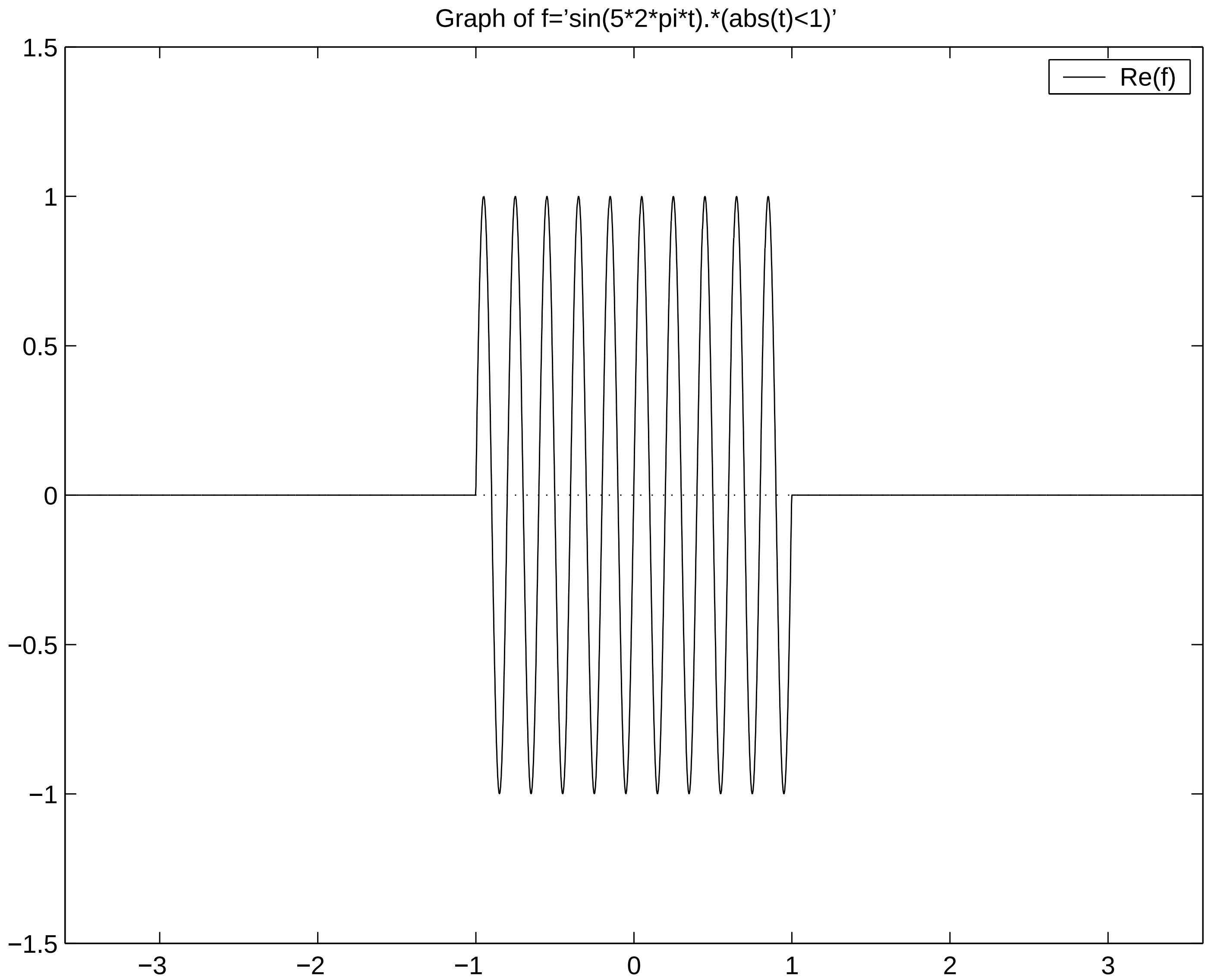
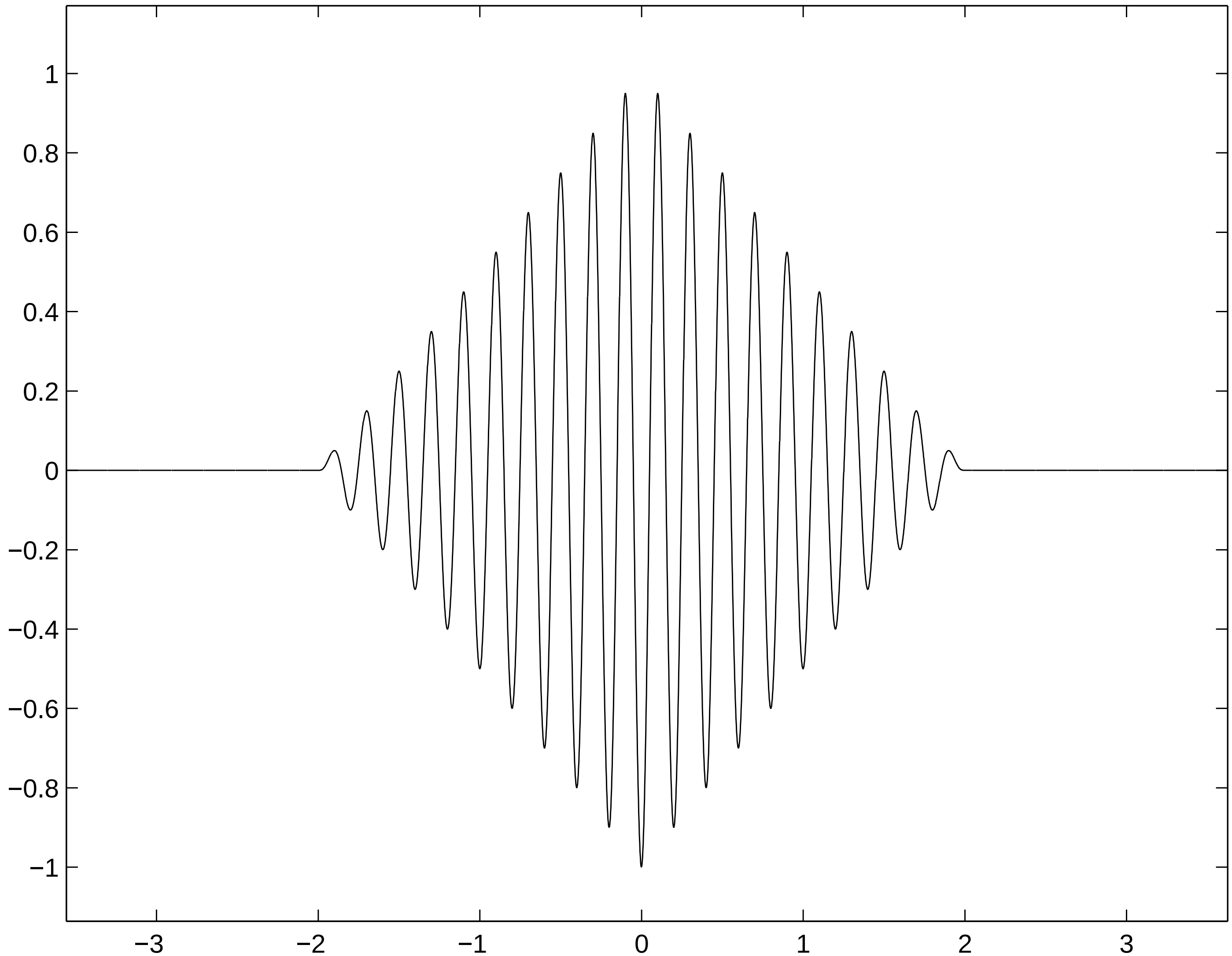
Wiener-Khintchini Theorem.

$$(f \odot h)^{\wedge} = \widehat{h}\, \overline{\widehat{f}}, \qquad (f \odot f)^{\wedge} = |\widehat{f}|^2.$$



f



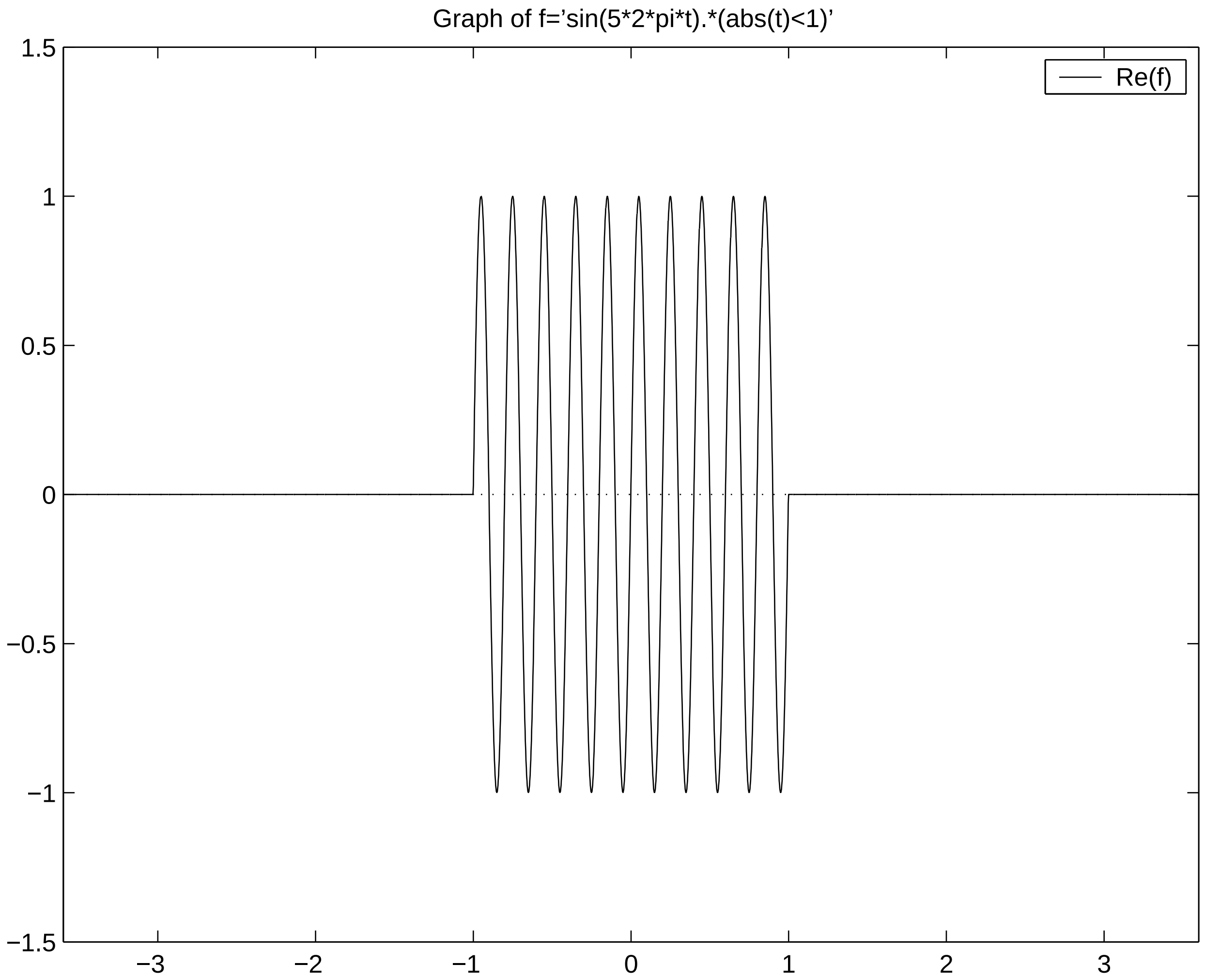
Auto correlation function $t \rightsquigarrow (f, f_t)$ for f.

Program

- Convolution products
- Correlation
- Radar

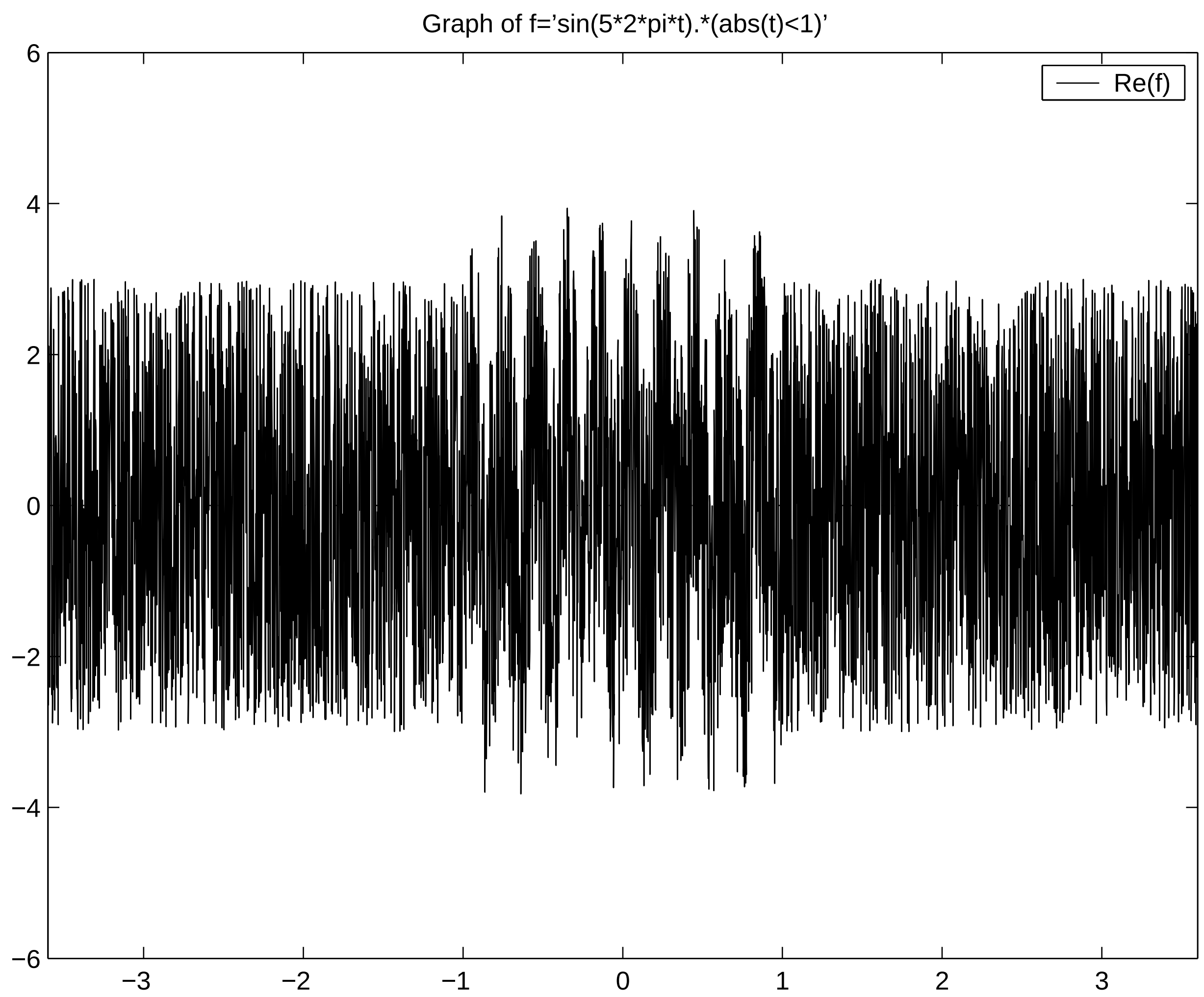
Application: Radar

Radar



The radar signal f is a short sine pulse.

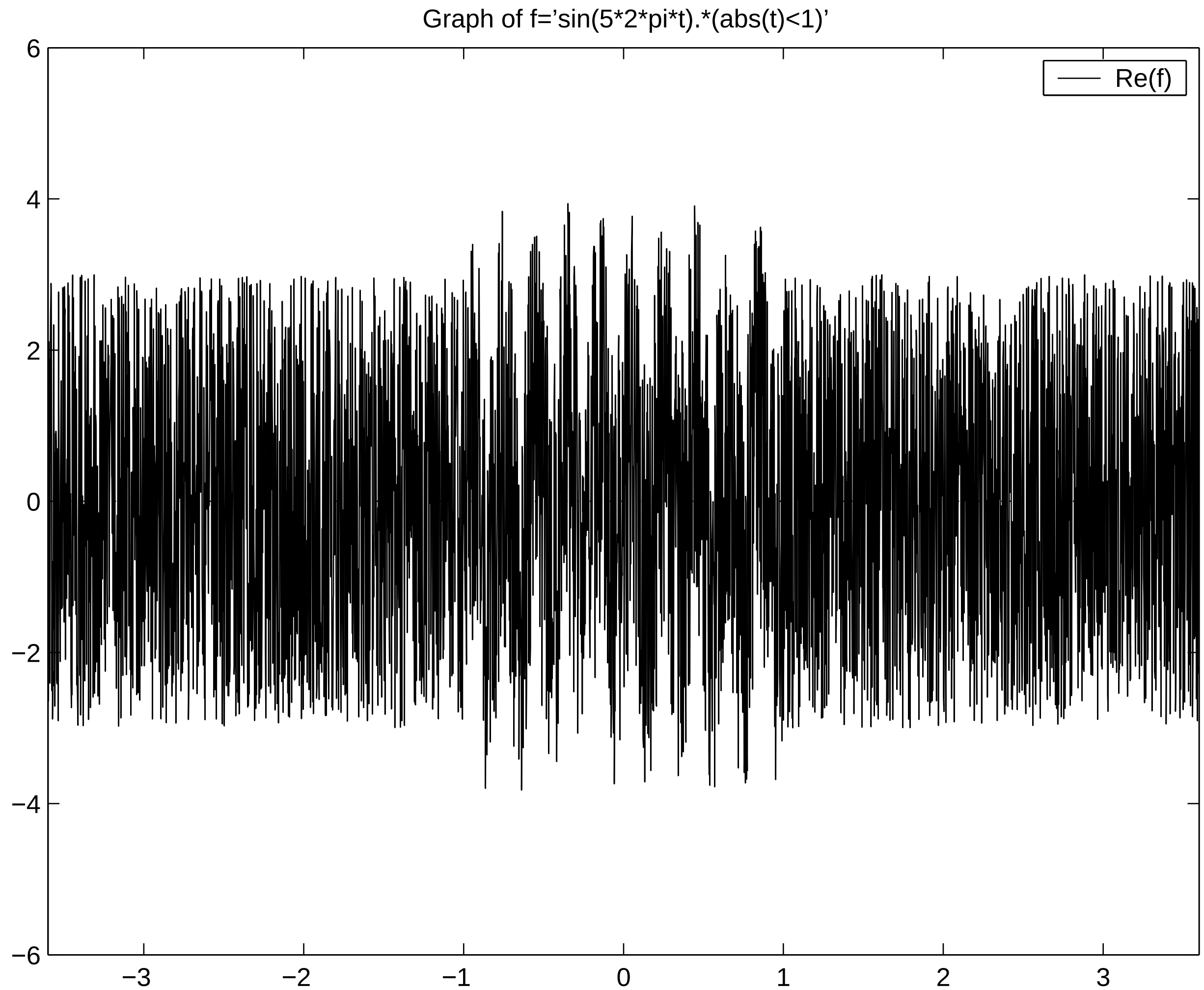
Radar



$\delta f_s + n$: the reflected signal arrives with delay s.

Radar: find s.

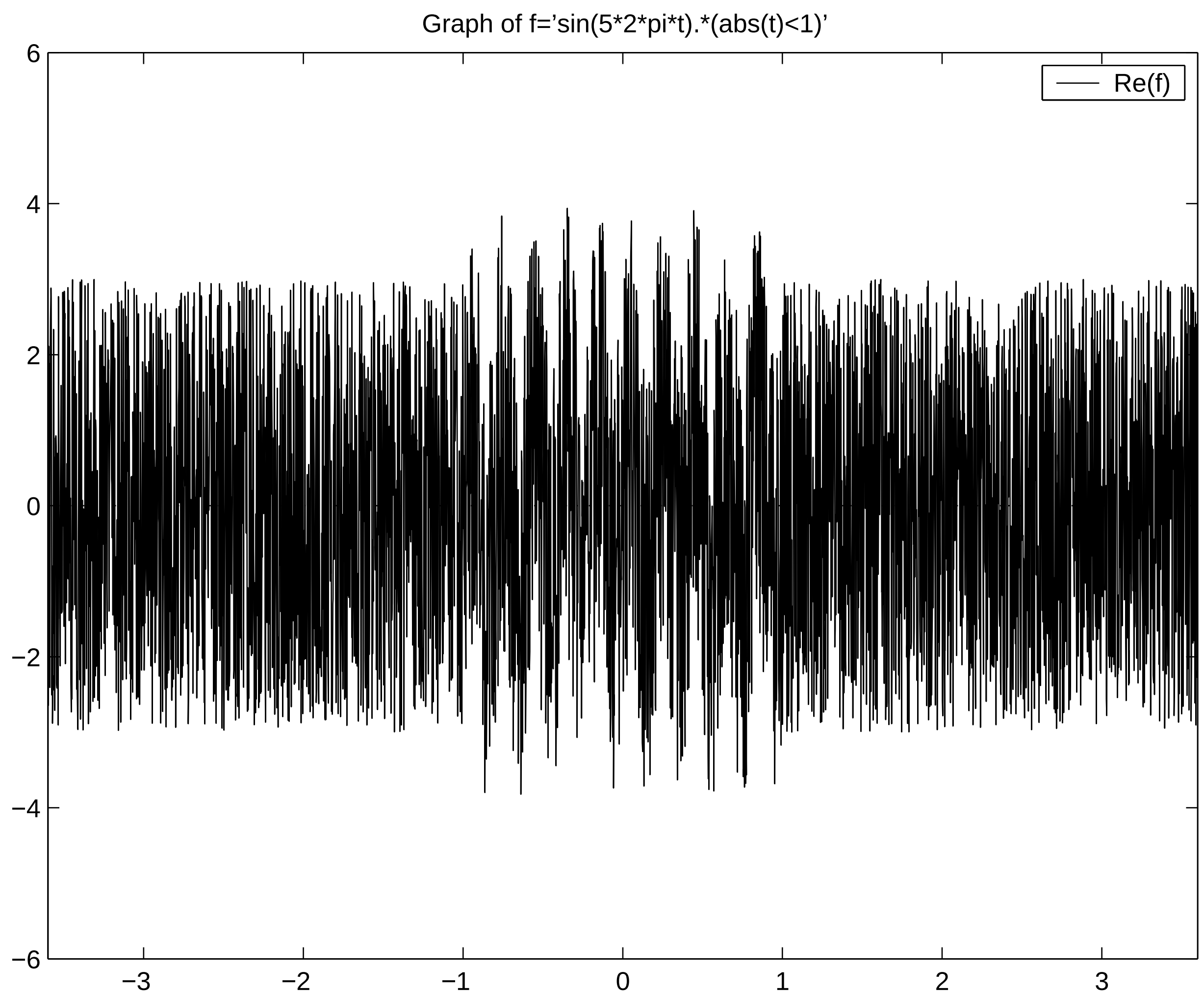
Radar



$\delta f_s + n$: the reflected signal is weakened by δ.

the reflected signal is polluted by noise n.

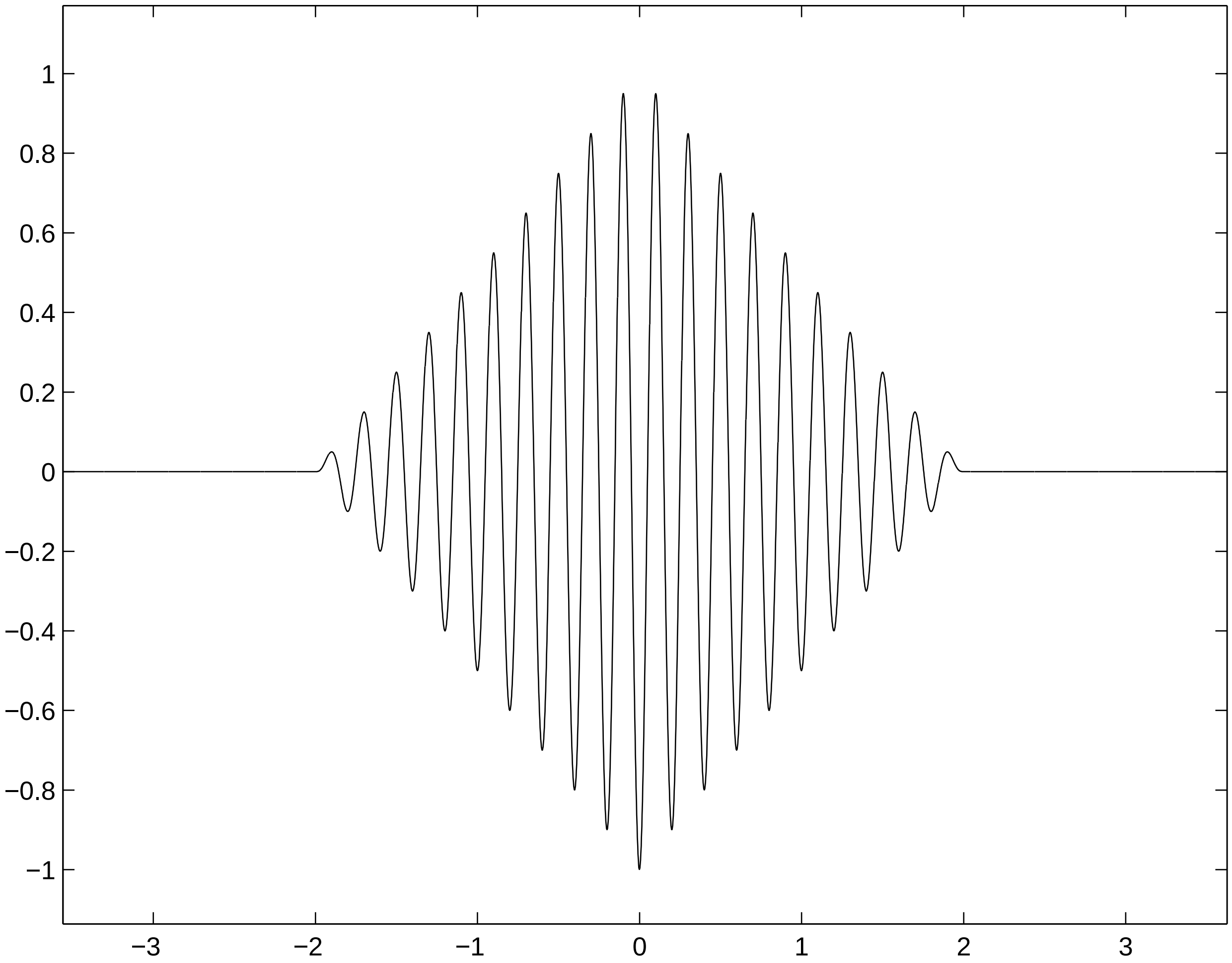
Radar



$\widetilde{f} \equiv f + n/\delta$. In picture: $s = 0$,

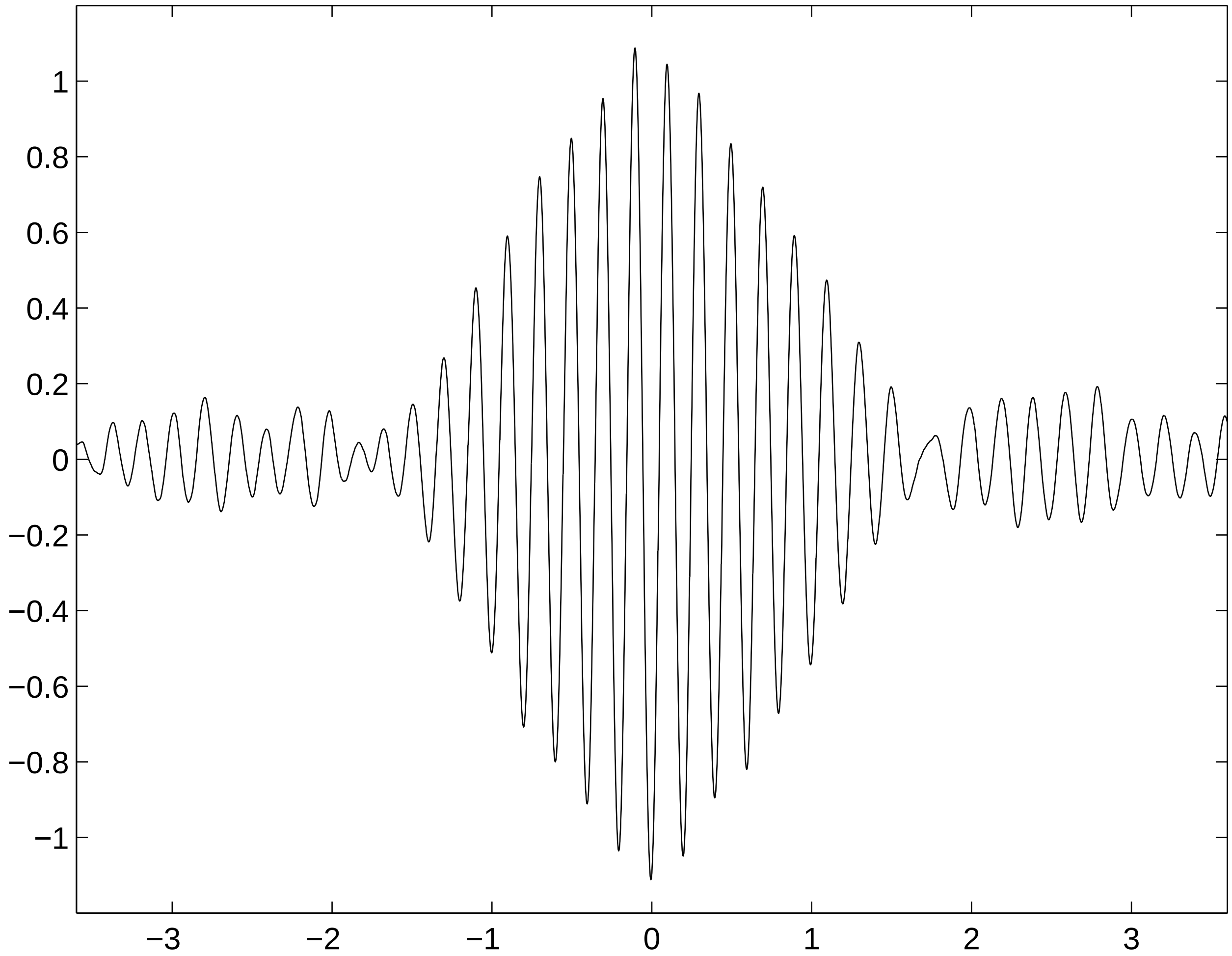
reflected signal is scaled.

Radar



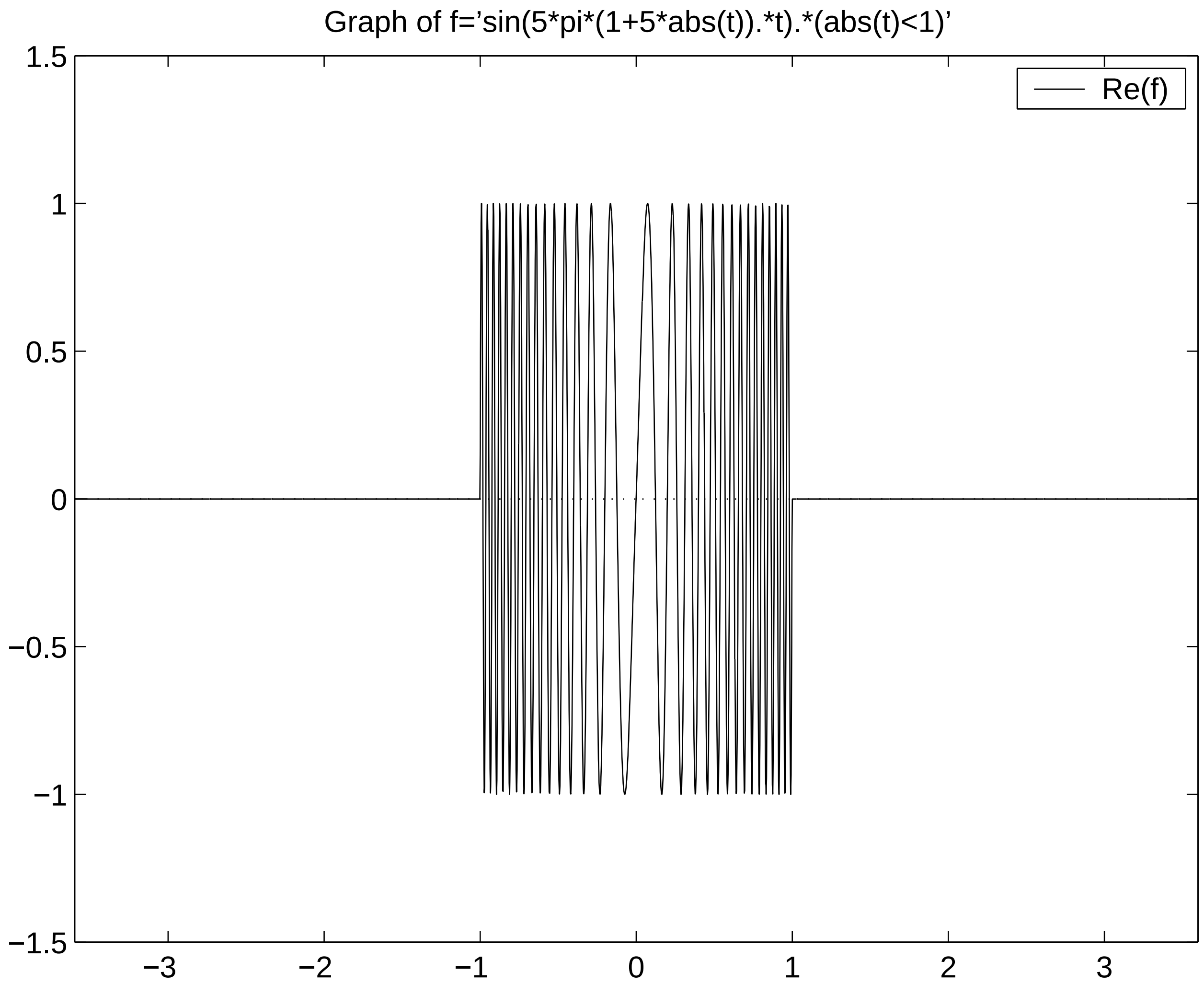
Auto correlation function $t \rightsquigarrow (f, f_t)$ for f.

Radar



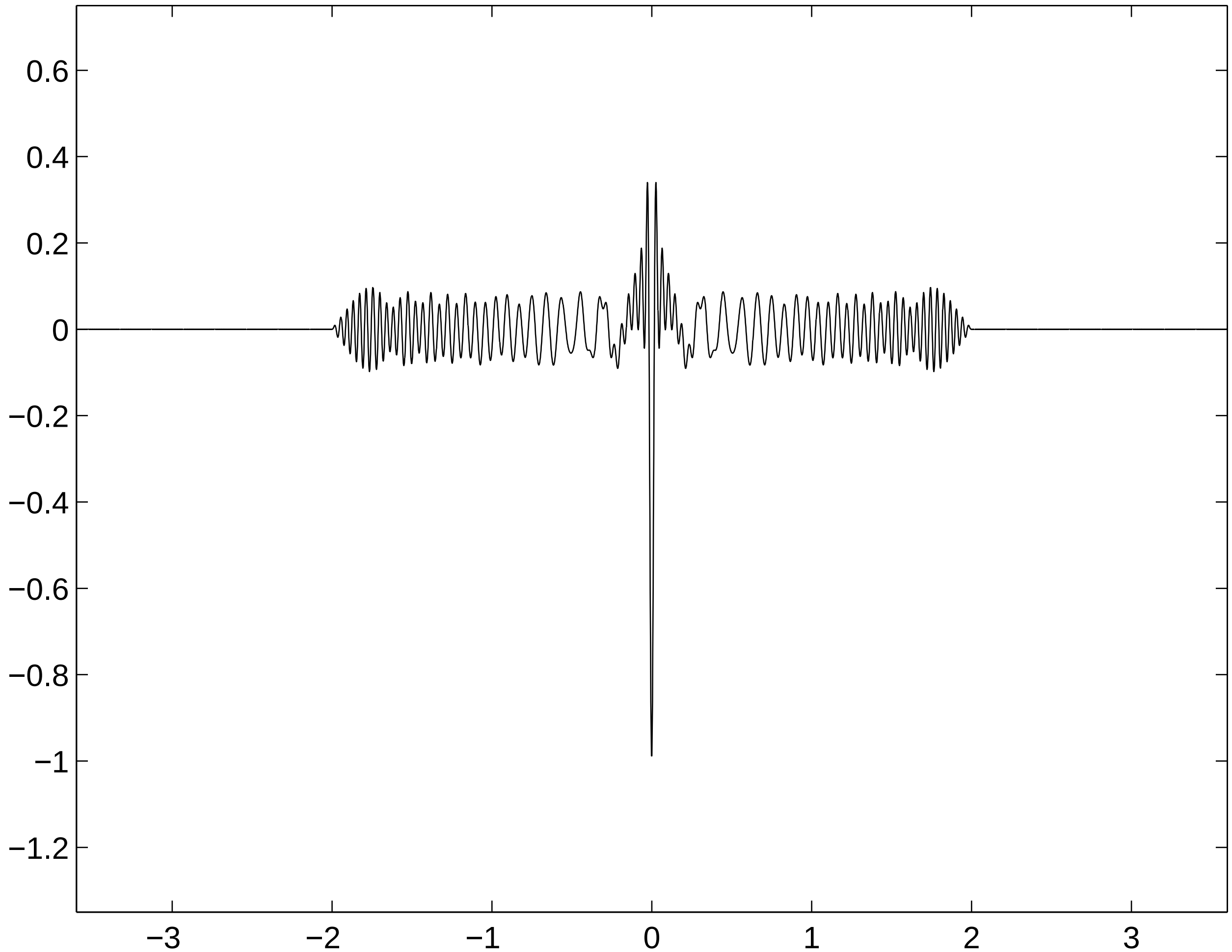
Polluted signal tested against the pure signal: $t \rightsquigarrow (\widetilde{f}, f_t)$

Radar



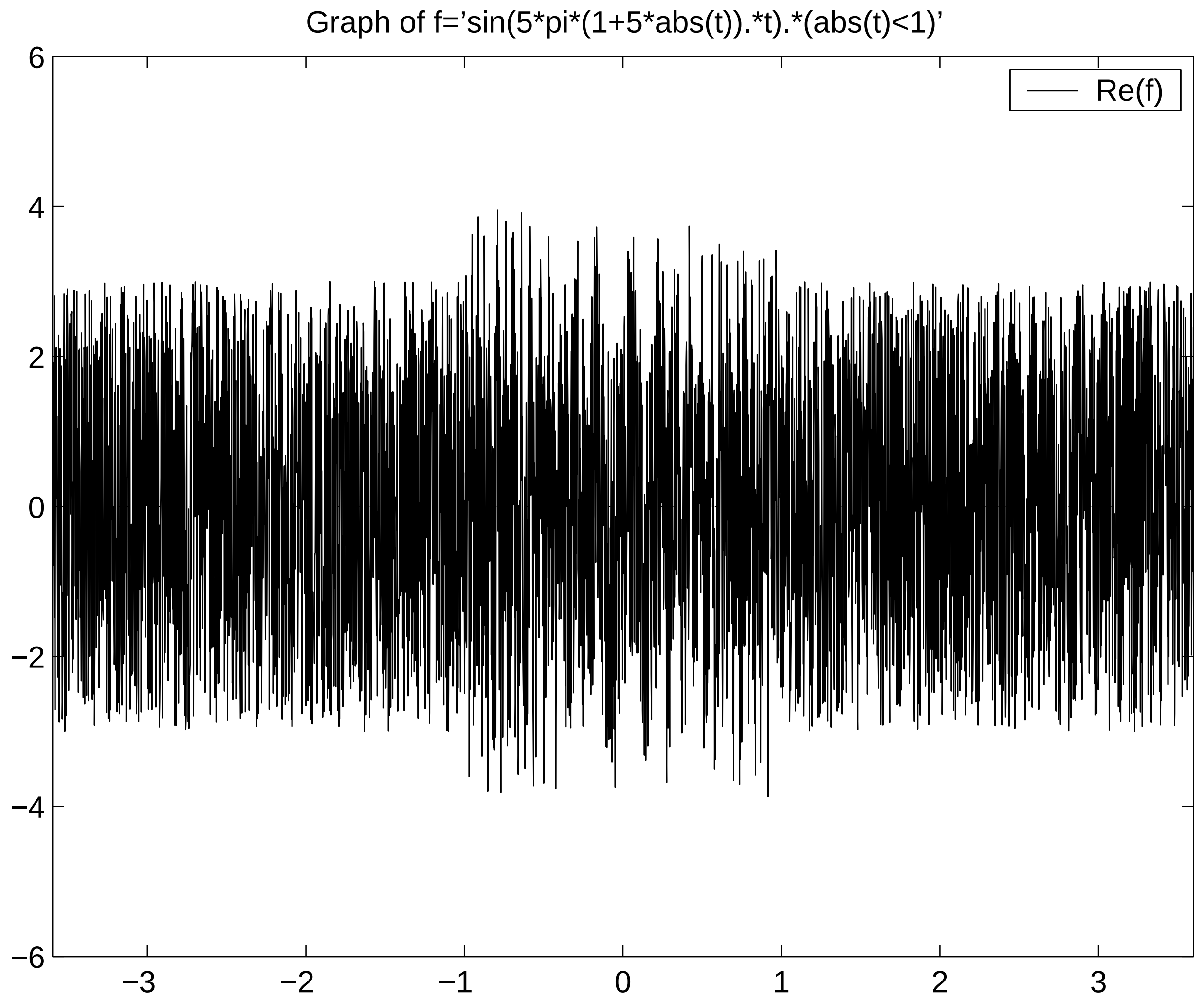
Radar signal f is a **chirp**.

Radar



Auto correlation function $t \rightsquigarrow (f, f_t)$ of the chirp.

Radar



$$\widetilde{f} \equiv f + n/\delta$$

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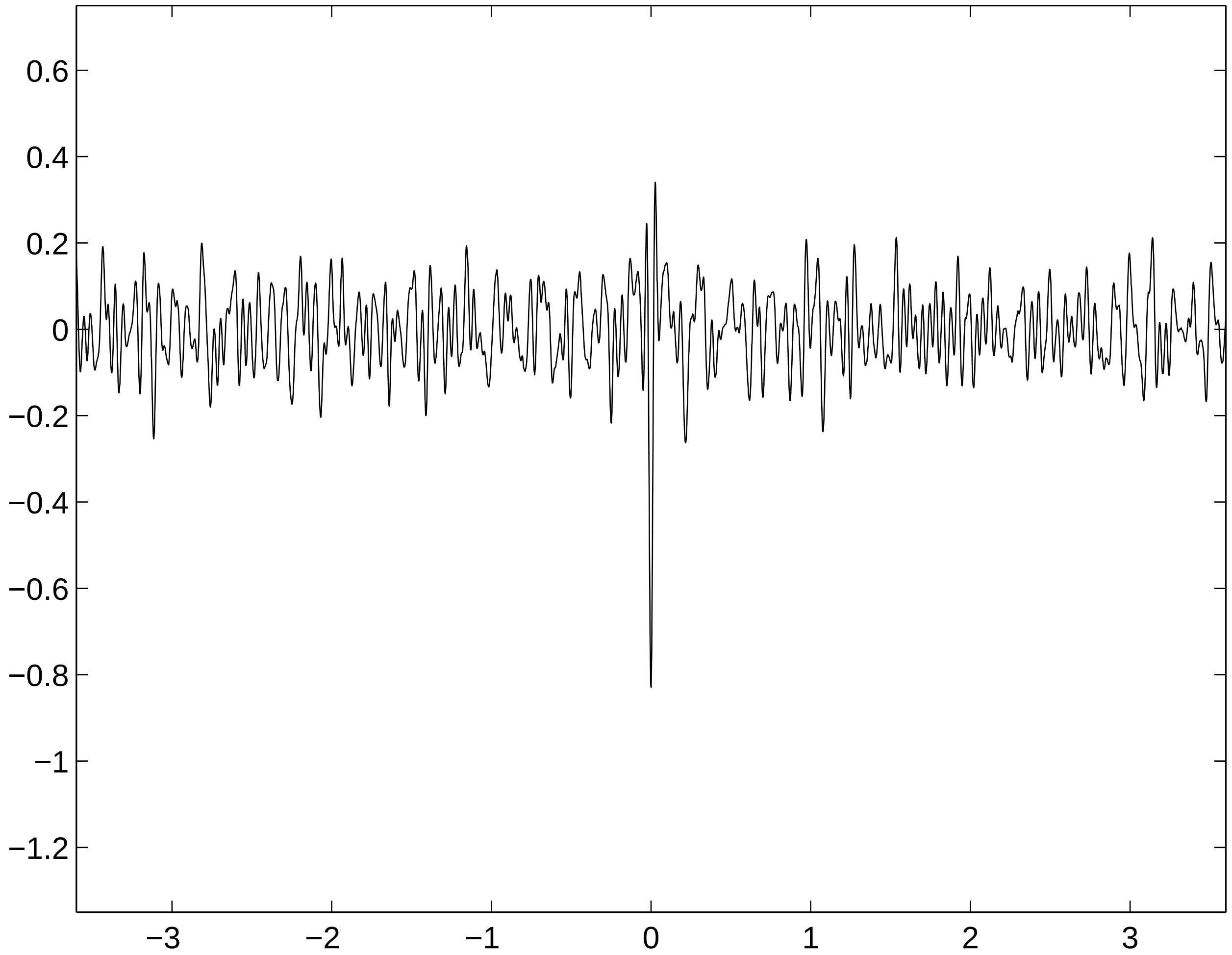

$$t \rightsquigarrow (\widetilde{f}, f_t)$$