Scientific Computing, Utrecht, March 24, 2014

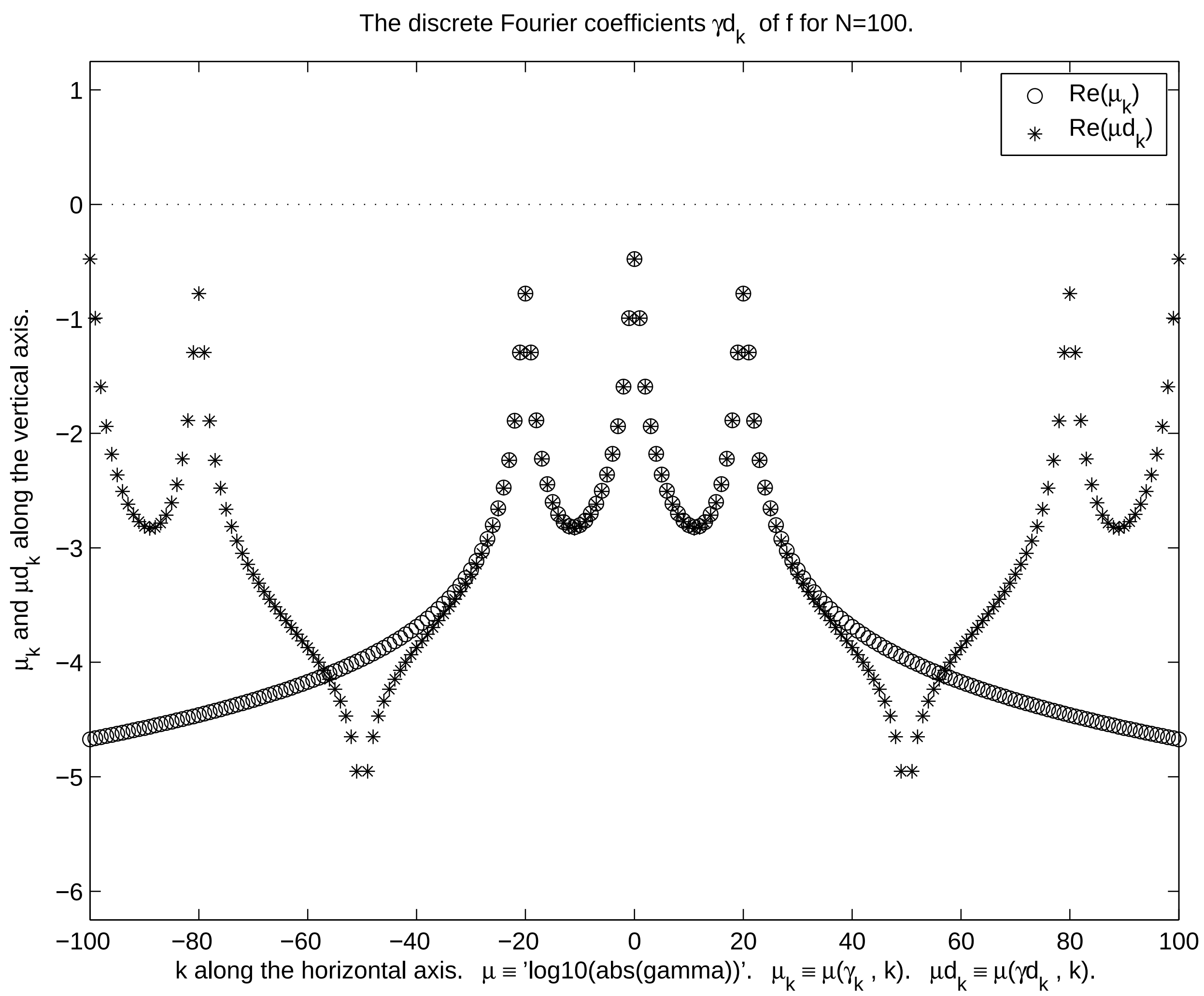
Fourier Transforms
Wavelets
Theory and Applications

Gerard Sleijpen

Universiteit Utrecht
Department of Mathematics

http://www.staff.science.uu.nl/~sleij101/

Computing Fourier Transforms



Program

- Computing Fourier Coefficients
- Discrete Fourier Transform
- Discrete Cosine Transform
- Fast Fourier Transform
- Computing Fourier Integrals

Program

- Computing Fourier Coefficients
- Discrete Fourier Transform
- Discrete Cosine Transform
- Fast Fourier Transform
- Computing Fourier Integrals

Fourier coefficients

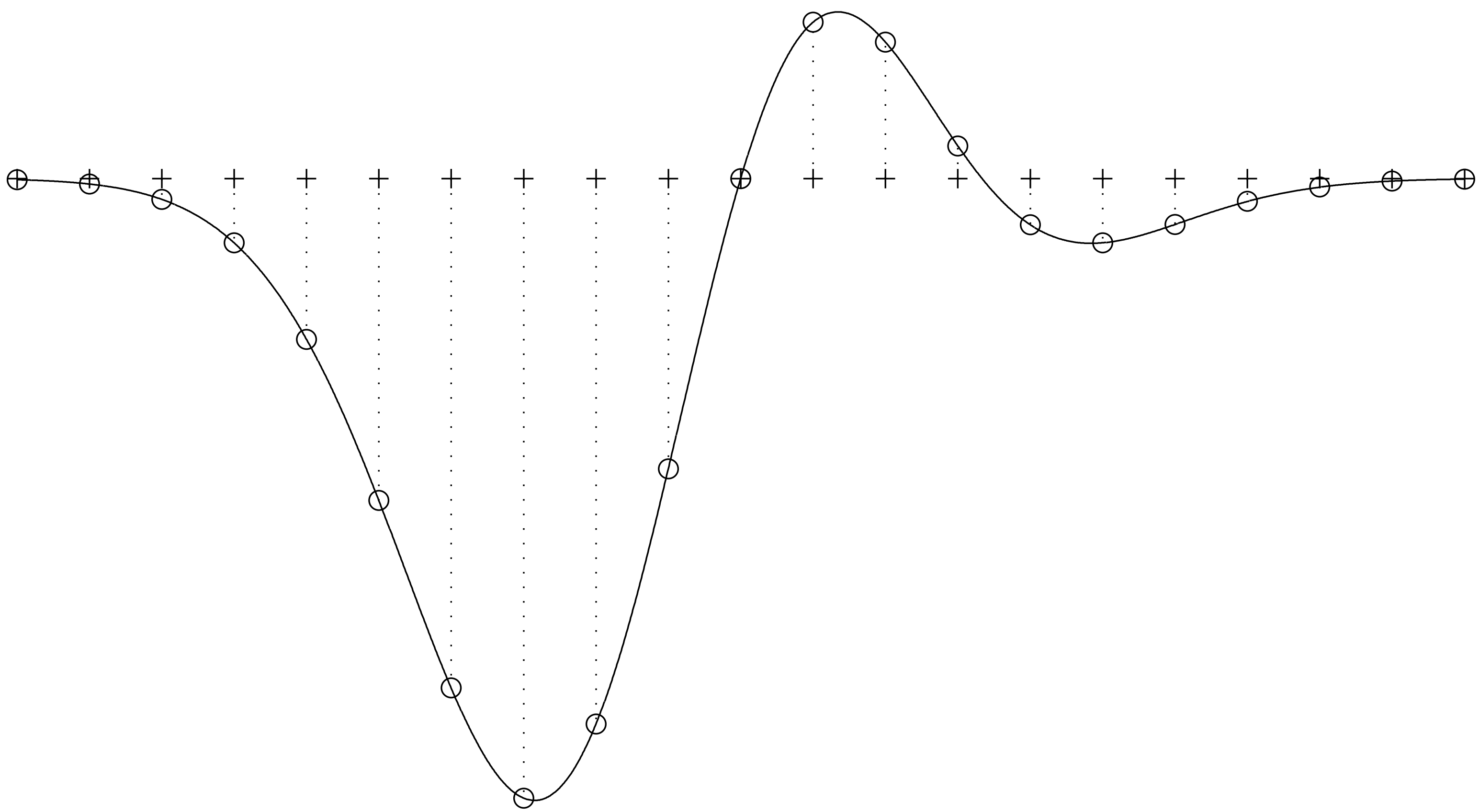
Let $f : \mathbb{R} \to \mathbb{C}$ be T-periodic and sufficiently smooth.

$$\gamma_k(f) = \frac{1}{T}\int_0^T f(t)\, e^{-2\pi i t \frac{k}{T}}\, \mathrm{d}t, \qquad f(t) = \sum_{k\in\mathbb{Z}} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

Suppose f is sampled at t_n with $t_n \equiv n\Delta t$ and $\Delta t \equiv \frac{T}{N}$.

- $1/\Delta t$ is the **sample frequency**,
- $f_n \equiv f(t_n)$ are the **sampled function values**.

We approximate γ_k with a Riemann integral using the sampled fucntion values.



Fourier coefficients

Let $f : \mathbb{R} \to \mathbb{C}$ be T-periodic and sufficiently smooth.

$$\gamma_k(f) = \frac{1}{T}\int_0^T f(t)\, e^{-2\pi i t \frac{k}{T}}\, \mathrm{d}t, \qquad f(t) = \sum_{k\in\mathbb{Z}} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

Suppose f is sampled at t_n with $t_n \equiv n\Delta t$ and $\Delta t \equiv \frac{T}{N}$.

$$\widetilde{\gamma}_k \equiv \frac{\Delta t}{T}\sum_{n=0}^{N-1} f(t_n)\, e^{-2\pi i t_n \frac{k}{T}} = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-2\pi i \frac{nk}{N}}$$

Fourier coefficients

Let $f : \mathbb{R} \to \mathbb{C}$ be T-periodic and sufficiently smooth.

$$\gamma_k(f) = \frac{1}{T} \int_0^T f(t)\, e^{-2\pi i t \frac{k}{T}}\, \mathrm{d}t, \qquad f(t) = \sum_{k \in \mathbb{Z}} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

Suppose f is sampled at t_n with $t_n \equiv n\Delta t$ and $\Delta t \equiv \frac{T}{N}$.

$$\widetilde{\gamma}_k \equiv \frac{\Delta t}{T} \sum_{n=0}^{N-1} f(t_n)\, e^{-2\pi i t_n \frac{k}{T}} = \frac{1}{N} \sum_{n=0}^{N-1} f_n e^{-2\pi i \frac{nk}{N}}$$

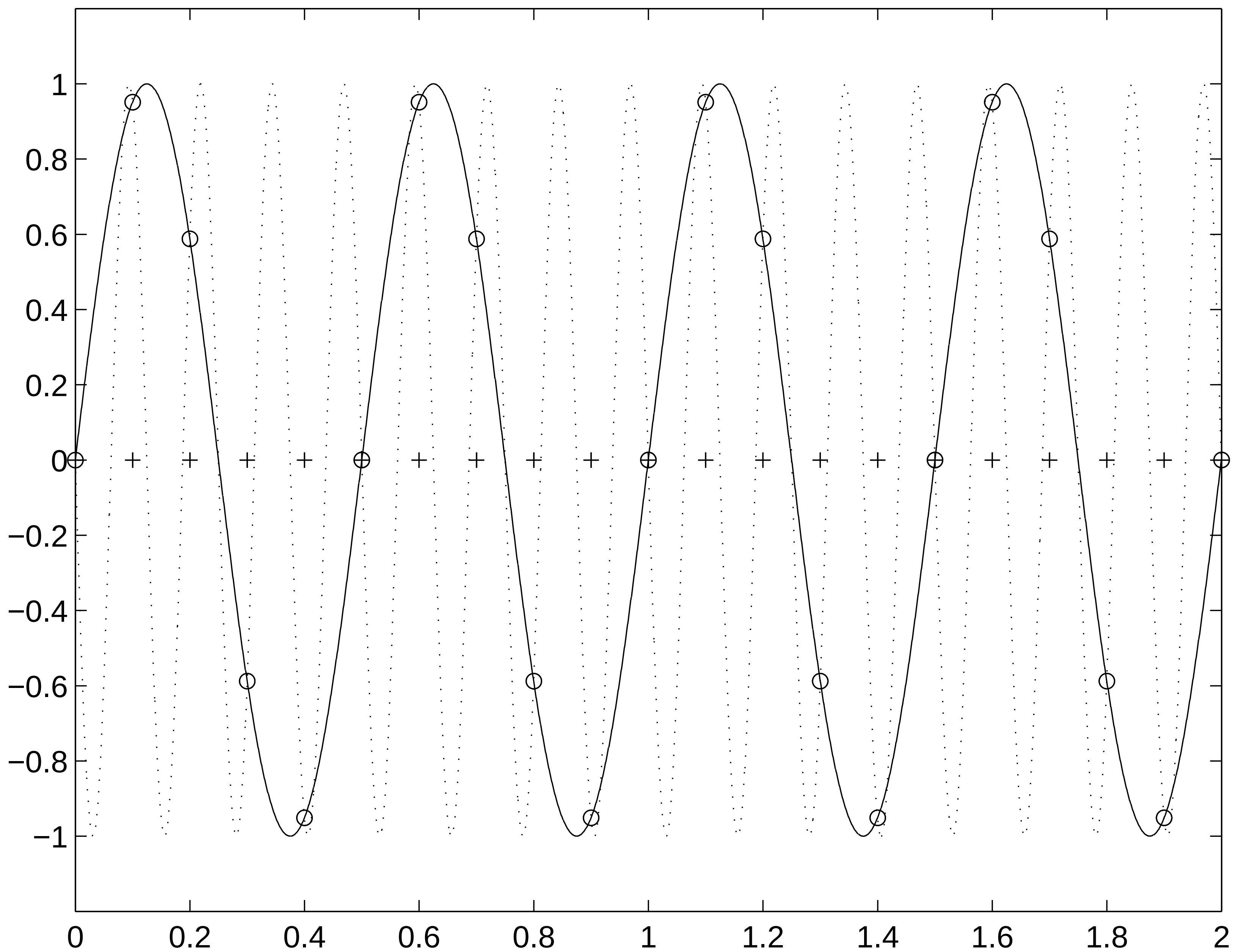
Note. The harmonic oscillations

$$t \rightsquigarrow e^{2\pi i t \frac{k}{T}} \quad \text{and} \quad t \rightsquigarrow e^{2\pi i t \frac{k+N}{T}}$$

coincide at the sample points t_n.

The second oscillation is an **alias** of the first.

This phenomenon of **aliasing** has many consequences in discretised Fourier series.



1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Fourier coefficients

Let $f : \mathbb{R} \to \mathbb{C}$ be T-periodic and sufficiently smooth.

$$\gamma_k(f) = \frac{1}{T}\int_0^T f(t)\, e^{-2\pi i t \frac{k}{T}}\, \mathrm{d}t, \qquad f(t) = \sum_{k\in\mathbb{Z}} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

Suppose f is sampled at t_n with $t_n \equiv n\Delta t$ and $\Delta t \equiv \frac{T}{N}$.

$$\widetilde{\gamma}_k \equiv \frac{\Delta t}{T}\sum_{n=0}^{N-1} f(t_n)\, e^{-2\pi i t_n \frac{k}{T}} = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-2\pi i \frac{nk}{N}}$$

$$\widetilde{\gamma}_k = \widetilde{\gamma}_{k+jN} \qquad (k, j \in \mathbb{Z}).$$

Fourier coefficients

Let $f : \mathbb{R} \to \mathbb{C}$ be T-periodic and sufficiently smooth.

$$\gamma_k(f) = \frac{1}{T}\int_0^T f(t)\, e^{-2\pi i t \frac{k}{T}}\, \mathrm{d}t, \qquad f(t) = \sum_{k\in\mathbb{Z}} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

Suppose f is sampled at t_n with $t_n \equiv n\Delta t$ and $\Delta t \equiv \frac{T}{N}$.

$$\widetilde{\gamma}_k \equiv \frac{\Delta t}{T}\sum_{n=0}^{N-1} f(t_n)\, e^{-2\pi i t_n \frac{k}{T}} = \frac{1}{N}\sum_{n=0}^{N-1} f_n e^{-2\pi i \frac{nk}{N}}$$

$$\widetilde{\gamma}_k = \widetilde{\gamma}_{k+jN} \qquad (k, j \in \mathbb{Z}).$$

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k\, e^{2\pi i \frac{nk}{N}}, \quad \text{where} \quad \mu_k \equiv \sum_{j\in\mathbb{Z}} \gamma_{k+jN}(f).$$

Fourier coefficients

Let $f:\mathbb{R}\to\mathbb{C}$ be T-periodic and sufficiently smooth.

$$\gamma_k(f)=\frac{1}{T}\int_0^T f(t)\,e^{-2\pi i t\frac{k}{T}}\,\mathrm{d}t,\qquad f(t)=\sum_{k\in\mathbb{Z}}\gamma_k(f)\,e^{2\pi i t\frac{k}{T}}$$

Suppose f is sampled at t_n with $t_n\equiv n\Delta t$ and $\Delta t\equiv\frac{T}{N}$.

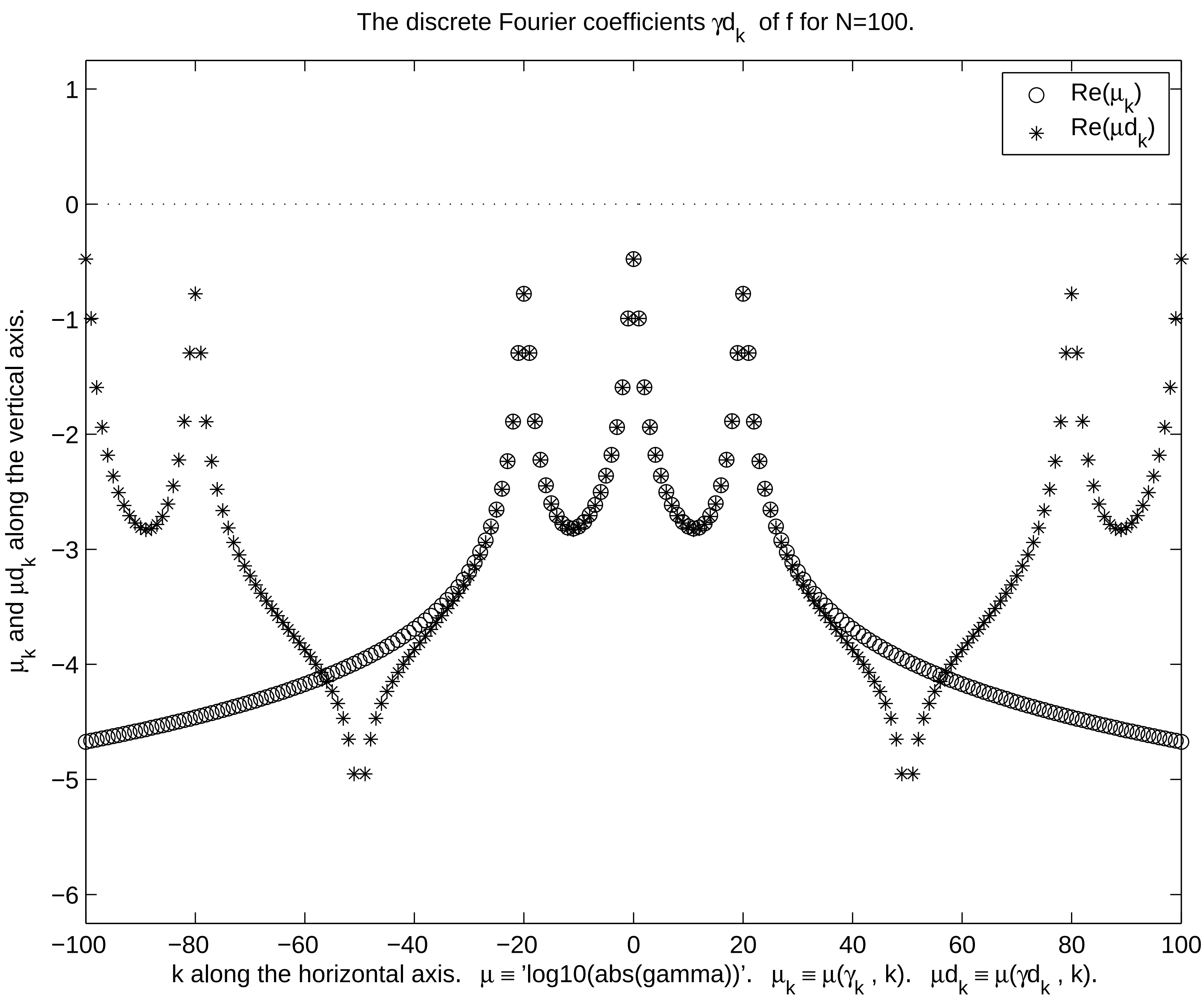
$$\widetilde{\gamma}_k\equiv\frac{\Delta t}{T}\sum_{n=0}^{N-1}f(t_n)\,e^{-2\pi i t_n\frac{k}{T}}=\frac{1}{N}\sum_{n=0}^{N-1}f_n e^{-2\pi i\frac{nk}{N}}$$

$$\widetilde{\gamma}_k=\widetilde{\gamma}_{k+jN}\qquad (k,j\in\mathbb{Z}).$$

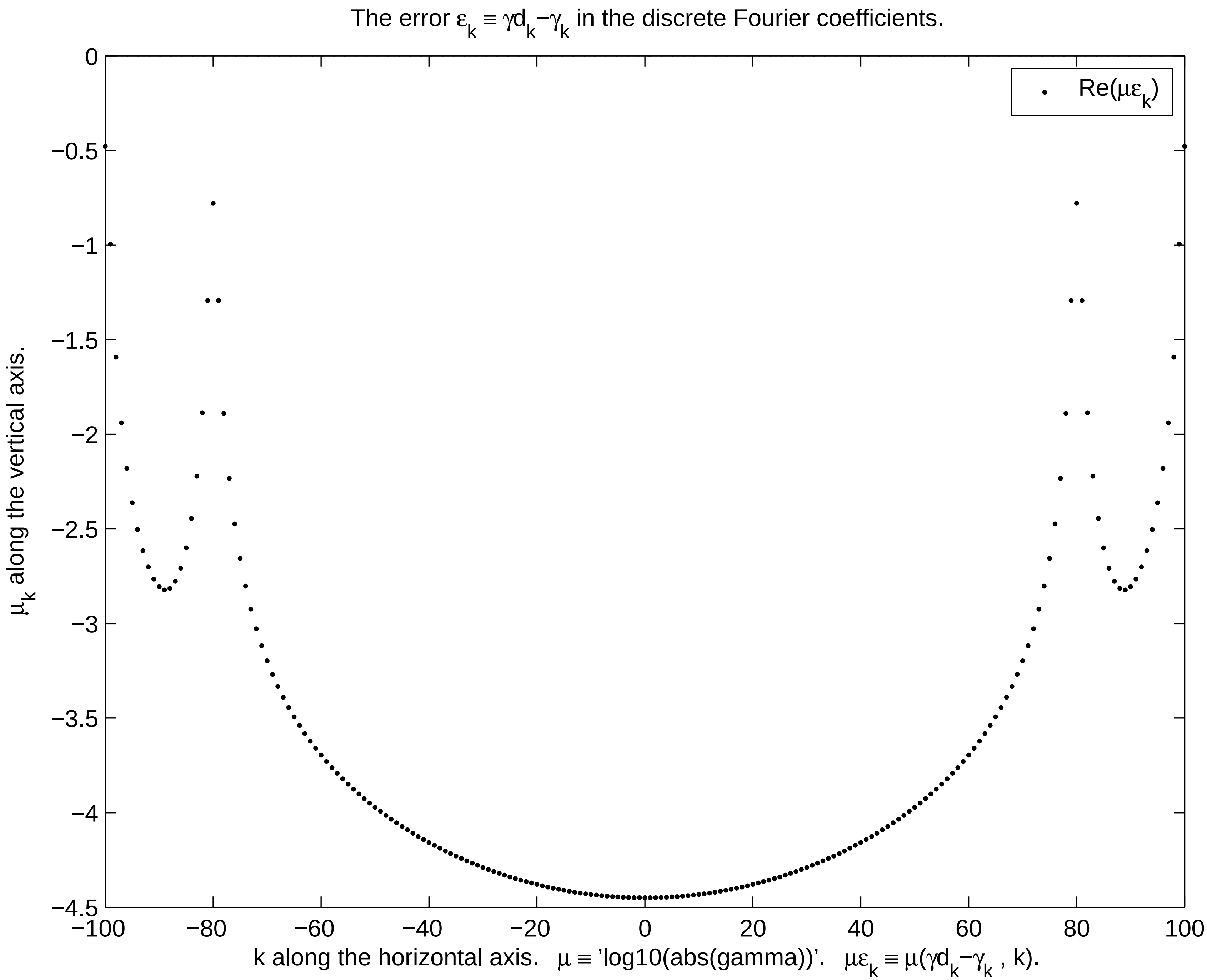
$$f_n=\sum_{k\in\mathbb{Z}}\mu_k\,e^{2\pi i\frac{nk}{N}},\quad\text{where}\quad \mu_k\equiv\sum_{j\in\mathbb{Z}}\gamma_{k+jN}(f).$$

Theorem. $\widetilde{\gamma}_k=\mu_k=\gamma_k(f)+\sum_{j\neq 0}\gamma_{k+jN}(f)$.

Proof. Apply next theorem.



The discrete Fourier coefficients γd_k of f for N=100.
Re(μ_k)
Re(μd_k)
μ_k and μd_k along the vertical axis.
k along the horizontal axis. μ ≡ 'log10(abs(gamma))'. μ_k ≡ μ(γ_k , k). μd_k ≡ μ(γd_k , k).
1
0
−1
−2
−3
−4
−5
−6
−100
−80
−60
−40
−20
0
20
40
60
80
100



The error εk ≡ γdk−γk in the discrete Fourier coefficients.
Re(μεk)
μk along the vertical axis.
0
−0.5
−1
−1.5
−2
−2.5
−3
−3.5
−4
−4.5
−100
−80
−60
−40
−20
0
20
40
60
80
100
k along the horizontal axis. μ ≡ 'log10(abs(gamma))'. μεk ≡ μ(γdk−γk , k).

Program

- Computing Fourier Coefficients
- Discrete Fourier Transform
- Discrete Cosine Transform
- Fast Fourier Transform
- Computing Fourier Integrals

Discrete Fourier Transform

Theorem. Let $(f_0, f_1, \ldots, f_{N-1})$ be a sequence of complex numbers. Define the sequence $(\widetilde{\gamma}_0, \ldots, \widetilde{\gamma}_{N-1})$ by

$$\widetilde{\gamma}_k \equiv \frac{1}{N} \sum_{n=0}^{N-1} f_n \, e^{-2\pi i \frac{nk}{N}} \qquad (k = 0, \ldots, N-1).$$

Then
$$f_n = \sum_{k=0}^{N-1} \widetilde{\gamma}_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

Discrete Fourier Transform

Theorem. Let $(f_0, f_1, \ldots, f_{N-1})$ be a sequence of complex numbers. Define the sequence $(\widetilde{\gamma}_0, \ldots, \widetilde{\gamma}_{N-1})$ by

$$\widetilde{\gamma}_k \equiv \frac{1}{N} \sum_{n=0}^{N-1} f_n \, e^{-2\pi i \frac{nk}{N}} \qquad (k = 0, \ldots, N-1).$$

Then

$$f_n = \sum_{k=0}^{N-1} \widetilde{\gamma}_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

Note. Except for the minus-sign in the exponential and the scaling $\frac{1}{N}$ in the definition of the $\widetilde{\gamma}_k$, the formulae are the same. *Some text books scale both formulae with* $\frac{1}{\sqrt{N}}$.

The sequence $(\widetilde{\gamma}_k)$ is the **Discrete Fourier Transform** of the sequence (f_n). The theorem gives the inverse **DFT**.

Discrete Fourier Transform

Theorem. $\widetilde{\gamma}_k \equiv \dfrac{1}{N}\displaystyle\sum_{n=0}^{N-1} f_n\, e^{-2\pi i \frac{nk}{N}} \;\Rightarrow\; f_n = \displaystyle\sum_{k=0}^{N-1} \widetilde{\gamma}_k\, e^{2\pi i \frac{kn}{N}}.$

Proof. Let $\ell(N)$ be the space of sequences $\mathbf{f} \equiv (f_0, \ldots, f_{N-1})$ of N complex numbers with inner product

$$<\mathbf{f}, \mathbf{g}> \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, \overline{g_n} \qquad (\mathbf{f}, \mathbf{g} \in \ell(N)).$$

For each $k = 0, \ldots, N-1$, consider

$$\phi_k(n) \equiv e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

The collection of ϕ_k forms an orthonormal basis of $\ell(N)$.

Discrete Fourier Transform

Theorem. $\widetilde{\gamma}_k \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n \, e^{-2\pi i \frac{nk}{N}} \;\Rightarrow\; f_n = \sum_{k=0}^{N-1} \widetilde{\gamma}_k \, e^{2\pi i \frac{kn}{N}}.$

Proof. Let $\ell(N)$ be the space of sequences $\mathbf{f} \equiv (f_0, \ldots, f_{N-1})$ of N complex numbers with inner product

$$<\mathbf{f}, \mathbf{g}> \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n \, \overline{g_n} \qquad (\mathbf{f}, \mathbf{g} \in \ell(N)).$$

For each $k = 0, \ldots, N-1$, consider

$$\phi_k(n) \equiv e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

The collection of ϕ_k forms an orthonormal basis of $\ell(N)$:

$$<\phi_k, \phi_m> = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i \frac{(k-m)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \zeta^n \quad \text{with} \quad \zeta \equiv e^{2\pi i \frac{k-m}{N}}.$$

Discrete Fourier Transform

Theorem. $\widetilde{\gamma}_k \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-2\pi i\frac{nk}{N}} \;\Rightarrow\; f_n = \sum_{k=0}^{N-1} \widetilde{\gamma}_k\, e^{2\pi i\frac{kn}{N}}.$

Proof. Let $\ell(N)$ be the space of sequences $\mathbf{f} \equiv (f_0, \ldots, f_{N-1})$ of N complex numbers with inner product

$$<\mathbf{f}, \mathbf{g}> \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, \overline{g_n} \qquad (\mathbf{f}, \mathbf{g} \in \ell(N)).$$

For each $k = 0, \ldots, N-1$, consider

$$\phi_k(n) \equiv e^{2\pi i\frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

The collection of ϕ_k forms an orthonormal basis of $\ell(N)$:

$$<\phi_k, \phi_m> = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i\frac{(k-m)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \zeta^n \quad \text{with} \quad \zeta \equiv e^{2\pi i\frac{k-m}{N}}.$$

If $k = m$, then $\zeta = 1$ and $<\phi_k, \phi_k> = 1$.

Discrete Fourier Transform

Theorem. $\widetilde{\gamma}_k \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-2\pi i \frac{nk}{N}} \;\Rightarrow\; f_n = \sum_{k=0}^{N-1} \widetilde{\gamma}_k\, e^{2\pi i \frac{kn}{N}}.$

Proof. Let $\ell(N)$ be the space of sequences $\mathbf{f} \equiv (f_0, \ldots, f_{N-1})$ of N complex numbers with inner product

$$<\mathbf{f}, \mathbf{g}> \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, \overline{g_n} \qquad (\mathbf{f}, \mathbf{g} \in \ell(N)).$$

For each $k = 0, \ldots, N-1$, consider

$$\phi_k(n) \equiv e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

The collection of ϕ_k forms an orthonormal basis of $\ell(N)$:

$$<\phi_k, \phi_m> = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i \frac{(k-m)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \zeta^n \quad \text{with} \quad \zeta \equiv e^{2\pi i \frac{k-m}{N}}.$$

Note $-N < k - m < N$. Hence, if $k - m \neq 0$, then $\zeta \neq 1$.

Discrete Fourier Transform

Theorem. $\widetilde{\gamma}_k \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-2\pi i \frac{nk}{N}} \;\Rightarrow\; f_n = \sum_{k=0}^{N-1} \widetilde{\gamma}_k\, e^{2\pi i \frac{kn}{N}}.$

Proof. Let $\ell(N)$ be the space of sequences $\mathbf{f} \equiv (f_0, \ldots, f_{N-1})$ of N complex numbers with inner product

$$<\mathbf{f}, \mathbf{g}> \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, \overline{g_n} \qquad (\mathbf{f}, \mathbf{g} \in \ell(N)).$$

For each $k = 0, \ldots, N-1$, consider

$$\phi_k(n) \equiv e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

The collection of ϕ_k forms an orthonormal basis of $\ell(N)$:

$$<\phi_k, \phi_m> = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i \frac{(k-m)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \zeta^n \quad \text{with} \quad \zeta \equiv e^{2\pi i \frac{k-m}{N}}.$$

If $k \neq m$, then $\zeta \neq 1$, $\zeta^N = 0$, and $<\phi_k, \phi_m> = \frac{1}{N}\frac{\zeta^N - 1}{\zeta - 1} = 0$.

Discrete Fourier Transform

Theorem. $\widetilde{\gamma}_k \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-2\pi i \frac{nk}{N}} \;\Rightarrow\; f_n = \sum_{k=0}^{N-1} \widetilde{\gamma}_k\, e^{2\pi i \frac{kn}{N}}.$

Proof. Let $\ell(N)$ be the space of sequences $\mathbf{f} \equiv (f_0, \ldots, f_{N-1})$ of N complex numbers with inner product

$$<\mathbf{f}, \mathbf{g}> \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, \overline{g_n} \qquad (\mathbf{f}, \mathbf{g} \in \ell(N)).$$

For each $k = 0, \ldots, N-1$, consider

$$\phi_k(n) \equiv e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

The collection of ϕ_k forms an orthonormal basis of $\ell(N)$:

$$<\phi_k, \phi_m> = \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i \frac{(k-m)n}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} \zeta^n \quad \text{with} \quad \zeta \equiv e^{2\pi i \frac{k-m}{N}}.$$

This proves orthonormality of the ϕ_k, i.p., linear independence.

A dimension argument ($\dim(\ell(N)) = N$) shows that the ϕ_k form a basis.

Discrete Fourier Transform

Theorem. $\tilde{\gamma}_k \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-2\pi i \frac{nk}{N}} \;\Rightarrow\; f_n = \sum_{k=0}^{N-1} \tilde{\gamma}_k\, e^{2\pi i \frac{kn}{N}}.$

Proof. Let $\ell(N)$ be the space of sequences $\mathbf{f} \equiv (f_0, \ldots, f_{N-1})$ of N complex numbers with inner product

$$<\mathbf{f}, \mathbf{g}> \equiv \frac{1}{N}\sum_{n=0}^{N-1} f_n\, \overline{g_n} \qquad (\mathbf{f}, \mathbf{g} \in \ell(N)).$$

For each $k = 0, \ldots, N-1$, consider

$$\phi_k(n) \equiv e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

The collection of ϕ_k forms an orthonormal basis of $\ell(N)$.

In particular, $$\mathbf{f} = \sum_{k=0}^{N-1} <\mathbf{f}, \phi_k>\, \phi_k$$

The def. of the inner product reveals that $\tilde{\gamma}_k = <\mathbf{f}, \phi_k>$.

Discrete Fourier Transform

Theorem. $\widetilde{\gamma}_k \equiv \dfrac{1}{N}\sum_{n=0}^{N-1} f_n\, e^{-2\pi i \frac{nk}{N}} \;\Rightarrow\; f_n = \sum_{k=0}^{N-1} \widetilde{\gamma}_k\, e^{2\pi i \frac{kn}{N}}.$

Exercise. For $\widetilde{\gamma} \equiv (\widetilde{\gamma}_0, \ldots, \widetilde{\gamma}_{N-1}) \in \ell(N)$, put

$$\mathcal{F}(\widetilde{\gamma})_n \equiv \mathcal{F}_N(\widetilde{\gamma})_n \equiv f_n \equiv \sum_{k=0}^{N-1} \widetilde{\gamma}_k\, e^{2\pi i \frac{nk}{N}} \qquad (k \in \mathbb{Z}).$$

With $\mathbf{f} \equiv (f_0, \ldots, f_{N-1})$, prove that

$$\widetilde{\gamma}_k = \tfrac{1}{N}\,\mathcal{F}(\mathbf{f})_{N-k} \qquad (k = 0, \ldots, N-1).$$

Note that the **DFT** $\mathcal{F}_N$ produces N-periodic sequences.

Conclusion. The inverse **DFT** can easily be obtained from the **DFT** and visa versa.

Program

- Computing Fourier Coefficients
- Discrete Fourier Transform
- Discrete Cosine Transform
- Fast Fourier Transform
- Computing Fourier Integrals

Discrete Cosine Transform

The **DFT** requires complex arithmetic.

Moreover, as we know form Fourier series, the series converge slowly if the periodic function is discontinuous.
Any function f on a bounded interval, say, $[0, T]$ can be extended to a T-periodic function. The obvious extension

$$f(t) \equiv f(t - jT) \qquad (t \in \mathbb{R}, j \in \mathbb{Z}),$$

may lead to a discontinuous function on $\mathbb{R}$ (if $f(0) \neq f(T)$) with slowly decreasing Fourier coefficients $\gamma_k(f)$.
With the even extension first

$$f(t) \equiv f(-t), \quad f(t) \equiv f(t - 2jT) \qquad (t \in \mathbb{R}, j \in \mathbb{Z})$$

we have an even $2T$-periodic function that is continuous whenever f is. In particular, γ_k of this function are real.

Discrete Cosine Transform

Similarly, if $\mathbf{f} \in \ell(N)$, then complex arithmetic is avoided and at the same time faster decreasing discrete Fourier coefficients $\widetilde{\gamma}_k$ are obtained by extending $\mathbf{f}$ first to an even function before extending to a periodic function.

For ease of notation, we put γ_k instead of $\widetilde{\gamma}_k$.

Discrete Cosine Transform

Example. Suppose $\mathbf{f} = (f_0, \ldots, f_N) \in \ell(N+1)$.

Extend $\mathbf{f}$ to an function that is even (around $n = N$):

$$\begin{aligned} \mathbf{g} &\equiv (f_0, f_1, \ldots, f_{N-1}, f_N, f_{N-1}, \ldots, f_2, \quad f_1) \\ &= (g_0, g_1, \ldots, g_{N-1}, g_N, g_{N+1}, \ldots, g_{2N-2}, g_{2N-1}) \end{aligned}$$

Note that the extension to a $2N$-periodic function is even also around $n = 0$.

The extended sequence is $(\ldots, \mathbf{g}, \mathbf{g}, \mathbf{g}, \ldots)$, to which we also shall refer to as $\mathbf{g}$.

Discrete Cosine Transform

Example. Suppose $\mathbf{f} = (f_0, \ldots, f_N) \in \ell(N+1)$.

Extend $\mathbf{f}$ to an function that is even (around $n = N$):

$$\begin{aligned} \mathbf{g} &\equiv (f_0, f_1, \ldots, f_{N-1}, f_N, f_{N-1}, \ldots, f_2, \quad f_1) \\ &= (g_0, g_1, \ldots, g_{N-1}, g_N, g_{N+1}, \ldots, g_{2N-2}, g_{2N-1}) \end{aligned}$$

The **DFT** of $\mathbf{g}$ is

$$\begin{aligned} \gamma_k &= \frac{1}{2N} \sum_{n=0}^{2N-1} g_n \, e^{-2\pi i \frac{kn}{2N}} \\ &= \frac{1}{2N}[f_0 + (-1)^k f_N] + \frac{1}{N} \sum_{n=1}^{N-1} f_n \cos(2\pi \tfrac{kn}{N}) \end{aligned}$$

Discrete Cosine Transform

Example. Suppose $\mathbf{f} = (f_0, \ldots, f_N) \in \ell(N+1)$.

Extend $\mathbf{f}$ to an function that is even (around $n = N$):

$$\begin{aligned} \mathbf{g} &\equiv (f_0, f_1, \ldots, f_{N-1}, f_N, f_{N-1}, \ldots, f_2, \quad f_1) \\ &= (g_0, g_1, \ldots, g_{N-1}, g_N, g_{N+1}, \ldots, g_{2N-2}, g_{2N-1}) \end{aligned}$$

The **DFT** of $\mathbf{g}$ is

$$\gamma_k = \frac{1}{2N}[f_0 + (-1)^k f_N] + \frac{1}{N}\sum_{n=1}^{N-1} f_n \cos(2\pi \tfrac{kn}{N})$$

Note that, as $\mathbf{g}$, (γ_k) is even around $k = 0$ and $k = N$.

In particular, γ_k has to be computed for $k = 0, \ldots, N$ only.

Discrete Cosine Transform

Example. Suppose $\mathbf{f} = (f_0, \ldots, f_N) \in \ell(N+1)$.

Extend **f** to an function that is even (around $n = N$):

$$\begin{aligned} \mathbf{g} &\equiv (f_0, f_1, \ldots, f_{N-1}, f_N, f_{N-1}, \ldots, f_2, \quad f_1) \\ &= (g_0, g_1, \ldots, g_{N-1}, g_N, g_{N+1}, \ldots, g_{2N-2}, g_{2N-1}) \end{aligned}$$

The **DFT** of **g** is

$$\gamma_k = \frac{1}{2N}[f_0 + (-1)^k f_N] + \frac{1}{N}\sum_{n=1}^{N-1} f_n \cos(2\pi \tfrac{kn}{N})$$

Note that, as **g**, (γ_k) is even around $k = 0$ and $k = N$.
Therefore, the inverse **DFT**, for $n = 0, \ldots, N$, is

$$g_n = f_n = [\gamma_0 + (-1)^n \gamma_N] + 2\sum_{k=1}^{N-1} \gamma_k \cos(2\pi \tfrac{kn}{N})$$

Discrete Cosine Transform

There are a number of ways to extend a finite sequence to a sequence of length $2N$ that is even.

Example. Suppose $\mathbf{f} = (f_0, \ldots, f_{N-1}) \in \ell(N)$.
Then the extension

$$\mathbf{g} \equiv (\mathbf{f}, \mathbf{f}^\top) \quad \text{with} \quad \mathbf{f}^\top \equiv (f_{N-1}, f_{N-2}, \ldots, f_1, f_0)$$

leads to an $2N$-periodic function $\mathbf{g}$ that is even around $n = -\frac{1}{2}$ and $n = N - \frac{1}{2}$.
This leads to the so-called **DCT-II** transform:

DCT-II. With $\phi_{n,k} \equiv \cos\left(\pi(n+\frac{1}{2})\frac{k}{N}\right)$,

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n\,\phi_{n,k}, \qquad f_n = \gamma_0 + 2\sum_{k=1}^{N-1} \gamma_k\,\phi_{n,k}$$

Discrete Cosine Transform

There are a number of ways to extend a finite sequence to a sequence of length $2N$ that is even. The first extension that we considered (even around 0 and N) is called **DCT-I**, the second (even around $-\frac{1}{2}$, $N-\frac{1}{2}$) is **DCT-II**. The **DCT-II** seems to be the most popular one in practice and is often simple called **the DCT**.

Odd extensions lead to sines rather than cosines. However, sinus are cosines up to some phase shift and with some simple manipulation, odd extensions also lead to transforms involving cosines only, to the so called **DCT-III** and **DCT-IV**. **DCT-IV** is the standard **DCT** in Matlab:

DCT-IV. With $\phi_{n,k} \equiv \cos\left(\frac{\pi}{N}(n+\frac{1}{2})(k+\frac{1}{2})\right)$,

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{N-1} f_n\,\phi_{n,k}, \qquad f_n = 2\sum_{k=0}^{N-1}\gamma_k\,\phi_{n,k}.$$

Discrete Cosine Transform

There are a number of ways to extend a finite sequence to a sequence of length $2N$ that is even. The first extension that we considered (even around 0 and N) is called **DCT-I**, the second (even around $-\frac{1}{2}$, $N - \frac{1}{2}$) is **DCT-II**. The **DCT-II** seems to be the most popular one in practice and is often simple called **the DCT**.

In the above **DCT**s, we extended to an even sequence $\mathbf{g}$ of length $2N$. Extension to an even sequence $\mathbf{g}$ of length $2N-1$ leads to **DCT**s of type **V**, **VI**, **VII** and **VIII**. These **DCT** seem to be rarely used in practice.

Applications of DCT

- **Image compression.**

Goal. Compression.

2-dimensional (and 3-d) **DCT-II** is used with N low.

JPEG, MJPEG, MPEG use **DCT-II** on 8×8 blocks

- **Audio compression.**

Goal. Compression and **spectral information**: the techniques in audio compression exploit psygological facts on how we hear combinations of harmonic oscillations, that is, compression depends on the distribution of frequencies.

A related transform, **Modified DCT**, is used in
AAC, Vorbis, MP3.

- **Partial Differential Equations.** **DCT**s are used for solving PDEs, where the variants of **DCT** correspond to (slightly) different boundary conditions.

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values. Then it is not feasible to compute the **FT**. As an alternative a part $(f_0, \ldots, f_{N-1})$ is considered. However **DFT** implicitly extends periodically. This will (probably) introduce 'jumps' in the function, implying slowly decreasing **DFT** coefficients.

Formally this argument does not apply to a discrete function, which, in some sense, has 'jumps' in all points t_n. But the argument **does** apply to a (T-periodic) function on $\mathbb{R}$ And discretization carries over the properties, in some approximate sense, to the discretized version.

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values. Then it is not feasible to compute the **FT**. As an alternative a part $(f_0, \ldots, f_{N-1})$ is considered. However **DFT** implicitly extends periodically. This will (probably) introduce 'jumps' in the function, implying slowly decreasing **DFT** coefficients.

- The **DFT** coefficients may form bad approximations of the Fourier coefficients of interest,
- The compressibility properties may seriously deteriorate.

The even extension first, as is incorporated in **DCT**, weakens this effect. But, since the even extension, still introduces 'jumps' in the 'derivative', the effect is still noticable. The odd extension (as in **DCT-IV**) may need special attention.

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values.

'Discontinuities' at the ends are 'softened' by multiplying with a smooth function that is zero at the ends. For instance, with

$$\mathbf{S} \equiv (s_0, \ldots, s_{2N-1}) \quad \text{with} \quad s_j \equiv \sin^2\left(\frac{\pi}{2N}(j+\tfrac{1}{2})\right),$$

apply the **DCT** to

$$(s_0 f_0, s_1 f_1, \ldots, s_{2N-1} f_{2N-1}), \quad (s_0 f_{2N}, \ldots, s_{2N-1}, f_{4N-1}), \ \ldots.$$

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values.
'Discontinuities' at the ends are 'softened' by multiplying with a smooth function that is zero at the ends. For instance, with

$$\mathbf{S} \equiv (s_0, \ldots, s_{2N-1}) \quad \text{with} \quad s_j \equiv \sin^2\left(\frac{\pi}{2N}(j+\tfrac{1}{2})\right),$$

apply the **DCT** to

$$(s_0 f_0, s_1 f_1, \ldots, s_{2N-1} f_{2N-1}), \quad (s_0 f_{2N}, \ldots, s_{2N-1}, f_{4N-1}), \ \ldots.$$

To avoid loss of information, apply the **DCT** also to the middle parts, to

$$(s_0 f_N, s_1 f_{N+1}, \ldots, s_{2N-1} f_{3N-1}), \quad (s_0 f_{3N}, \ldots, s_{2N-1}, f_{5N-1}), \ \ldots.$$

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values.
'Discontinuities' at the ends are 'softened' by multiplying with a smooth function that is zero at the ends. For instance, with

$$\mathbf{S} \equiv (s_0, \ldots, s_{2N-1}) \quad \text{with} \quad s_j \equiv \sin^2\left(\frac{\pi}{2N}(j+\tfrac{1}{2})\right),$$

apply the **DCT** to

$$s_0 f_0, s_1 f_1, \ldots, s_{2N-1} f_{2N-1}\ , \qquad s_0 f_{2N}, \ldots, s_{2N-1}, f_{4N-1}\ , \ \ldots .$$

To avoid loss of information, apply the **DCT** also to the middle parts, to

$$s_0 f_N, s_1 f_{N+1}, \ldots, s_{2N-1} f_{3N-1}\ , \qquad s_0 f_{3N}, \ldots, s_{2N-1}, f_{5N-1}\ , \ \ldots .$$

Note that the two sequences (when the parts are grouped in a long sequences) add to the original sequence:

$$\sin^2\phi + \cos^2\phi = 1.$$

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values.
'Discontinuities' at the ends are 'softened' by multiplying with a smooth function that is zero at the ends. For instance, with

$$\mathbf{S} \equiv (s_0, \ldots, s_{2N-1}) \quad \text{with} \quad s_j \equiv \sin^2\left(\frac{\pi}{2N}(j+\tfrac{1}{2})\right),$$

apply the **DCT** to

$$s_0 f_0, s_1 f_1, \ldots, s_{2N-1} f_{2N-1} \ , \qquad s_0 f_{2N}, \ldots, s_{2N-1}, f_{4N-1} \ , \ \ldots .$$

To avoid loss of information, apply the **DCT** also to the middle parts, to

$$s_0 f_N, s_1 f_{N+1}, \ldots, s_{2N-1} f_{3N-1} \ , \qquad s_0 f_{3N}, \ldots, s_{2N-1}, f_{5N-1} \ , \ \ldots .$$

Note that the two sequences (when the parts are grouped in a long sequences) add to the original sequence: **MDCT** is designed to deal with this overlapping type of grouping.

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values. Assume $M \equiv N/2$ is an integer.

With $\quad \phi_{n,k} \equiv \cos\left(\frac{\pi}{N}(k+\frac{1}{2})(n+M+\frac{1}{2})\right), \quad$ we have

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{2N-1} f_n\,\phi_{n,k} \quad (k=0,\ldots,N-1), \quad \widetilde{f}_n = \sum_{k=0}^{N-1} \gamma_k\,\phi_{n,k} \quad (n=0,\ldots,2N-1)$$

Using sequences of length N, we summerize this as

$$(F_1, F_2) \rightsquigarrow \Gamma_1, \qquad \Gamma_1 \rightsquigarrow (\widetilde{F}_1, \widetilde{G}_2).$$

Theorem. $\quad (\widetilde{F}_1, \widetilde{G}_2) = (F_1 - F_1^{\mathsf{T}}, F_2 + F_2^{\mathsf{T}}).$

Proof. MDCT is a form of DCT-IV.

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values. Assume $M \equiv N/2$ is an integer.

With $\qquad \phi_{n,k} \equiv \cos\left(\frac{\pi}{N}(k+\frac{1}{2})(n+M+\frac{1}{2})\right), \qquad$ we have

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{2N-1} f_n\,\phi_{n,k} \quad (k=0,\ldots,N-1), \quad \widetilde{f}_n = \sum_{k=0}^{N-1} \gamma_k\,\phi_{n,k} \quad (n=0,\ldots,2N-1)$$

Using sequences of length N, we summerize this as

$$(F_1, F_2) \rightsquigarrow \Gamma_1, \qquad \Gamma_1 \rightsquigarrow (\widetilde{F}_1, \widetilde{G}_2).$$

Theorem. $\qquad (\widetilde{F}_1, \widetilde{G}_2) = (F_1 - F_1^{\mathsf{T}}, F_2 + F_2^{\mathsf{T}}).$

Modified DCT applies this to long sequences as

$$(F_1, F_2, F_3, F_4 \ldots, F_k) \quad \rightsquigarrow \quad (\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{k-1})$$

$$\rightsquigarrow \quad (\widetilde{F}_1, \widetilde{G}_2 + \widetilde{F}_2, \widetilde{G}_3 + \widetilde{F}_3, \widetilde{G}_4 + \widetilde{F}_4, \ldots, \widetilde{G}_{k-1} + \widetilde{F}_{k-1}, \widetilde{G}_k)$$

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values. Assume $M \equiv N/2$ is an integer.

With $\qquad \phi_{n,k} \equiv \cos\left(\frac{\pi}{N}(k+\frac{1}{2})(n+M+\frac{1}{2})\right), \qquad$ we have

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{2N-1} f_n\,\phi_{n,k} \quad (k=0,\ldots,N-1), \quad \widetilde{f}_n = \sum_{k=0}^{N-1} \gamma_k\,\phi_{n,k} \quad (n=0,\ldots,2N-1)$$

Using sequences of length N, we summerize this as

$$(F_1, F_2) \rightsquigarrow \Gamma_1, \qquad \Gamma_1 \rightsquigarrow (\widetilde{F}_1, \widetilde{G}_2).$$

Theorem. $\quad (\widetilde{F}_1, \widetilde{G}_2) = (F_1 - F_1^{\mathsf{T}}, F_2 + F_2^{\mathsf{T}}).$

Modified DCT applies this to long sequences as

$$(F_1, F_2, F_3, F_4 \ldots, F_k) \quad \rightsquigarrow \quad (\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{k-1})$$

$$\rightsquigarrow \quad (\widetilde{F}_1, \widetilde{G}_2 + \widetilde{F}_2, \widetilde{G}_3 + \widetilde{F}_3, \widetilde{G}_4 + \widetilde{F}_4, \ldots, \widetilde{G}_{k-1} + \widetilde{F}_{k-1}, \widetilde{G}_k)$$

Modifed DCT

Let $(\dots, f_1, f_0, \dots, f_{N-1}, f_N, \dots)$ be a long sequence of (sampled) function values. Assume $M \equiv N/2$ is an integer.

With $\quad \phi_{n,k} \equiv \cos\left(\frac{\pi}{N}(k+\frac{1}{2})(n+M+\frac{1}{2})\right), \quad$ we have

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{2N-1} f_n\,\phi_{n,k} \quad (k = 0,\dots,N-1), \qquad \widetilde{f}_n = \sum_{k=0}^{N-1}\gamma_k\,\phi_{n,k} \quad (n = 0,\dots,2N-1)$$

Using sequences of length N, we summerize this as

$$(F_1, F_2) \rightsquigarrow \Gamma_1, \qquad \Gamma_1 \rightsquigarrow (\widetilde{F}_1, \widetilde{G}_2).$$

Theorem. $\quad (\widetilde{F}_1, \widetilde{G}_2) = (F_1 - F_1^{\top}, F_2 + F_2^{\top}).$

Modified DCT applies this to long sequences as

$$(F_1, F_2, F_3, F_4 \dots, F_k) \quad \rightsquigarrow \quad (\Gamma_1, \Gamma_2, \Gamma_3, \dots, \Gamma_{k-1})$$

$$\rightsquigarrow \quad (\widetilde{F}_1, \widetilde{G}_2 + \widetilde{F}_2, \widetilde{G}_3 + \widetilde{F}_3, \widetilde{G}_4 + \widetilde{F}_4, \dots, \widetilde{G}_{k-1} + \widetilde{F}_{k-1}, \widetilde{G}_k)$$

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values. Assume $M \equiv N/2$ is an integer.

With $\quad \phi_{n,k} \equiv \cos\left(\frac{\pi}{N}(k+\frac{1}{2})(n+M+\frac{1}{2})\right), \quad$ we have

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{2N-1} f_n\,\phi_{n,k} \quad (k=0,\ldots,N-1), \quad \widetilde{f}_n = \sum_{k=0}^{N-1} \gamma_k\,\phi_{n,k} \quad (n=0,\ldots,2N-1)$$

Using sequences of length N, we summerize this as

$$(F_1, F_2) \rightsquigarrow \Gamma_1, \qquad \Gamma_1 \rightsquigarrow (\widetilde{F}_1, \widetilde{G}_2).$$

Theorem. $\quad (\widetilde{F}_1, \widetilde{G}_2) = (F_1 - F_1^{\mathsf{T}}, F_2 + F_2^{\mathsf{T}}).$

Modified DCT applies this to long sequences as

$$(F_1, F_2, F_3, F_4 \ldots, F_k) \quad \rightsquigarrow \quad (\Gamma_1, \Gamma_2, \Gamma_3, \ldots, \Gamma_{k-1})$$
$$\rightsquigarrow \quad (\widetilde{F}_1, \widetilde{G}_2 + \widetilde{F}_2, \widetilde{G}_3 + \widetilde{F}_3, \widetilde{G}_4 + \widetilde{F}_4, \ldots, \widetilde{G}_{k-1} + \widetilde{F}_{k-1}, \widetilde{G}_k)$$

Note that we have exact reconstruction except for the first and last block (usual choice: $F_1 = \mathbf{0}$, $F_k = \mathbf{0}$).

Modifed DCT

Let $(\ldots, f_1, f_0, \ldots, f_{N-1}, f_N, \ldots)$ be a long sequence of (sampled) function values. Assume $M \equiv N/2$ is an integer.

With $\quad \phi_{n,k} \equiv \cos\left(\frac{\pi}{N}(k+\frac{1}{2})(n+M+\frac{1}{2})\right), \quad$ we have

$$\gamma_k = \frac{1}{N}\sum_{n=0}^{2N-1} f_n\,\phi_{n,k} \quad (k=0,\ldots,N-1), \quad \widetilde{f}_n = \sum_{k=0}^{N-1}\gamma_k\,\phi_{n,k} \quad (n=0,\ldots,2N-1)$$

Including windows, with $C = S^{\mathsf{T}}$, $\mathbf{S} = (S, C)$,

$$(F_1, F_2) \rightsquigarrow (SF_1, CF_2) \rightsquigarrow \widetilde{\Gamma}_1 \rightsquigarrow (\widetilde{F}_1, \widetilde{G}_2) \rightsquigarrow (S\widetilde{F}_1, C\widetilde{G}_2).$$

Theorem.

$$(S\widetilde{F}_1, C\widetilde{G}_2) = (S^2F_1 - SCF_1^{\mathsf{T}}, C^2F_2 + SCF_2^{\mathsf{T}}).$$

With $S^2 + C^2 = 1$, application to long sequences leads to exact reconstruction.

Program

- Computing Fourier Coefficients
- Discrete Fourier Transform
- Discrete Cosine Transform
- Fast Fourier Transform
- Computing Fourier Integrals

Fast Fourier Transform

Suppose the sequence $\gamma \equiv (\gamma_0, \ldots, \gamma_{N-1}) \in \ell(N)$ is available. The naive way of computing the **DFT**

$$\mathcal{F}(\gamma)_n \equiv f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

requires more than $2N^2$ **floating point operations** (additions, multiplications): for each of the N ns, $2N$ **flop**.

In practice N is huge.

N of the order of $10^6 \sim 10^8$ is not exceptional.

Gauss **[first half of the 19th century]**, *Runge* **[1903]** *and Cooley & Tukey* **[1965]** *in the most cited mathematical paper ever, proposed a computational scheme,* **FFT**, *that reduces the computational costs to* $2N\log_2(N)$ **flop**.
For, e.g., $n = 2^{20} \approx 10^6$, *this makes a difference with* $2N^2$ *of 1 sec versus 3:30 hours.*

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

We split the sum into one with even indices and one with odd indices.

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

$$f_n = \left(\sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}} \right) + \left(\sum_{2k+1<N} \gamma_{2k+1} \, e^{2\pi i \frac{kn}{M}} \right) e^{2\pi \frac{n}{N}}.$$

or

$$f_n = f_{e,n} + f_{o,n} \, e^{2\pi \frac{n}{N}} \quad \text{with} \quad f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}$$

and $f_{o,n}$ defined similarly:

$$f_{e,n} \equiv \sum_{2k+1<N} \gamma_{2k+1} \, e^{2\pi i \frac{kn}{M}}.$$

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

$$f_n = f_{e,n} + f_{o,n} \, e^{2\pi i \frac{n}{N}} \quad \text{with} \quad f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}$$

Note that $f_{e,n} = f_{e,n+M}$. Similarly, $f_{o,n} = f_{o,n+M}$.

Moreover, $\exp(\pi i \frac{n+M}{M}) = -\exp(\pi i \frac{n}{M})$. Therefore,

$$\begin{aligned} f_n &= f_{e,n} + f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1), \\ f_{n+M} &= f_{e,n} - f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1). \end{aligned}$$

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

$$f_n = f_{e,n} + f_{o,n} \, e^{2\pi i \frac{n}{N}} \quad \text{with} \quad f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}$$

Note that $f_{e,n} = f_{e,n+M}$. Similarly, $f_{o,n} = f_{o,n+M}$.

Moreover, $\exp(\pi i \frac{n+M}{M}) = -\exp(\pi i \frac{n}{M})$. Therefore,

$$\begin{aligned} f_n &= f_{e,n} + f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1), \\ f_{n+M} &= f_{e,n} - f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1). \end{aligned}$$

To compute f_n for all $n = 0, \ldots, N-1$, we need to compute two Fourier transforms $f_{e,n}$ and $f_{o,n}$ for sequences of coefficients of half of the length and for only half of the number of ns ($n = 0, \ldots, M-1$).

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

With $\displaystyle f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}, \qquad f_{o,n} \equiv \sum_{2k+1<N} \gamma_{2k+1} \, e^{2\pi i \frac{kn}{M}},$

we have

$$\begin{aligned} f_n &= f_{e,n} + f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1), \\ f_{n+M} &= f_{e,n} - f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1). \end{aligned}$$

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

With $\quad f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}, \qquad f_{o,n} \equiv \sum_{2k+1<N} \gamma_{2k+1} \, e^{2\pi i \frac{kn}{M}},$

we have

$$\begin{aligned} f_n &= f_{e,n} + f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1), \\ f_{n+M} &= f_{e,n} - f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1). \end{aligned}$$

Let κ_ℓ be the number of **flop** required to compute the **DFT** of length $M = 2^\ell$. Then, the above implies that

$$\kappa_\ell = 2\kappa_{\ell-1} + 1.5\, N.$$

We need N additions (subtractions), M multiplications; For now, we neglected the costs for computing $e^{\pi i \frac{n}{M}}$.

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

With $f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}, \quad f_{o,n} \equiv \sum_{2k+1<N} \gamma_{2k+1} \, e^{2\pi i \frac{kn}{M}},$

we have

$$\begin{aligned} f_n &= f_{e,n} + f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1), \\ f_{n+M} &= f_{e,n} - f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1). \end{aligned}$$

Let κ_ℓ be the number of **flop** required to compute the **DFT** of length $M = 2^\ell$. Then, the above implies that

$$\kappa_\ell = 2\kappa_{\ell-1} + 1.5\, N.$$

We need N additions (subtractions), M multiplications;
For now, we neglected the costs for computing $e^{\pi i \frac{n}{M}}$.

We can repeat the partitioning trick to $f_{e,n}$ and $f_{o,n}$.

Fast Fourier Transform

Suppose $N=2^\ell$ for some $\ell\in\mathbb{N}$. Put $M=2^{\ell-1}=\frac{1}{2}N$.

$$f_n=\sum_{k=0}^{N-1}\gamma_k\, e^{2\pi i\frac{kn}{N}}\qquad (n=0,\ldots,N-1).$$

With $f_{e,n}\equiv\sum_{2k<N}\gamma_{2k}\, e^{2\pi i\frac{kn}{M}},\quad f_{o,n}\equiv\sum_{2k+1<N}\gamma_{2k+1}\, e^{2\pi i\frac{kn}{M}}$,

we have

$$\begin{aligned} f_n &= f_{e,n}+f_{o,n}\, e^{\pi i\frac{n}{M}} \quad (n=0,\ldots,M-1),\\ f_{n+M} &= f_{e,n}-f_{o,n}\, e^{\pi i\frac{n}{M}} \quad (n=0,\ldots,M-1). \end{aligned}$$

Let κ_ℓ be the number of **flop** required to compute the **DFT** of length $M=2^\ell$. Then, the above implies that

$$\kappa_\ell=2\kappa_{\ell-1}+1.5\,N=2(2\kappa_{\ell-2}+1.5\,M)+1.5\,N.$$

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

With $\quad f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}, \qquad f_{o,n} \equiv \sum_{2k+1<N} \gamma_{2k+1} \, e^{2\pi i \frac{kn}{M}},$

we have

$$\begin{aligned} f_n &= f_{e,n} + f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1), \\ f_{n+M} &= f_{e,n} - f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1). \end{aligned}$$

Let κ_ℓ be the number of **flop** required to compute the **DFT** of length $M = 2^\ell$. Then, the above implies that

$$\kappa_\ell = 2\kappa_{\ell-1} + 1.5\,N = 4\kappa_{\ell-2} + 2\,1.5\,N.$$

Repeating, the partitioning trick to $f_{e,n}$ and $f_{o,n}$, etc., and using the fact that $\kappa_0 = 0$, shows that

$$\kappa_\ell = 2\kappa_{\ell-1} + 1.5\,N = 1.5\,\ell N.$$

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

With $f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}, \quad f_{o,n} \equiv \sum_{2k+1<N} \gamma_{2k+1} \, e^{2\pi i \frac{kn}{M}},$

we have

$$\begin{aligned} f_n &= f_{e,n} + f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1), \\ f_{n+M} &= f_{e,n} - f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1). \end{aligned}$$

Repeating this partitioning trick recursively down to level $\ell = 0$ is **Fast Fourier Transform**.

Theorem. **FFT** requires $(1.5\,\ell + 0.5)N$ **flop**.

Fast Fourier Transform

Suppose $N = 2^\ell$ for some $\ell \in \mathbb{N}$. Put $M = 2^{\ell-1} = \frac{1}{2}N$.

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

With $f_{e,n} \equiv \sum_{2k<N} \gamma_{2k} \, e^{2\pi i \frac{kn}{M}}, \quad f_{o,n} \equiv \sum_{2k+1<N} \gamma_{2k+1} \, e^{2\pi i \frac{kn}{M}},$

we have

$$\begin{aligned} f_n &= f_{e,n} + f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1), \\ f_{n+M} &= f_{e,n} - f_{o,n} \, e^{\pi i \frac{n}{M}} \quad (n = 0, \ldots, M-1). \end{aligned}$$

Repeating this partitioning trick recursively down to level $\ell = 0$ is **Fast Fourier Transform**.

Theorem. **FFT** requires $(1.5\,\ell + 0.5)N$ **flop**.

Proof. The $0.5\,N$ comes from the computation of $e^{\pi i \frac{n}{M}}$, which can be computed as $\zeta^n = \zeta^{n-1}\zeta$ with $\zeta = e^{\pi i \frac{1}{M}}$. Note that, $e^{\pi i n 2^{-\ell+j}} = e^{\pi i (2^j n) 2^{-\ell}}$.

DFT & FFT as matrix multiplication

Let $\mathbf{f} = (f_0, \ldots, f_{N-1})^T$ and $\gamma = (\gamma_0, \ldots, \gamma_{N-1})^T$ be such that

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

We will represent the **DFT** and **FFT** with respect to column vectors here: $.^T$ is the transpose for vectors.

Note that, for consistency of notation, we selected the first index of the vectors to be 0 rather than 1.
Our matrices will also be indexed from 0 on: the left top matrix element will be the $(0,0)$-entry.

DFT & FFT as matrix multiplication

Let $\mathbf{f} = (f_0, \ldots, f_{N-1})^T$ and $\gamma = (\gamma_0, \ldots, \gamma_{N-1})^T$ be such that $\quad f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$

Let $\mathbf{F}$ be the $N \times N$ matrix with (n,k)-entry $e^{2\pi i \frac{kn}{N}}$. Then

$$\mathbf{f} = \mathbf{F}\gamma$$

This represents the **DFT** (Discrete Fourier transform) as a **MV** (matrix-vector multipliciation).

- Note that $\frac{1}{N}\mathbf{F}^*\mathbf{F} = \mathbf{I}$: except for the scaling N, the **DFT** matrix $\mathbf{F}$ is unitary. Here $\mathbf{F}^* \equiv \overline{\mathbf{F}}^T$.
- The inverse $\frac{1}{N}\mathbf{F}^*$ of $\mathbf{F}$ represents the inverse **DFT**.
- The **DFT** matrix $\mathbf{F}$ is full: non of its entries is zero. This makes the **MV** expensive.

DFT & FFT as matrix multiplication

Let $\mathbf{f} = (f_0, \ldots, f_{N-1})^T$ and $\gamma = (\gamma_0, \ldots, \gamma_{N-1})^T$ be such that $f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1)$.

Let $\mathbf{F}$ be the $N \times N$ matrix with (n,k)-entry $e^{2\pi i \frac{kn}{N}}$. Then

$$\mathbf{f} = \mathbf{F}\gamma$$

FFT. Now suppose $N = 2^\ell$. Put $M \equiv 2^{\ell-1}$.
Let $\mathbf{F}_\ell \equiv \mathbf{F}$ be the **DFT** for level ℓ, i.e., for $N = 2^\ell$.
The first step in **FFT** can be written as

$$\mathbf{f} = \begin{bmatrix} \mathbf{f}' \\ \mathbf{f}'' \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\ell-1} & +\mathbf{D}_{\ell-1} \\ \mathbf{I}_{\ell-1} & -\mathbf{D}_{\ell-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_e \\ \mathbf{f}_o \end{bmatrix}.$$

Here, $\mathbf{f}' \equiv (f_0, \ldots, f_{M-1})^T$, $\mathbf{f}'' \equiv (f_M, \ldots, f_{N-1})^T$,
$\mathbf{I}_{\ell-1}$ is the $M \times M$ identity matrix,
$\mathbf{D}_{\ell-1}$ is the $M \times M$ diagonal matrix with (n,n)-entry $e^{\pi i \frac{n}{M}}$.

Note that the even diagonal entries of $D_{\ell-1}$ form $D_{\ell-2}$, ...

DFT & FFT as matrix multiplication

Let $\mathbf{f} = (f_0, \ldots, f_{N-1})^T$ and $\gamma = (\gamma_0, \ldots, \gamma_{N-1})^T$ be such that
$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$
Let $\mathbf{F}$ be the $N \times N$ matrix with (n,k)-entry $e^{2\pi i \frac{kn}{N}}$. Then
$$\mathbf{f} = \mathbf{F}\gamma$$

FFT. Now suppose $N = 2^\ell$. Put $M \equiv 2^{\ell-1}$.
Let $\mathbf{F}_\ell \equiv \mathbf{F}$ be the **DFT** for level ℓ, i.e., for $N = 2^\ell$.
The first step in **FFT** can be written as
$$\mathbf{f} = \begin{bmatrix} \mathbf{I}_{\ell-1} & +\mathbf{D}_{\ell-1} \\ \mathbf{I}_{\ell-1} & -\mathbf{D}_{\ell-1} \end{bmatrix} \begin{bmatrix} \mathbf{f}_e \\ \mathbf{f}_o \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{\ell-1} & +\mathbf{D}_{\ell-1} \\ \mathbf{I}_{\ell-1} & -\mathbf{D}_{\ell-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{\ell-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{\ell-1} \end{bmatrix} \begin{bmatrix} \gamma_e \\ \gamma_o \end{bmatrix}$$
where $\gamma_e = (\gamma_0, \gamma_2, \ldots)^T$, $\gamma_o = (\gamma_1, \gamma_3, \ldots)^T$.

DFT & FFT as matrix multiplication

Let $\mathbf{f} = (f_0, \ldots, f_{N-1})^T$ and $\gamma = (\gamma_0, \ldots, \gamma_{N-1})^T$ be such that

$$f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$$

Let $\mathbf{F}$ be the $N \times N$ matrix with (n,k)-entry $e^{2\pi i \frac{kn}{N}}$. Then

$$\mathbf{f} = \mathbf{F}\gamma$$

FFT. Now suppose $N = 2^\ell$. Put $M \equiv 2^{\ell-1}$.
Let $\mathbf{F}_\ell \equiv \mathbf{F}$ be the **DFT** for level ℓ, i.e., for $N = 2^\ell$.
The first step in **FFT** can be written as

$$\mathbf{f} = \begin{bmatrix} \mathbf{I}_{\ell-1} & +\mathbf{D}_{\ell-1} \\ \mathbf{I}_{\ell-1} & -\mathbf{D}_{\ell-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{\ell-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{\ell-1} \end{bmatrix} \gamma(P)$$

where P is the permutation that puts the even indexed γ_k first: following Matlab $\quad \gamma(P) \equiv (\gamma_{P(0)}, \ldots, \gamma_{P(N-1)})^T$.
$P(k) \equiv 2k \ (k < M)$, $P(k) \equiv 2(k-M)+1 \ (k > M)$.

DFT & FFT as matrix multiplication

Let $\mathbf{f} = (f_0, \ldots, f_{N-1})^T$ and $\gamma = (\gamma_0, \ldots, \gamma_{N-1})^T$ be such that $f_n = \sum_{k=0}^{N-1} \gamma_k \, e^{2\pi i \frac{kn}{N}} \qquad (n = 0, \ldots, N-1).$

Let $\mathbf{F}$ be the $N \times N$ matrix with (n,k)-entry $e^{2\pi i \frac{kn}{N}}$. Then

$$\mathbf{f} = \mathbf{F}\gamma$$

FFT. Now suppose $N = 2^\ell$. Put $M \equiv 2^{\ell-1}$.
Let $\mathbf{F}_\ell \equiv \mathbf{F}$ be the **DFT** for level ℓ, i.e., for $N = 2^\ell$.
The first step in **FFT** can be written as

$$\mathbf{f} = \begin{bmatrix} \mathbf{I}_{\ell-1} & +\mathbf{D}_{\ell-1} \\ \mathbf{I}_{\ell-1} & -\mathbf{D}_{\ell-1} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{\ell-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{\ell-1} \end{bmatrix} \gamma(P)$$

This decomposes $\mathbf{F}$ as a product of a very sparse unitary matrix (only 2 non-zeros per row), two **DFT**s of half size and a permutation. Repeating this decomposition gives **FFT**: $\mathbf{F}$ is a product of only ℓ very sparse unitary matrices and a permutation.

FFT for sequences of any length?

Suppose $N = q^\ell$ for some $q \in \mathbb{N}, q > 2$.

Then we can design a **FFT** algorithm similar to the one for $q = 2$. For instance, if $q = 3$, and $(\gamma_0, \ldots, \gamma_{N-1})$ is a sequence of length N, then we can group the coefficients in three classes (γ_{3k}), (γ_{3k+1}) and (γ_{3k+2}) instead of the two as for $q = 2$ (the one with even indices and one with odd indices) and we can decompose the f_n accordingly.

q is the **radix** of the **FFT**.

The computational costs are in the order of $N \log_3 N$:

$$\text{Comp. Costs} \approx \mathcal{C}_q \, N \log_3 N \quad \text{for some} \quad \mathcal{C}_q > 0.$$

Property. **FFT** with radix 4 allows the most efficient implementation (i.e., $4 = \operatorname{argmin}_q \mathcal{C}_q \, N \log_q N$).

FFT for sequences of any length?

If, say, $N = 2^{\ell_1} 3^{\ell_2}$, then we can form a **FFT** by

- applying the **FFT** with radix 2 ℓ_1-times
- followed by ℓ_2-times the **FFT** with radix 3.

More generally,

FFT for sequences of any length?

We can factorise any $N \in \mathbb{N}$, that is, we can decompose any N into a product of prime factors and we can design a **FFT** for sequences of length N that is a mixture of **FFT**s of radix p_j with p_j the primes that occur in the factors.

However, computationally, this approach is not attractive:

- we have to factorise N
- coding of such a **FFT** with a mixture of **FFT**s with different radixes is messy
- if the primes are large (with the extremal situation where N itself is prime), then the **FFT** is not faster.

FFT for sequences of any length?

If, for instance, we have to compute Fourier coefficients of a T-periodic function $f : \mathbb{R} \to \mathbb{C}$, then we can select the sample frequency $1/\Delta t$ as we like (with the only restriction that it is sufficiently large), for instance,

$$\Delta t = T/N \quad \text{with} \quad N = 2^\ell.$$

Conclusion.
Some application allow to select N to be a power of 2.

FFT for sequences of any length?

Some application allow sequences $(\gamma_0,\ldots,\gamma_{M-1})$ of length M to be extended to sequences of length 2^ℓ (with ℓ such that $2^{\ell-1} < M \leq 2^\ell$) by appending with zeros.

Example. The **convolution product** $\alpha\star\beta$ of the sequence $\alpha = (\gamma_0,\ldots,\alpha_{M-1})$ and $\beta = (\beta_0,\ldots,\beta_{M-1})$ is defined by

$$(\alpha\star\beta)_k \equiv \sum_j \alpha_j\,\beta_{k-j} \qquad (k = 0,\ldots,2M-2),$$

where we sum over all $j\in\mathbb{Z}$ for which α_j and β_{k-j} exists, that is, j such that $j, k-j \in \{0,\ldots,M-1\}$.

Application.
If p is the polynomial $p(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_{M-1}x^{M-1}$ and $q(x) = \beta_0 + \beta_1 x + \ldots + \beta_{M-1}x^{M-1}$, then $(\alpha\star\beta)_k$ are the coefficients of the product polynomial pq.

FFT for sequences of any length?

Some application allow sequences $(\gamma_0, \ldots, \gamma_{M-1})$ of length M to be extended to sequences of length 2^ℓ (with ℓ such that $2^{\ell-1} < M \leq 2^\ell$) by appending with zeros.

Example. The **convolution product** $\alpha \star \beta$ of the sequence $\alpha = (\gamma_0, \ldots, \alpha_{M-1})$ and $\beta = (\beta_0, \ldots, \beta_{M-1})$ is defined by

$$(\alpha \star \beta)_k \equiv \sum_j \alpha_j \, \beta_{k-j} \qquad (k = 0, \ldots, 2M-2).$$

Note that the value of $\alpha \star \beta$ does not change if we extend α and β by appending with zeros. Therefore, we may assume that the length of α and β is $M = 2^\ell$.

FFT for sequences of any length?

Some application allow sequences $(\gamma_0, \ldots, \gamma_{M-1})$ of length M to be extended to sequences of length 2^ℓ (with ℓ such that $2^{\ell-1} < M \leq 2^\ell$) by appending with zeros.

Example. The **convolution product** $\alpha \star \beta$ of the sequence $\alpha = (\gamma_0, \ldots, \alpha_{M-1})$ and $\beta = (\beta_0, \ldots, \beta_{M-1})$ is defined by

$$(\alpha \star \beta)_k \equiv \sum_j \alpha_j \, \beta_{k-j} \qquad (k = 0, \ldots, 2M-2).$$

Assume that the length of α and β is $M = 2^\ell$.

FFT for sequences of any length?

Some application allow sequences $(\gamma_0, \ldots, \gamma_{M-1})$ of length M to be extended to sequences of length 2^ℓ (with ℓ such that $2^{\ell-1} < M \leq 2^\ell$) by appending with zeros.

Example. The **convolution product** $\alpha \star \beta$ of the sequence $\alpha = (\gamma_0, \ldots, \alpha_{M-1})$ and $\beta = (\beta_0, \ldots, \beta_{M-1})$ is defined by

$$(\alpha \star \beta)_k \equiv \sum_j \alpha_j \, \beta_{k-j} \qquad (k = 0, \ldots, 2M-2).$$

Assume that the length of α and β is $M = 2^\ell$. Append α and β with zeros to sequences of length $N \equiv 2^{\ell+1}$. Next, extend α and β periodically (period N) and define $\star_N$:

$$(\alpha \star_N \beta)_k \equiv \sum_{j=0}^{N-1} \alpha_j \, \beta_{k-j} \qquad (k = 0, \ldots, N-1).$$

Note that the definitions of $\alpha \star \beta$ are consistent (lead to the same values for $k < N$).

Discrete Convolution Products

Definition. For $\alpha, \beta \in \ell(N)$, let

$$(\alpha \star_N \beta)_k \equiv \sum_{j=0}^{N-1} \alpha_j \, \beta_{k-j} \qquad (k = 0, \ldots, N-1),$$

where $\beta_{k-j} \equiv \beta_{N+k-j}$ if $k - j < 0$ (periodic extension).

Theorem. $\mathcal{F}_N(\alpha *_N \beta) = \mathcal{F}_N(\alpha) \cdot \mathcal{F}_N(\beta)$, where the $\cdot$-product is coordinate wise (the **Hadamard product**).

Suppose $2^{\ell-1} < N < 2^{\ell}$. Put $L \equiv 2\,2^{\ell}$.
Form $\widetilde{\beta} \equiv (\beta, \mathbf{0}, \beta)$ to a sequence of length L.
Form $\widetilde{\alpha} \equiv (\alpha, \mathbf{0}, \mathbf{0})$ to a sequence of length L.

Property. $(\alpha \star_N \beta)_k = (\widetilde{\alpha} \star_L \widetilde{\beta})_k$ for $k = 0, \ldots, N-1$.

Corollary. $(\alpha *_N \beta)_k = (\mathcal{F}_L^{-1}[\mathcal{F}_L(\widetilde{\alpha}) \cdot \mathcal{F}_L(\widetilde{\beta})])_k \qquad (k < N)$.

$\alpha \star_L \beta$ can be computed with three **DFT** of radix 2 plus L mult.. Costs: $\leq 24N(\ell + 2)$ **flop** rather than $0.5\,N^2$.

FFT for sequences of any length?

Some application allow sequences $(\gamma_0, \ldots, \gamma_{M-1})$ of length M to be extended to sequences of length 2^ℓ (with ℓ such that $2^{\ell-1} < M \leq 2^\ell$) by appending with zeros.

Example. The **convolution product** $\alpha \star \beta$ of the sequence $\alpha = (\gamma_0, \ldots, \alpha_{M-1})$ and $\beta = (\beta_0, \ldots, \beta_{M-1})$ is defined by

$$(\alpha \star \beta)_k \equiv \sum_j \alpha_j \, \beta_{k-j} \qquad (k = 0, \ldots, 2M-2).$$

Conclusion. In these applications the FFT is nothing more than an efficient computational tool. The quantities to be computed are in same domain as the inputs (time-domain rather than in frequency domain).

Appending with zeros

Consider $\gamma = (\gamma_0, \ldots, \gamma_{M-1}) \in \ell(M)$ with $2^{\ell-1} < M < N \equiv 2^\ell$. Append γ with zeros to a sequence γ^+ of length N:

$$\gamma^+ \equiv (\gamma_0, \ldots, \gamma_{M-1}, 0, \ldots, 0).$$

Observe that $\quad \mathcal{F}_M(\gamma) \neq \mathcal{F}_N(\gamma^+), \quad$ because

$$\sum_{k=0}^{M-1} \gamma_k \, e^{2\pi i \frac{kn}{M}} = \sum_{k=0}^{N-1} \gamma_k^+ \, e^{2\pi i \frac{kn}{M}} \neq \sum_{k=0}^{N-1} \gamma_k^+ \, e^{2\pi i \frac{kn}{N}}.$$

Heuristics. Appending with zeros introduces 'discontinuities', thus quickly decreasing Fourier coefficients maybe changed into slowly decreasing ones.

Appending with zeros

Consider $\gamma = (\gamma_0, \ldots, \gamma_{M-1}) \in \ell(M)$ with $2^{\ell-1} < M < N \equiv 2^{\ell}$.
Append γ with zeros to a sequence γ^+ of length N:

$$\gamma^+ \equiv (\gamma_0, \ldots, \gamma_{M-1}, 0, \ldots, 0).$$

Observe that $\quad \mathcal{F}_M(\gamma) \neq \mathcal{F}_N(\gamma^+), \quad$ because

$$\sum_{k=0}^{M-1} \gamma_k \, e^{2\pi i \frac{kn}{M}} = \sum_{k=0}^{N-1} \gamma_k^+ \, e^{2\pi i \frac{kn}{M}} \neq \sum_{k=0}^{N-1} \gamma_k^+ \, e^{2\pi i \frac{kn}{N}}.$$

Conclusion. If the quantities of interest are in the 'dual' domain (frequency rather than time, or time rather than frequency), then appending zeros is not allowed.

FFT for sequences of any length?

Consider $\gamma = (\gamma_0, \ldots, \gamma_{M-1}) \in \ell(M)$ with $2^{\ell-1} < M < N \equiv 2^\ell$. Put

$$\beta_k \equiv e^{-\pi i \frac{k^2}{M}} \qquad (k = 0, \ldots, M-1).$$

Note that $\quad e^{2\pi i \frac{kn}{M}} = \overline{\beta}_k \, \beta_{n-k} \, \overline{\beta}_n.\quad$ Hence,

$$\mathcal{F}_M(\gamma)_n = \sum_{k=0}^{M-1} \gamma_k \, e^{2\pi i \frac{kn}{M}} = \left(\sum_{k=0}^{M-1} (\gamma_k \overline{\beta}_k) \, \beta_{n-k} \right) \overline{\beta}_n.$$

Apparently,

$$\mathcal{F}_M(\gamma) = ((\gamma\overline{\beta}) \star_M \beta)\overline{\beta}.$$

Here the convolution product is defined for M-periodic sequences.

FFT for sequences of any length?

Consider $\gamma = (\gamma_0, \ldots, \gamma_{M-1}) \in \ell(M)$ with $2^{\ell-1} < M < N \equiv 2^{\ell}$.

Property. With $\beta_k \equiv e^{-\pi i \frac{k^2}{M}}$, we have that

$$\mathcal{F}_M(\gamma) = \mu\,\overline{\beta} \quad \text{with} \quad \mu \equiv (\gamma\,\overline{\beta}) \star_M \beta$$

As we saw before, the convolution product can be computed with three **DFT** of radix 2 (and length $L \equiv 2N$), plus L multiplications. The multiplications $\gamma\overline{\beta}$ and $\mu\overline{\beta}$ require an additional $2M$ multiplications.

Program

- Computing Fourier Coefficients
- Discrete Fourier Transform
- Discrete Cosine Transform
- Fast Fourier Transform
- Computing Fourier Integrals

Computing Fourier integrals

f sampled at $t_n = t_0 + n\,\Delta t$. $1/\Delta t$ *sample frequency*.
For ease of notation, take $t_0 = 0$ (otherwise shift by t_0).

$$\widehat{f}(\omega) \approx \int_{t_0}^{t_0+T} f(t)\, e^{-2\pi i t\omega}\,\mathrm{d}t \approx \Delta t \sum_{n=0}^{N-1} f_n\, e^{-2\pi i\, n\,\Delta t\,\omega}$$

Here, $T = N\Delta t$ and $f_n = f(t_n)$.
Of interest for $\omega = \frac{k}{T}$ $(k = 0, \ldots, N-1)$.

$\widehat{f}(\omega)$ to be computed by **DFT**.

Two 'discretizations'! How accurate is this?

$$\widehat{f}(\omega) \approx \int_{t_0}^{t_0+T} f(t)\, e^{-2\pi i t\omega}\, \mathrm{d}t$$

If $f \in L^1(\mathbb{R})$ then,
for each $\varepsilon > 0$, there is a t_0 and a $T > 0$ such that

$$\int_{-\infty}^{t_0} |f(t)|\, \mathrm{d}t < \varepsilon \quad \text{and} \quad \int_{t_0+T}^{\infty} |f(t)|\, \mathrm{d}t < \varepsilon$$

However,
this observation is often only of theoretical interest.
In practice T can be large and huge values of N may be required, or spectral information is requested before all relevant function values f are available.

Windowing

$$\widehat{f}(\omega) \approx \int_{t_0}^{t_0+T} f(t)\, e^{-2\pi i t\omega}\, \mathrm{d}t$$

Actually, we are computing the Fourier transform of

fW_{t_0}, where $W(t) = 1$ if $1 \leq t \leq T$, and
$W(t) = 0$ elsewhere

and $W_{t_0}(t) \equiv W(t - t_0)$.

W is a **time-window**.
Of interest: the difference between $\widehat{f}(\omega)$ and $\widehat{(fW_{t_0})}(\omega)$.

$\Phi(t, \omega) \equiv \widehat{(fW_t)}(\omega)$ is called a **spectogram** of f.

Effects of windowing

Computing Fourier integrals

f sampled at $t_n = t_0 + n\,\Delta t$. $1/\Delta t$ *sample frequency*.
For ease of notation, take $t_0 = 0$ (otherwise shift by t_0).

$$\widehat{f}(\omega) \approx \int_{t_0}^{t_0+T} f(t)\, e^{-2\pi i t\omega}\, \mathrm{d}t \approx \Delta t \sum_{n=0}^{N-1} f_n\, e^{-2\pi i\, n\, \Delta t\, \omega}$$

Two 'discretizations'! How accurate is this?

Computing Fourier integrals

f sampled at $t_n = t_0 + n\,\Delta t$. $1/\Delta t$ *sample frequency*.
For ease of notation, take $t_0 = 0$ (otherwise shift by t_0).

$$\widehat{f}(\omega) \approx \int_{t_0}^{t_0+T} f(t)\, e^{-2\pi i t\omega}\,\mathrm{d}t \approx \Delta t \sum_{n=0}^{N-1} f_n\, e^{-2\pi i\, n\,\Delta t\,\omega}$$

Two ‘discretizations’! How accurate is this?

Analysis. Other order: first discretize, then window.

Discretization

$$F(\omega) \equiv \Delta t \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \, n \, \Delta t \, \omega}$$

Relation $\widehat{f}(\omega)$ and $F(\omega)$? Does this depend on ω?

$B_f \equiv \{\omega \in \mathbb{R} \mid |\widehat{f}(\omega)| \neq 0\}$ is the **frequency band** of f.
f is of **bounded bandwidth** if $B_f \subset [-\Omega, +\Omega]$
for some $\Omega > 0$: smallest Ω is the **bandwidth**.

Suppose f is of bandwidth $\leq \Omega$.

$$f(t) = \sum_{k=-\infty}^{\infty} \gamma_k \, e^{2\pi i \frac{t}{T} k} \quad \Leftrightarrow \quad \gamma_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{-2\pi i \frac{t}{T} k} \, \mathrm{d}t$$

Take $\Delta t = \frac{1}{2\Omega}$, change $-t \leftrightarrow \omega$, $T \leftrightarrow 2\Omega$, $n \leftrightarrow k$ ...

$$F(\omega) \equiv \Delta t \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \, n \, \Delta t \, \omega} \;\Leftrightarrow\; f_n = \int_{-\Omega}^{\Omega} F(\omega) \, e^{2\pi i \frac{\omega}{2\Omega} n} \, \mathrm{d}\omega$$

f of bandwidth $\leq \Omega \;\Rightarrow$

$$f(t) = \int_{-\infty}^{\infty} \widehat{f}(\omega) \, e^{2\pi i \, t \, \omega} \, \mathrm{d}\omega = \int_{-\Omega}^{\Omega} \widehat{f}(\omega) \, e^{2\pi i \, t \, \omega} \, \mathrm{d}\omega$$

In particular, $\quad f_n = f(t_n) = \int_{-\Omega}^{\Omega} \widehat{f}(\omega) \, e^{2\pi i \frac{n \, \omega}{2\Omega}} \, \mathrm{d}\omega.$

$$f(t) = \sum_{k=-\infty}^{\infty} \gamma_k \, e^{2\pi i \frac{t}{T} k} \quad \Leftrightarrow \quad \gamma_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{-2\pi i \frac{t}{T} k} \, \mathrm{d}t$$

Take $\Delta t = \frac{1}{2\Omega}$, change $-t \leftrightarrow \omega$, $T \leftrightarrow 2\Omega$, $n \leftrightarrow k$ . . .

$$F(\omega) \equiv \Delta t \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \, n \, \Delta t \, \omega} \;\Leftrightarrow\; f_n = \int_{-\Omega}^{\Omega} F(\omega) \, e^{2\pi i \frac{\omega}{2\Omega} n} \, \mathrm{d}\omega$$

f of bandwidth $\leq \Omega \;\; \Rightarrow$

$$\int_{-\Omega}^{\Omega} [\widehat{f}(\omega) - F(\omega)] e^{2\pi i \, \frac{n\,\omega}{2\Omega}} \, \mathrm{d}\omega = 0 \qquad \forall n \in \mathbb{Z}.$$

$$f(t) = \sum_{k=-\infty}^{\infty} \gamma_k \, e^{2\pi i \frac{t}{T} k} \quad \Leftrightarrow \quad \gamma_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \, e^{-2\pi i \frac{t}{T} k} \, \mathrm{d}t$$

Take $\Delta t = \frac{1}{2\Omega}$, change $-t \leftrightarrow \omega$, $T \leftrightarrow 2\Omega$, $n \leftrightarrow k$ . . .

$$F(\omega) \equiv \Delta t \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \, n \, \Delta t \, \omega} \; \Leftrightarrow \; f_n = \int_{-\Omega}^{\Omega} F(\omega) \, e^{2\pi i \frac{\omega}{2\Omega} n} \, \mathrm{d}\omega$$

f of bandwidth $\leq \Omega \;\; \Rightarrow$

$$\widehat{f}(\omega) = \Delta t \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \, n \, \Delta t \, \omega} \qquad \forall \omega \in [-\Omega, +\Omega].$$

$\Delta t = \frac{1}{2\Omega}$ is the **Nyquist rate**.

Theorem.

f of bandwidth $\leq \Omega$ & sample frequency $1/\Delta t \geq 2\Omega \;\Rightarrow$

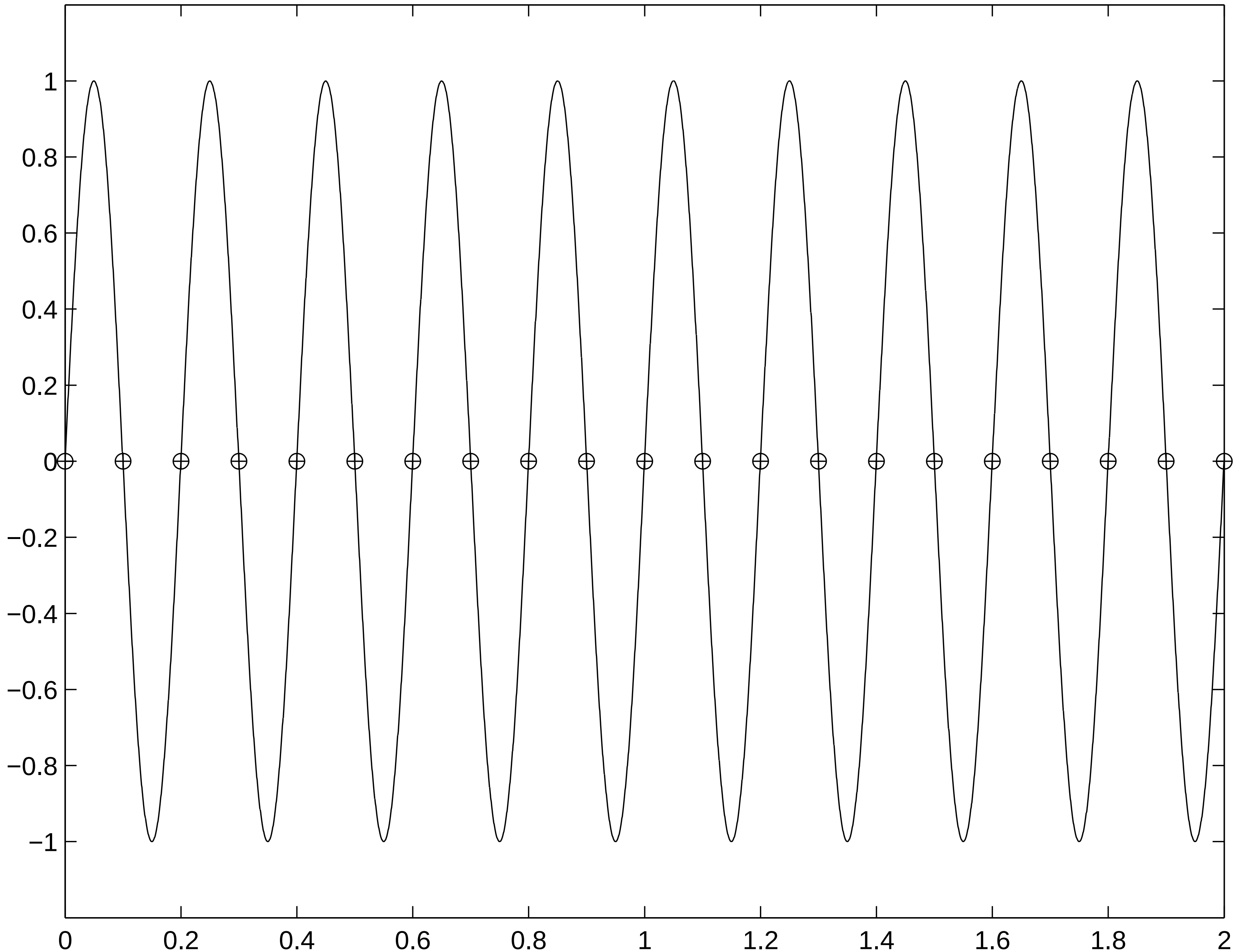
$$\widehat{f}(\omega) = \Delta t \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \, n \, \Delta t \, \omega} \qquad \forall \omega \in [-\Omega, +\Omega]$$

The discretization is exact if the bandwidth $\leq \Omega$ and the sample frequency $\geq 2\Omega$ $(\Delta t \leq 1/(2\Omega))$. Fourier transform of this result leads to

The Shannon–Whittakker Theorem.

f of bandwidth $\leq \Omega$ & sample frequency $1/\Delta t \geq 2\Omega \;\Rightarrow$

$$f(t) = \sum_{n=-\infty}^{\infty} f_n \, \mathrm{sinc}\left(\frac{t - t_n}{\Delta t}\right) \qquad \forall t \in \mathbb{R}.$$



1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Discussion. The Shannon–Whittakker theorem tells us that f can be reconstructed from its sample values, if f is of bounded bandwidth and the sample frequency is at least twice the maximal frequency of f. However, reconstruction requires values f_n from the (far) future as well as from the (far) past.

Application. Resampling (sampling at another sampling rate) is possible.

If the new sample rate is $\frac{p}{q}$ times the old sample rate Δt, then, in practice, resampling is achieved by

1) upsampling by p
2) filtering to get rid of frequencies $> \Omega$
3) downsampling by q.

(Details later)

Conclusions

- Discretization is fine provided f is of bounded bandwidth and the sample frequency is high enough.

- Perturbations by windowing can not be avoided. Effects include smearing and leakage. Effects can be diminished by a larger time-window. One effect can be diminished at the cost of others (by other time-windows).