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Fourier Transforms
Wavelets
Theory and Applications

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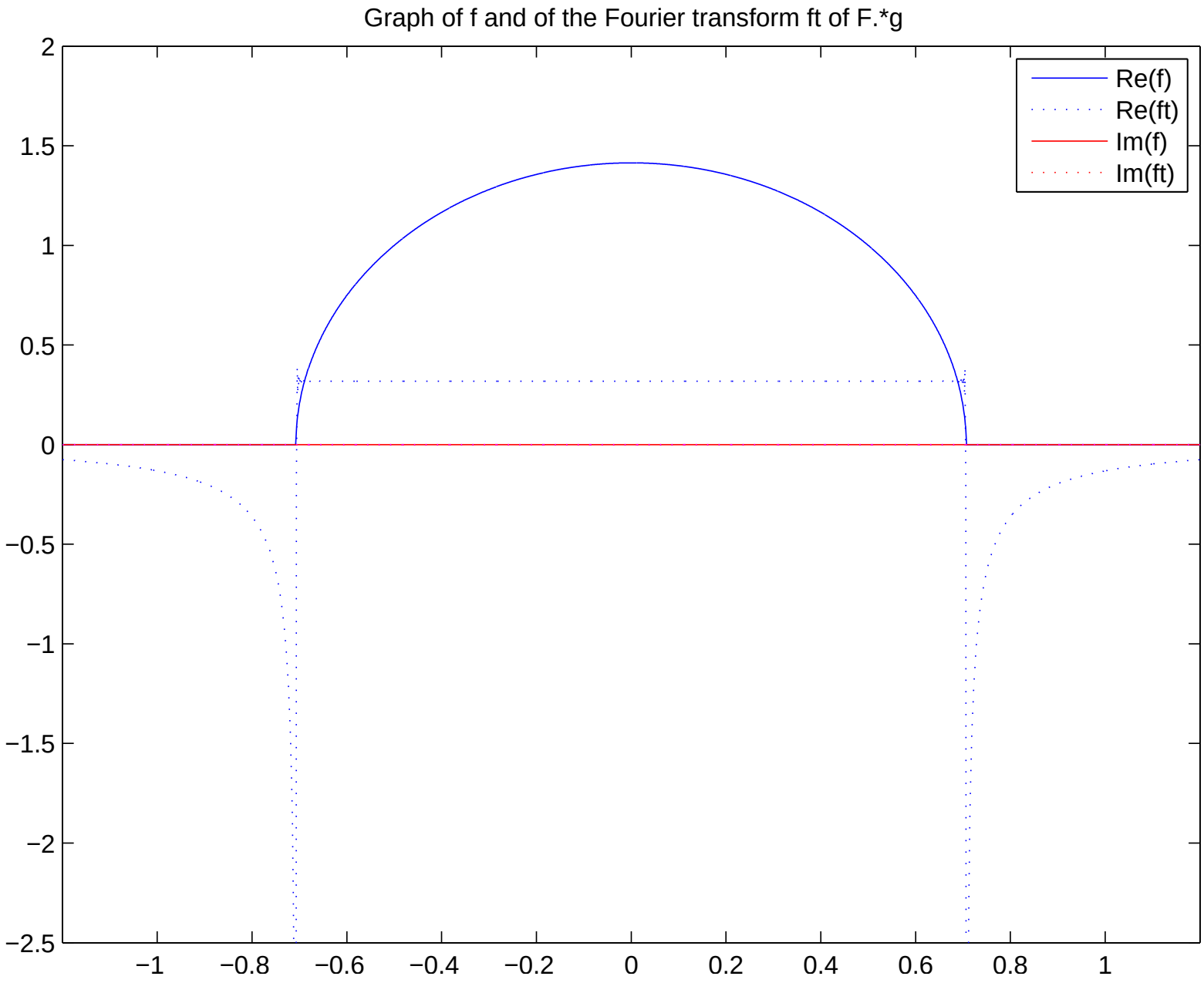
Applications: CT



Program

- Electronic Circuits
- MRI
- Diffraction
- CT

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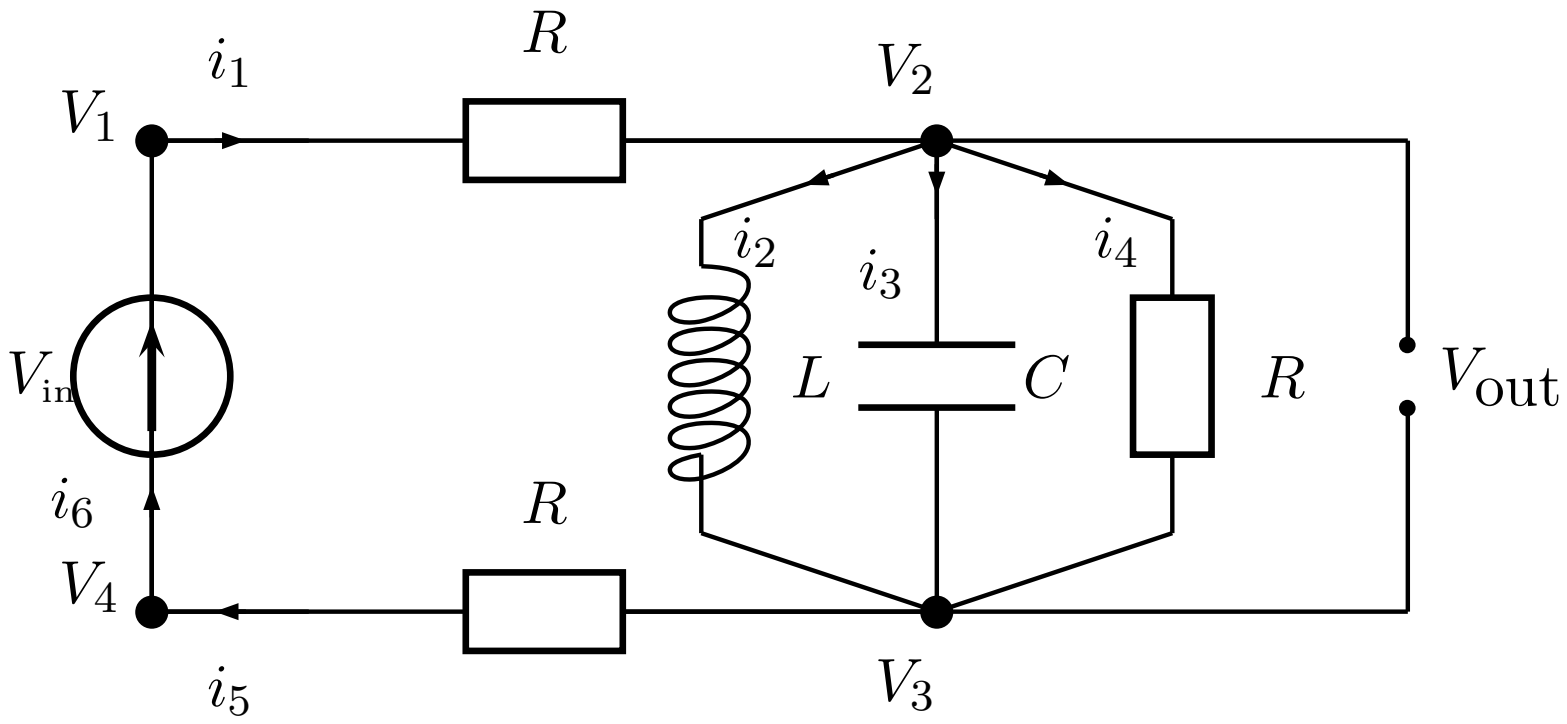
Electronic circuits

Example.



Larger examples in computer chips, with up to $5\,10^8$ electronic components (2011: Intel's dual-core i5).

Definition. A **directed graph** is a collection of vertices (v_i) (points) and edges (e_j) (lines) with a direction.

An electronic network can be described by a directed graph, where each edge contains exactly one electronic component, as a resistor, capacitor, inductor, etc.. At vertex v_i we have Voltage V_i, in edge e_j is an electrical current i_j, with i_j positive if the current is in line with the direction of edge e_j and negative if it is in opposite direction.

One way of describing a directed graph is by an
Definition. Incidence matrix G:
• the ith row of **G** corresponds to the ith vertex v_i;
• the jth column of G corresponds to the jth edge e_j;
• if edge i_j connects v_k and v_ℓ with v_k first, then G has value $+1$ at entry (k, j) and -1 at (ℓ, j), while all other entries in the jth column have value 0.

We collect the voltages in a vector **V**, and the currents in a vector **i**, with vector indices corresponding to the index of the vertices and edges, respectively.

Definition. A **directed graph** is a collection of vertices (v_i) (points) and edges (e_j) (lines) with a direction.

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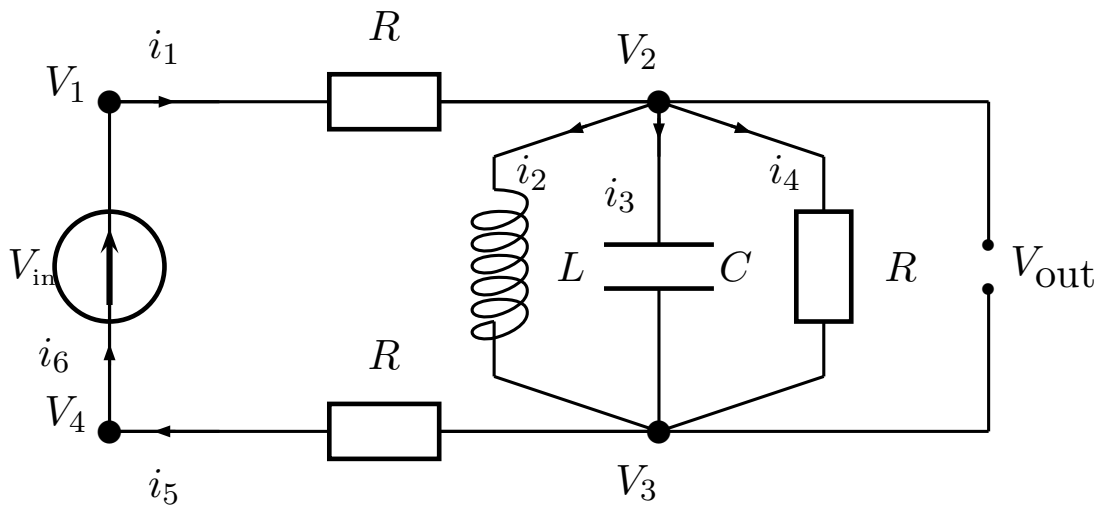
The Voltages $\mathbf{V} = \mathbf{V}(t)$ and the currents $\mathbf{i} = \mathbf{i}(t)$ change in time (are time dependent). The laws of electricity describe how they are related.

Example.



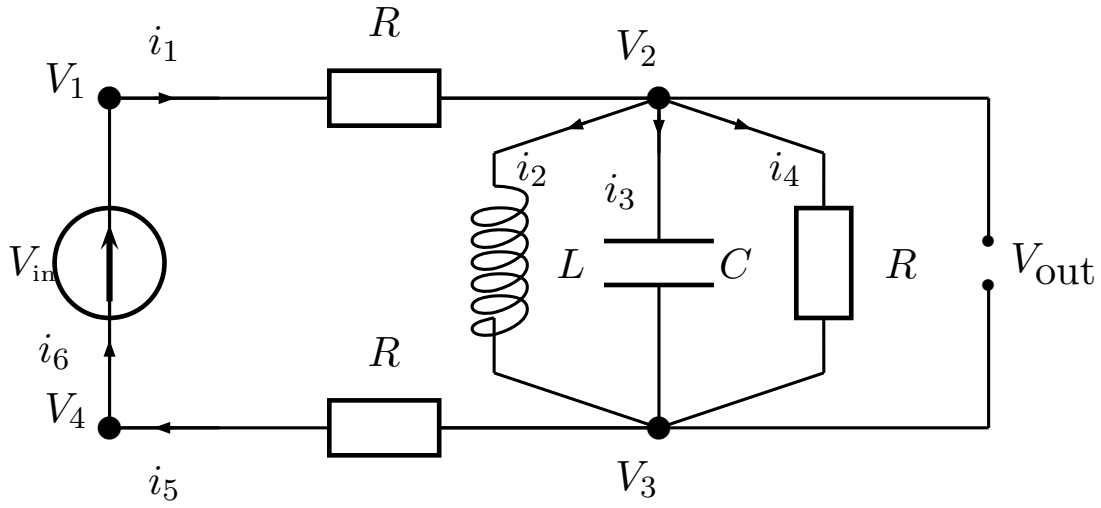
$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}$$

The size of the incidence matrix is $k \times n$, where k is the number of vertices and n the number of edges, that is, of electronic components. $k = 4$ and $n = 6$ in the present example. $n \approx 5\,10^8$ in Intels dual-core i5.

Kirchhoff's laws

Gi = 0 expresses **Kirchhoff's law of currents** stating that the inflow of the currents at a vertex equals the outflow at that vertex.

Example.



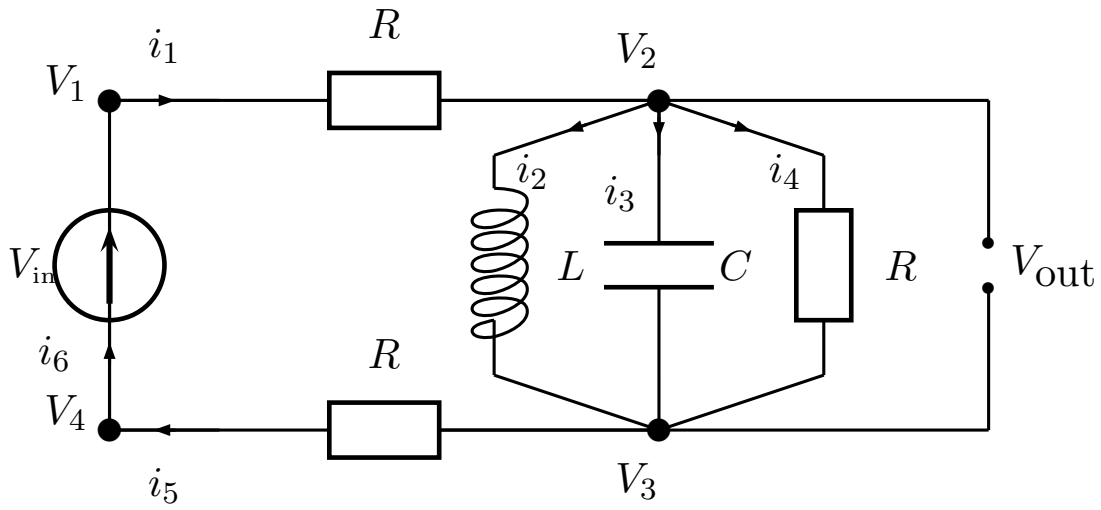
$$\mathbf{Gi} = \begin{bmatrix} i_1 - i_6 \\ -i_1 + i_2 + i_3 + i_4 \\ -i_2 - i_3 - i_4 + i_5 \\ -i_5 + i_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Kirchhoff's laws

Gi = 0 expresses **Kirchhoff's law of currents** stating that the inflow of the currents at a vertex equals the outflow at that vertex.

Kirchhoff's law of voltages is automatically fulfilled. This law states that in any closed loop (sub-circuit) the sum of the voltage differences is 0.

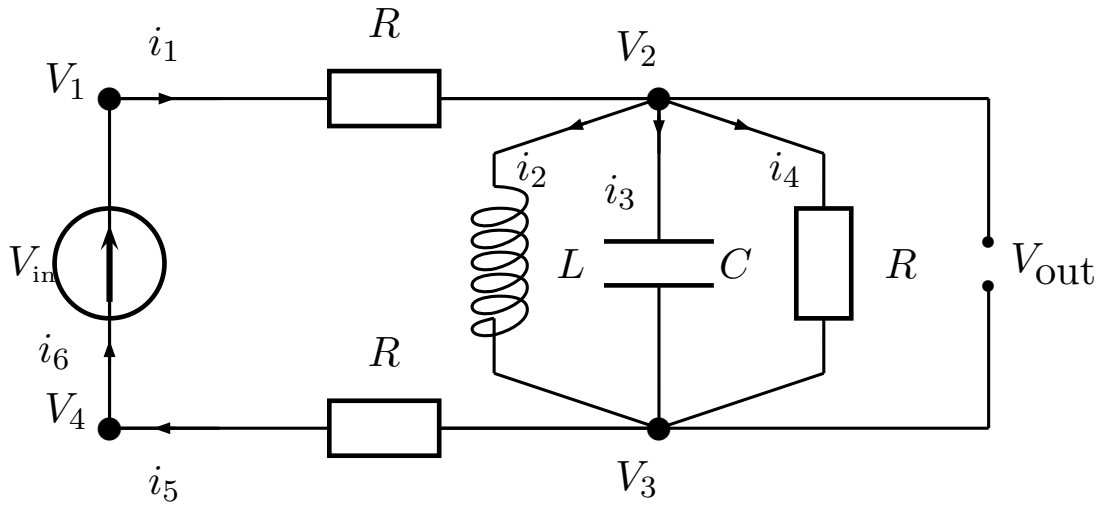
Example.



$$(V_1 - V_2) + (V_2 - V_3) + (V_3 - V_4) + (V_4 - V_1) = 0,$$
$$(V_2 - V_3) + (V_3 - V_2) = 0, \ldots$$

Note that $\mathbf{G}^\mathsf{T}\mathbf{V}$ is the vector of Voltage differences across the edges.

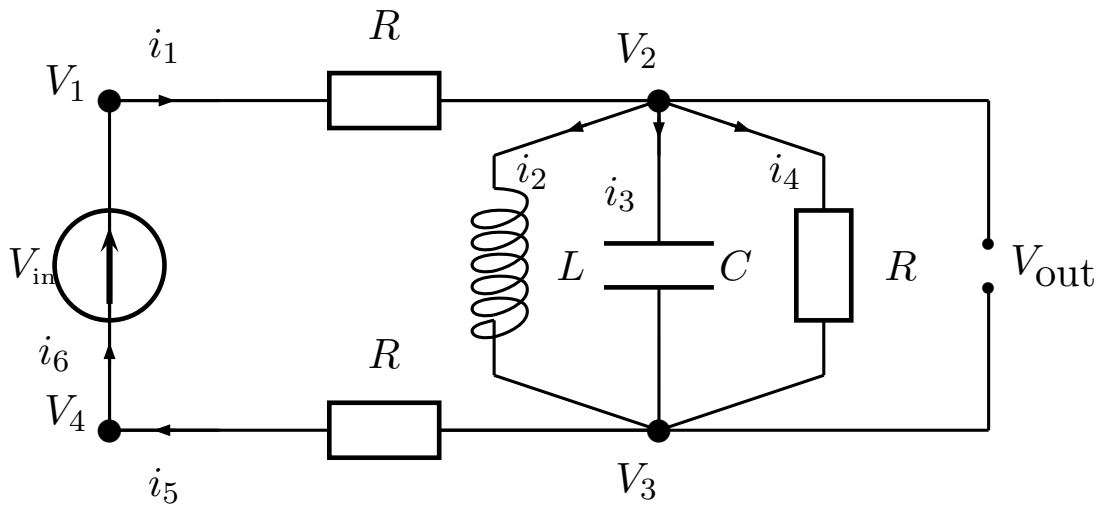
Example.



$$\mathbf{G}^\mathsf{T}\mathbf{V} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 \\ V_2 - V_3 \\ V_2 - V_3 \\ V_2 - V_3 \\ V_3 - V_4 \\ V_4 - V_1 \end{bmatrix}.$$

Note that $\mathbf{G}^\mathsf{T}\mathbf{V}$ is the vector of Voltage differences across the edges.

Example.

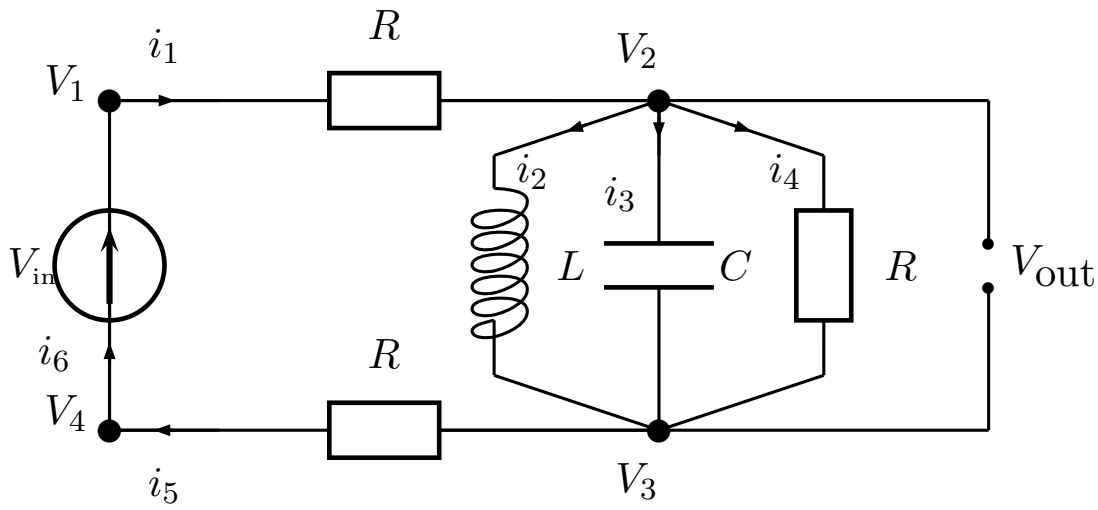


$$\mathbf{G}^\mathsf{T}\mathbf{V} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 \\ V_2 - V_3 \\ V_2 - V_3 \\ V_2 - V_3 \\ V_3 - V_4 \\ V_4 - V_1 \end{bmatrix}.$$

Note. $\mathbf{G}$ is not of full rank: $\text{rank}(\mathbf{G}) = k - 1$. This follows from the fact that $\mathbf{G}^\mathsf{T}\mathbf{1} = \mathbf{0}$: The value of $\mathbf{G}^\mathsf{T}\mathbf{V}$ does not change by adding the same constant to all V_i.

Note that $\mathbf{G}^{\mathsf{T}}\mathbf{V}$ is the vector of Voltage differences across the edges.

Example.



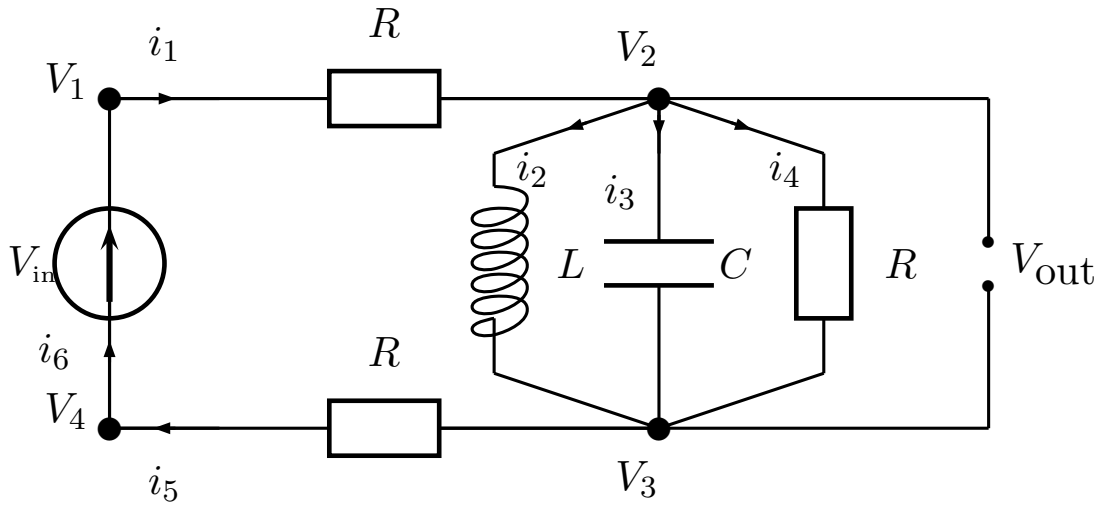
According to **Ohm's law**, we have, for instance,

$$V_1 - V_2 = R_1 i_1.$$

We will give the value of the electronic component at edge e_j (with current i_j) the same index as the edge: R_1 is the resistance of the resistor in edge e_1, R_5 is the resistance of the resistor in edge e_5, C_3 is the capacitance of the capacitor at edge e_3, etc..

Note that $\mathbf{G}^{\mathsf{T}}\mathbf{V}$ is the vector of Voltage differences across the edges.

Example.

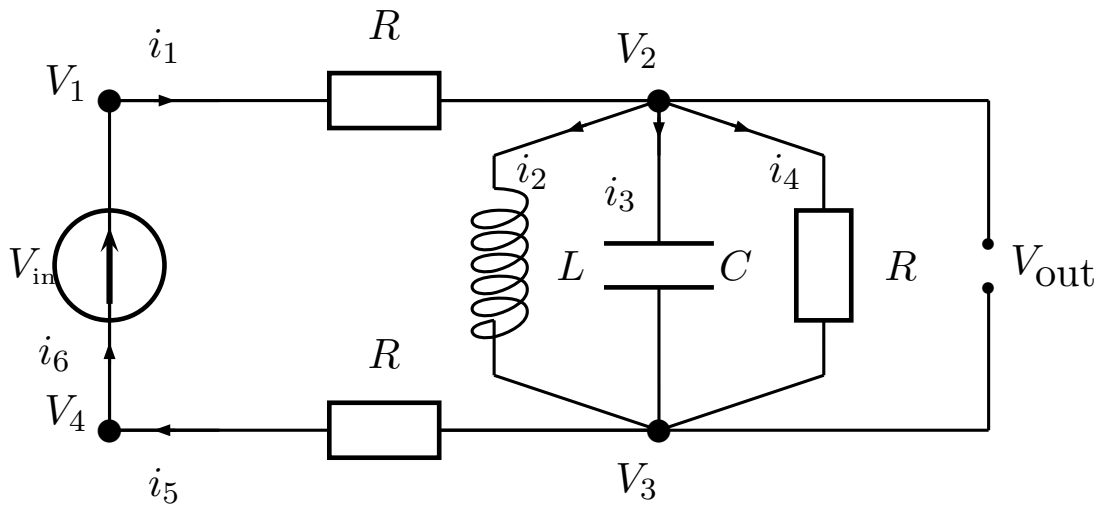


$$
\begin{aligned}
V_1 - V_2 &= R_1\, i_1, \\
V_2 - V_3 &= L_2\, i_2', \\
V_2' - V_3' &= \tfrac{1}{C_3}\, i_3, \\
V_2 - V_3 &= R_4\, i_4, \\
V_3 - V_4 &= R_5\, i_5, \\
V_4 - V_1 &= R_6\, i_6 + V_{\text{in}}.
\end{aligned}
$$

Here R_6 is the internal resistance of the input device.

Note that $\mathbf{G}^{\mathsf{T}}\mathbf{V}$ is the vector of Voltage differences across the edges.

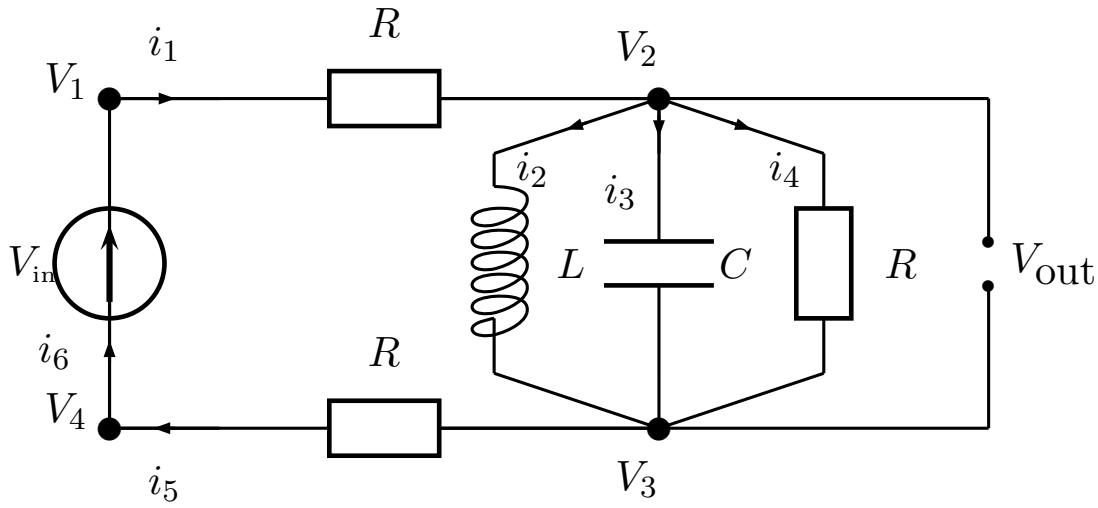
Example.



$$
\begin{aligned}
V_1' - V_2' &= R_1\, i_1' \\
V_2' - V_3' &= L_2\, i_2'' \\
V_2' - V_3' &= \tfrac{1}{C_3}\, i_3 \\
V_2' - V_3' &= R_4\, i_4' \\
V_3' - V_4' &= R_5\, i_5' \\
V_4' - V_1' &= R_6\, i_6' + V_{\text{in}}'
\end{aligned}
$$

Note that $\mathbf{G}^\mathsf{T}\mathbf{V}$ is the vector of Voltage differences across the edges.

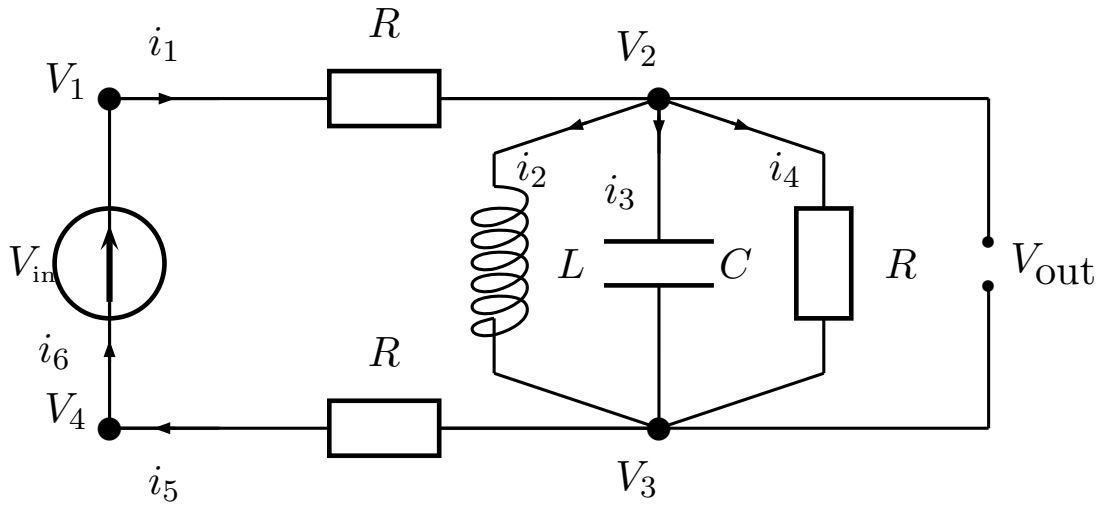
Example.



$$\mathbf{G}^\mathsf{T}\mathbf{V}' = \begin{bmatrix} V_1' - V_2' \\ V_2' - V_3' \\ V_2' - V_3' \\ V_2' - V_3' \\ V_3' - V_4' \\ V_4' - V_1' \end{bmatrix} = \begin{bmatrix} R_1\, i_1' \\ L_2\, i_2'' \\ \frac{1}{C_3}\, i_3 \\ R_4\, i_4' \\ R_5\, i_5' \\ R_6\, i_6' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_{\text{in}}' \end{bmatrix}$$

Note that $\mathbf{G}^{\mathsf{T}}\mathbf{V}$ is the vector of Voltage differences across the edges.

Example.

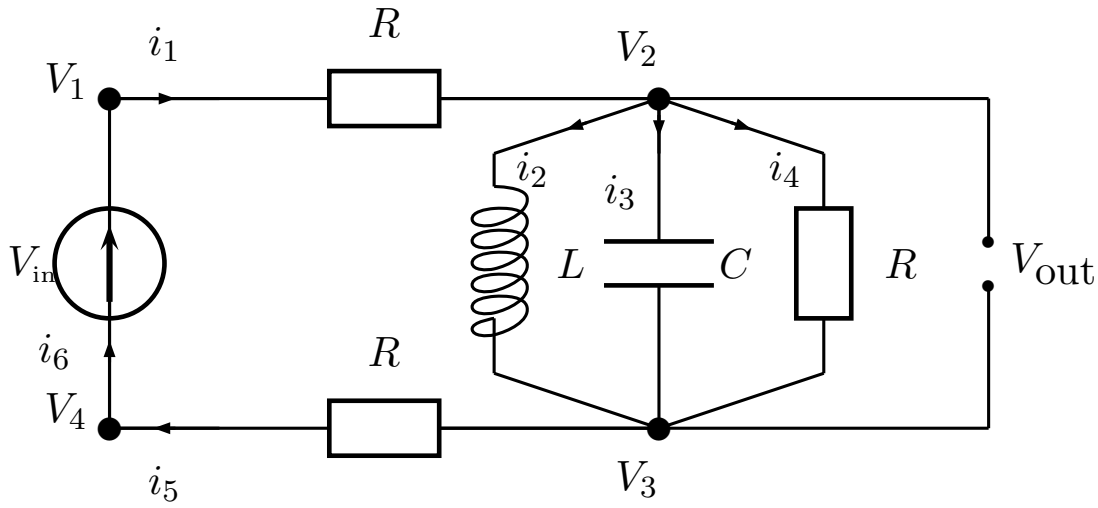


$$\mathbf{G}^{\mathsf{T}}\mathbf{V}' = \mathbf{R}\mathbf{I}' + \widetilde{\mathbf{C}}\,\mathbf{I} + \mathbf{L}\mathbf{I}'' + \mathbf{e}\,u, \qquad \text{where}$$

$$\mathbf{R} \equiv \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 \end{bmatrix}, \quad \widetilde{\mathbf{C}} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{L} = \ldots$$

Note that $\mathbf{G}^{\mathsf{T}}\mathbf{V}$ is the vector of Voltage differences across the edges.

Example.



$$\mathbf{G}^{\mathsf{T}}\mathbf{V}' = \mathbf{R}\mathbf{I}' + \widetilde{\mathbf{C}}\,\mathbf{I} + \mathbf{L}\mathbf{I}'' + \mathbf{e}\,u, \qquad \text{where}$$

$$\mathbf{L} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad \mathbf{e} \equiv \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \qquad u(t) \equiv V_{\text{in}}'(t)$$

The currents and the voltages in the electronic network satisfy the relations

$$\begin{cases} \mathbf{G}\,\mathbf{i} = \mathbf{0} \\ \mathbf{G}^{\top}\mathbf{V}' = \mathbf{R}\,\mathbf{i}' + \widetilde{\mathbf{C}}\,\mathbf{i} + \mathbf{L}\,\mathbf{i}'' + \mathbf{e}\,u \end{cases}$$

or, with $\mathbf{J} \equiv \mathbf{i}'$, we can turn the second order differential equation into two coupled first order differential equations:

$$\begin{cases} \mathbf{G}\,\mathbf{i}' = \mathbf{0} \\ \mathbf{G}^{\top}\mathbf{V}' - \mathbf{R}\,\mathbf{i}' - \mathbf{L}\,\mathbf{J}' = \widetilde{\mathbf{C}}\,\mathbf{i} + \mathbf{e}\,u \\ \mathbf{i}' = \mathbf{J} \end{cases}$$

Combine these three relations into one first order diff.eq.

$$\begin{bmatrix} \mathbf{0} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^{\top} & -\mathbf{R} & -\mathbf{L} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix}' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \\ \mathbf{0} \end{bmatrix} u$$

Here, a $\mathbf{0}$ in the block matrices represent a matrix of zeros of matching size, a $\mathbf{0}$ in the block vector is a vector of appropriate size, $\mathbf{I}$ is the $n \times n$ identity matrix.

The currents and the voltages in the electronic network satisfy the relation

$$\begin{bmatrix} \mathbf{0} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^\top & -\mathbf{R} & -\mathbf{L} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix}' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \\ \mathbf{0} \end{bmatrix} u$$

Uniqueness. If we have a solution, then adding a constant to the Voltages at all vertices (the same constant) is also a solution. We therefore, fix one of the Voltages to 0 (i.e., connect that vertex to the earth).
We incorporate this scaling into the model by replacing the $k \times k$ left upper block of $\mathbf{0}$ in the matrix at the left by $\mathbf{E}$, a $k \times k$ matrix of zeros except at the diagonal position (ℓ, ℓ) where $\mathbf{E}$ has entry 1. This means that we fix V_ℓ to 0.
Note that this does not affect the values of the i_j: because, since $\mathbf{G}$ does not have full rank, the other rows (other than the ℓth) determine the values of the i_j.

The currents and the voltages in the electronic network satisfy the relation

$$\begin{bmatrix} \mathbf{0} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^{\mathsf{T}} & -\mathbf{R} & -\mathbf{L} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix}' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \\ \mathbf{0} \end{bmatrix} u$$

Uniqueness. If we have a solution, then adding a constant to the Voltages at all vertices (the same constant) is also a solution. We therefore, fix one of the Voltages to 0 (i.e., connect that vertex to the earth).
As an alternative, this scaling can be incorporated by reducing the matrix $\mathbf{G}$ to a $(k-1) \times n$ matrix by deleting the ℓth rows of $\mathbf{G}$. Deleting the ℓth column of $\mathbf{G}^{\mathsf{T}}$ means elimination of V_ℓ (of which the value is 0). As before, eliminating, the ℓth row of $\mathbf{G}$ does not affect the values of the vector $\mathbf{i}$.

The currents and the voltages in the electronic network satisfy the relation $\mathbf{B}\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}\,u$, where

$$\mathbf{B} \equiv \begin{bmatrix} \mathbf{E} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^{\mathsf{T}} & -\mathbf{R} & -\mathbf{L} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \; \mathbf{A} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \; \mathbf{x} \equiv \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix}, \; \mathbf{b} \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \\ \mathbf{0} \end{bmatrix}.$$

B and **A** are square matrices of dimension $k + 2n$,

b is a $(k + 2n)$-vector,

x is a $(k + 2n)$-vector valued function of t,

u is a scalar-valued function of t.

The currents and the voltages in the electronic network satisfy the relation $\mathbf{B}\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}\,u$, where

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The output

We are interested in the voltage difference at two vertices. We can express this as an inner product $\mathbf{c}^{\mathsf{T}}\mathbf{x}$ of $\mathbf{x}$ and some $(k + 2n)$-vector $\mathbf{c}$.

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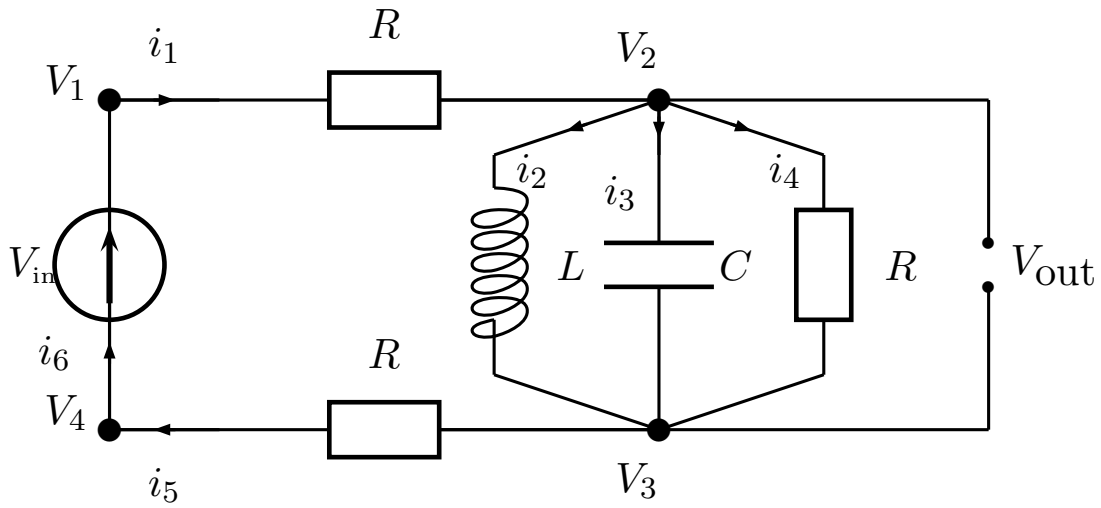
$$\mathbf{B} \equiv \begin{bmatrix} \mathbf{E} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^{\mathsf{T}} & -\mathbf{R} & -\mathbf{L} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \quad \mathbf{x} \equiv \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix}, \quad \mathbf{b} \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \\ \mathbf{0} \end{bmatrix}.$$

The output

Example.



With $\mathbf{c}^{\mathsf{T}} \equiv (0, 1, -1, 0, \mathbf{0}^{\mathsf{T}}, \mathbf{0}^{\mathsf{T}})^{\mathsf{T}}$ we have that

$$\mathbf{c}^{\mathsf{T}}\mathbf{x} = V_2 - V_3$$

Here $\mathbf{0}$ is the n-vector of zeros.

Dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

$k + 2n$ is the **number of states** or **order** of the system,

$t \rightsquigarrow \mathbf{x}(t)$ is the **state** of the system,

b is the **input** or **control** vector, **c** is the **output** vector,

$t \rightsquigarrow u(t)$ is the **control function**,

$t \rightsquigarrow y(t)$ is the **output of the system**.

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Suppose the control function $u(t) = e^{2\pi i\omega t}$ is an harmonic oscillation. We try to find a state function $\mathbf{x}$ of the form

$$\mathbf{x}(t) = \mathbf{x}_0\, e^{2\pi i\omega t}$$

for some $k + 2n$ vector $\mathbf{x}_0$. Substitution leads to

$$\mathbf{x}'(t) = \mathbf{B}\mathbf{x}_0(2\pi i\omega)(e^{2\pi i\omega t}) = \mathbf{A}\mathbf{x}_0(e^{2\pi i\omega t}) + \mathbf{b}(e^{2\pi i\omega t}).$$

Hence, $\mathbf{x}_0 = (\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$ and

$$y(t) = H(\omega)e^{2\pi i\omega t}, \quad \text{with} \quad H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$$

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Theorem. If $u(t) = e^{2\pi i\omega t}$, then $y(t) = H(\omega)e^{2\pi i\omega t}$

with $$H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$$

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Theorem. If $u(t) = e^{2\pi i\omega t}$, then $y(t) = H(\omega)e^{2\pi i\omega t}$

with $$H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$$

If $u \in L^2(\mathbb{R})$, then $u(t) = \int \widehat{u}(\omega)e^{2\pi i\omega t}\,\mathrm{d}t$ and

$$y(t) = \int H(\omega)\,\widehat{u}(\omega)e^{2\pi i\omega t}\,\mathrm{d}\omega$$

Here we used that fact that our system is linear.

Dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

Theorem. If $u \in L^2(\mathbb{R})$, then $y(t) = \int H(\omega)\,\widehat{u}(\omega) e^{2\pi i\omega t}\,\mathrm{d}\omega$

with $$H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$$

H is the **response** or **transfer** function. It describes the response of the system to an harmonic oscillation (at the input). The amplitude (at the output) of such an oscillation with frequency ω is amplified with $|H(\omega)|$ and the **phase is shifted** by $\phi(\omega)$ with $\phi(\omega) \in [0, 2\pi)$ such that

$$H(\omega) = |H(\omega)|\, e^{i\phi(\omega)}.$$

Dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

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with $$H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$$

H is the **response** or **transfer** function.

The transfer function reveals how the system works as a **filter**: some frequencies are damped, others are amplified.

Dynamical system

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$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

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with $$H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$$

H is the **response** or **transfer** function.

If all input is real, (all entries of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{b}$ and $\mathbf{c}$ are real), then H is even: $\overline{H(\omega)} = H(-\omega)$.

Dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

Theorem. If $u \in L^2(\mathbb{R})$, then $y(t) = \int H(\omega)\,\widehat{u}(\omega) e^{2\pi i\omega t}\,\mathrm{d}\omega$

with $$H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$$

H is the **response** or **transfer** function.

The graph of $\omega \rightsquigarrow |H(\omega)|$ ($\omega \in [0, \infty)$ along the horizontal axis, $|H(\omega)|$ along the vertical axis on **Decibel scale** (Db), i.e., $20\log_{10}$-scale) is called the **Bode plot** of the transfer function.

The Bode plot shows what frequencies (in the input) are 'amplified', and what frequencies are damped.

Dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

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The curve in the complex plain described by $\omega \rightsquigarrow H(\omega)$ also gives useful information. Note that a point on this curve does not reveal the corresponding value of ω: it relates $|H(\omega)|$ to $\phi(\omega)$.

Stability of dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t) \end{cases}$$

Consider an **eigenpair** $(\lambda, \mathbf{v})$ of the matrix pair $(\mathbf{A}, \mathbf{B})$:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{B}\mathbf{v}.$$

λ is an **eigenvalue** with **eigenvector** $\mathbf{v}$.

Stability of dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t) \end{cases}$$

Consider an **eigenpair** $(\lambda, \mathbf{v})$ of the matrix pair $(\mathbf{A}, \mathbf{B})$:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{B}\mathbf{v}.$$

Suppose that at time t_0 the solution $\mathbf{x}$ is perturbed by $\varepsilon\mathbf{v}$, i.e., $\widetilde{\mathbf{x}}$ satisfies

$$\begin{cases} \mathbf{B}\widetilde{\mathbf{x}}'(t) = \mathbf{A}\widetilde{\mathbf{x}}(t) + \mathbf{b}\,u(t), \\ \widetilde{\mathbf{x}}(t) = \mathbf{x} \text{ for } t < t_0, \\ \widetilde{\mathbf{x}}(t_0) = \mathbf{x}(t_0) + \varepsilon\mathbf{v}. \end{cases}$$

Then, the error $\mathbf{e} \equiv \widetilde{\mathbf{x}} - \mathbf{x}$ satisfies

$$\mathbf{B}\mathbf{e}' = \mathbf{A}\mathbf{e} \quad \text{and} \quad \mathbf{e}(t_0) = \varepsilon\mathbf{v}$$

Hence, $\mathbf{e}(t) = \varepsilon e^{\lambda t}\mathbf{v}$ for $t \geq t_0$.

Stability of dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t) \end{cases}$$

Consider an **eigenpair** $(\lambda, \mathbf{v})$ of the matrix pair $(\mathbf{A}, \mathbf{B})$:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{B}\mathbf{v}.$$

The system is unstable if some small perturbation has a large effect.

Stability of dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t) \end{cases}$$

Consider an **eigenpair** $(\lambda, \mathbf{v})$ of the matrix pair $(\mathbf{A}, \mathbf{B})$:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{B}\mathbf{v}.$$

The system is **stable** if all eigenvalues of $(\mathbf{A}, \mathbf{B})$ are in $\mathbb{C}^- \equiv \{\lambda \in \mathbb{C} \mid \text{Re}(\lambda) < 0\}$, the left half of the complex plane.

Then, all singularities of $\lambda \rightsquigarrow \mathbf{c}^{\mathsf{T}}(\mathbf{A} - \lambda\mathbf{B})^{-1}\mathbf{b}$ are in $\mathbb{C}^-$.

Dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

Theorem. If $u \in L^2(\mathbb{R})$, then $y(t) = \int H(\omega)\,\widehat{u}(\omega) e^{2\pi i\omega t}\,\mathrm{d}\omega$

with $$H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$$

Here we used that fact that our system is linear and we assumed that H is bounded, or, equivalently, the pair $(\mathbf{A}, \mathbf{B})$ does not have an eigenvalue on the imaginary axis.

The transfer function

The transfer function of the dynamical system

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

is given by $\quad H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b} \qquad (\omega \in \mathbb{R}).$

Properties.

- $k + 2n$ is huge
- **A** and **B** are sparse (only a few non-zeros in all rows).
- **A** and **B** are general matrices (not symmetric, . . .).
- The differences in the coefficients R_i, C_i and L_i can be many order of magnitudes.

The transfer function

The transfer function of the dynamical system

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

is given by $H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b} \qquad (\omega \in \mathbb{R}).$

Computational challenges • $N \equiv k + 2n$ is huge ($\approx 10^9$).
• $H(\omega)$ has to be computed for a large range of ω.
• The transfer function has to be computed for several (related) matrices $(\mathbf{A}, \mathbf{B})$ (in the design stage).
• Practical systems contain not only **passive** elements, like resistors, capacitors, and inductors, but also many **active** components (doides), which turn the problem into a non-linear one.
• Practical system do not have only one **S**ingle **I**nput vector and a **S**ingle **O**utput vector (SISO system), but they have multiple inputs and multiple outputs (MIMO):
$\mathbf{b}$ is $N \times \ell$, $\mathbf{c}$ is $N \times \ell'$.

The transfer function

The transfer function of the dynamical system

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

is given by $\quad H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b} \qquad (\omega \in \mathbb{R}).$

Dynamical systems show up in many other applications.
For instance,

- the stability of buildings (in response to an earthquake),
- aeroplanes and bridges (to eddies in the flow of the air),
- electrical power networks,
- music instruments,
- tidal waves in bays,
- ...

Program

- Electronic Circuits
- MRI
- Diffraction
- CT

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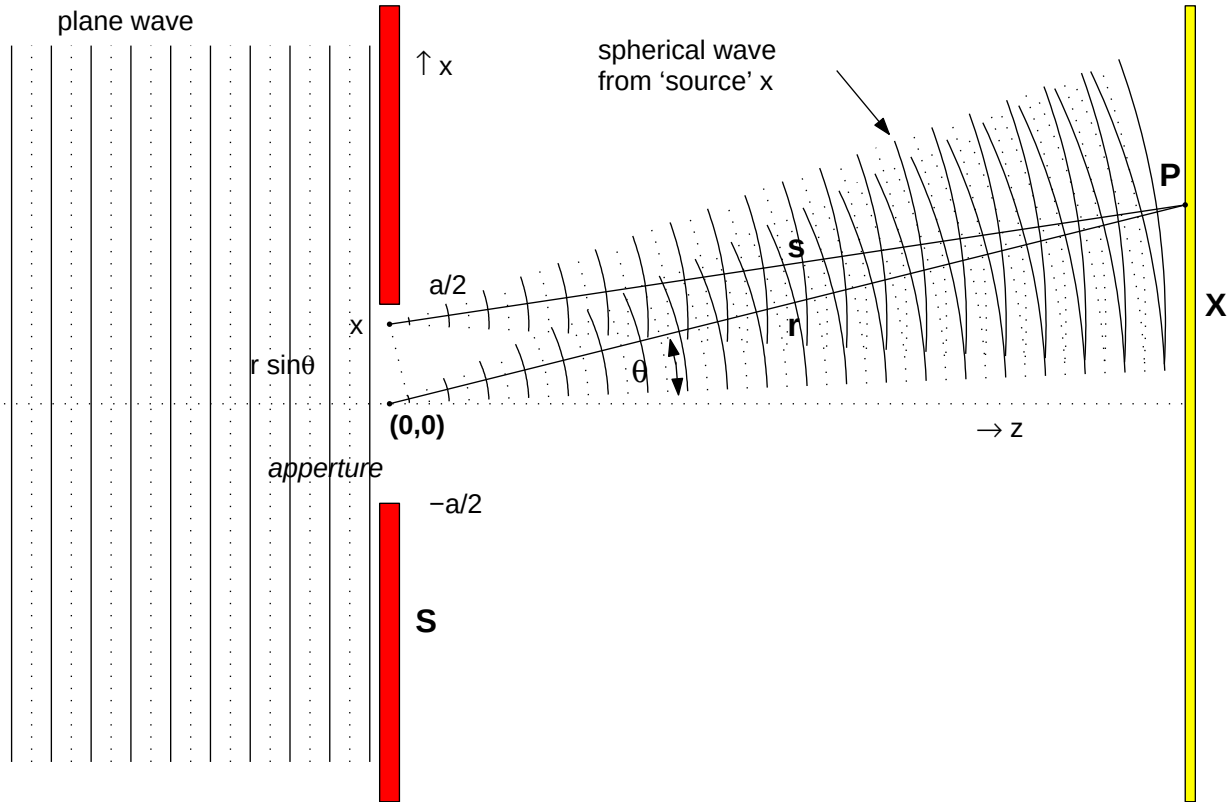
Diffraction, the stage



Diffraction, the actors

Light is a composition of electromagnetic waves.

An electromagnetic wave is combination of oscillating electric and magnetic fields in perpencdicular orientation to each other moving through space. The direction of the oscillations is perpendicular to the wave's direction of travel.

We assume a light source at location $-\infty$ on the z-axis emitting a mono-chromatic light, i.e.,

the electric field E at $z = 0$ (the location of a screen S) has the same magnitude and a contant phase at all points of the transparant part (slit) of S.
E is a plane wave for $z < 0$.

The electric field E is a **plane wave** if it is of the form

$$\vec{E}_0 \exp(2\pi i[(\vec{k}, \vec{x}) - \omega t]),$$

where, with $\vec{k} \equiv (k_1, k_2, k_3)$ and $\vec{x} \equiv (x, y, z)$,

$$(\vec{k}, \vec{x}) \equiv k_1 x + k_2 y + k_3 z.$$

$E_0 \equiv \|\vec{E}_0\|_2$ is the **amplitude**.

The wave travels in the direction of the **wave vector** $\vec{k}$.

$k \equiv \|\vec{k}\|_2$, or actually $2\pi k$, is the **wave number**,
$\lambda \equiv 1/k$ is the **wavelength**,
with c the speed of light, $\omega = \frac{c}{\lambda}$ is the **frequency**.

The vector $\vec{E}_0$ is perpendicular to $\vec{k} = (k_1, k_2, k_3)$: electromagnetic waves travel **transverse**, in contrast to, for instance, sound waves. They are **longitudinal**: $\vec{E}_0$ is a multiple of $\vec{k}$.

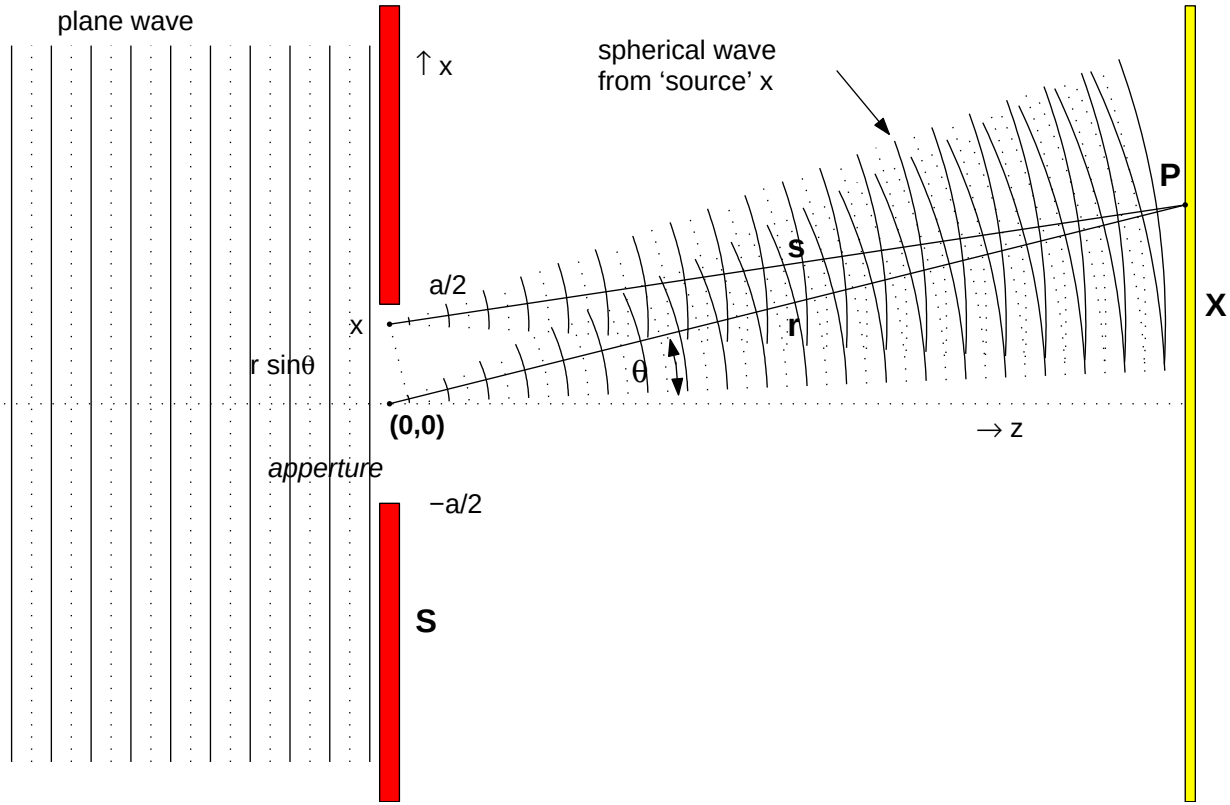
Here, we restrict ourselves to the (x, z) plane (we omit y), and we assume that the wave travels along the z-axis, i.e., $\vec{k} = (0, k_3)$. We write k instead of k_3 and we write E_0 instead of $\vec{E}_0$, thus omitting the direction of E_0.

Diffraction, the director

Huygens' principle. Every point on a wavefront which comes from asource can itself be regarded as a (secondary) source.

Frauenhofer diffraction. The diffraction obstruction (screen S with slit) is at many wavelength distance from the point at which the wave is measured (the "projection screen" X)

Diffraction, the stage



The field E at time t and location (x, z) is given by

$$E = E_0 \exp(2\pi i(kz - \omega t)) \quad \text{if} \quad z \leq 0.$$

According to Huygens' principle, the field at $P = (X, Z)$ induced from a strip of width $\mathrm{d}x$ at $(x, 0)$ will be

$$\Gamma\, A(x)\, E_0\, e^{2\pi iks}\, \mathrm{d}x,$$

where s is the distance from $(x, 0)$ to P, Γ is some proportionality factor that depends on the wavelength and Z, and $A(x)$ is the aperture function, which describes the transparent and opaque parts of the screen S.

Examples.

A is the top-hat function with width a in case of one slit.
If two slits: A is the sum of two shifted top-hat functions.

Huygens' principle is applicable if both X and the width a of the aperture are small relative to Z and that is what we will assume here.

$r \equiv$ distance$((0,0), P)$,
$\theta \equiv \angle$(line$[(0,0), P], z$-axis.
Then $P \equiv (X, Z) = (r \sin\theta, r\cos\theta)$ and $s = r - x\sin(\theta)$.
Note $\sin\theta \approx X/Z$. Hence, $|x| \ll r \;\Rightarrow$ field at P equals

$$\Gamma\, E_0\, e^{2\pi i k r} \int A(x)\, e^{-2\pi i k x \sin(\theta)}\, \mathrm{d}x = \Gamma\, E_0\, e^{2\pi i k r}\, \widehat{A}(k\sin(\theta)) :$$

the reduction of the 'strength' of the original field is $\sim \widehat{A}$.
The *intensity* $I(P)$ of the diffracted waves at P is the square of the absolute value:

$$I(P) = I_0\, (\widehat{A}(k\sin(\theta))^2, \quad \text{where} \quad I_0 \equiv |\Gamma\, E_0|^2.$$

The intensity of the wave is what we measure/observe: the absolute value of the Fourier transform can be measured. There is no information on the phase. This fact is known as **the missing phase problem** *in crystallography.*

Example.

In case of a single slit, A is the top-hat function $\Pi_{a/2}$ and

$$I(P) = I_0\, a^2 \mathrm{sinc}^2(a\,\omega), \quad \text{where} \quad \omega \equiv \frac{\sin\theta}{\lambda}.$$

Program

- Electronic Circuits
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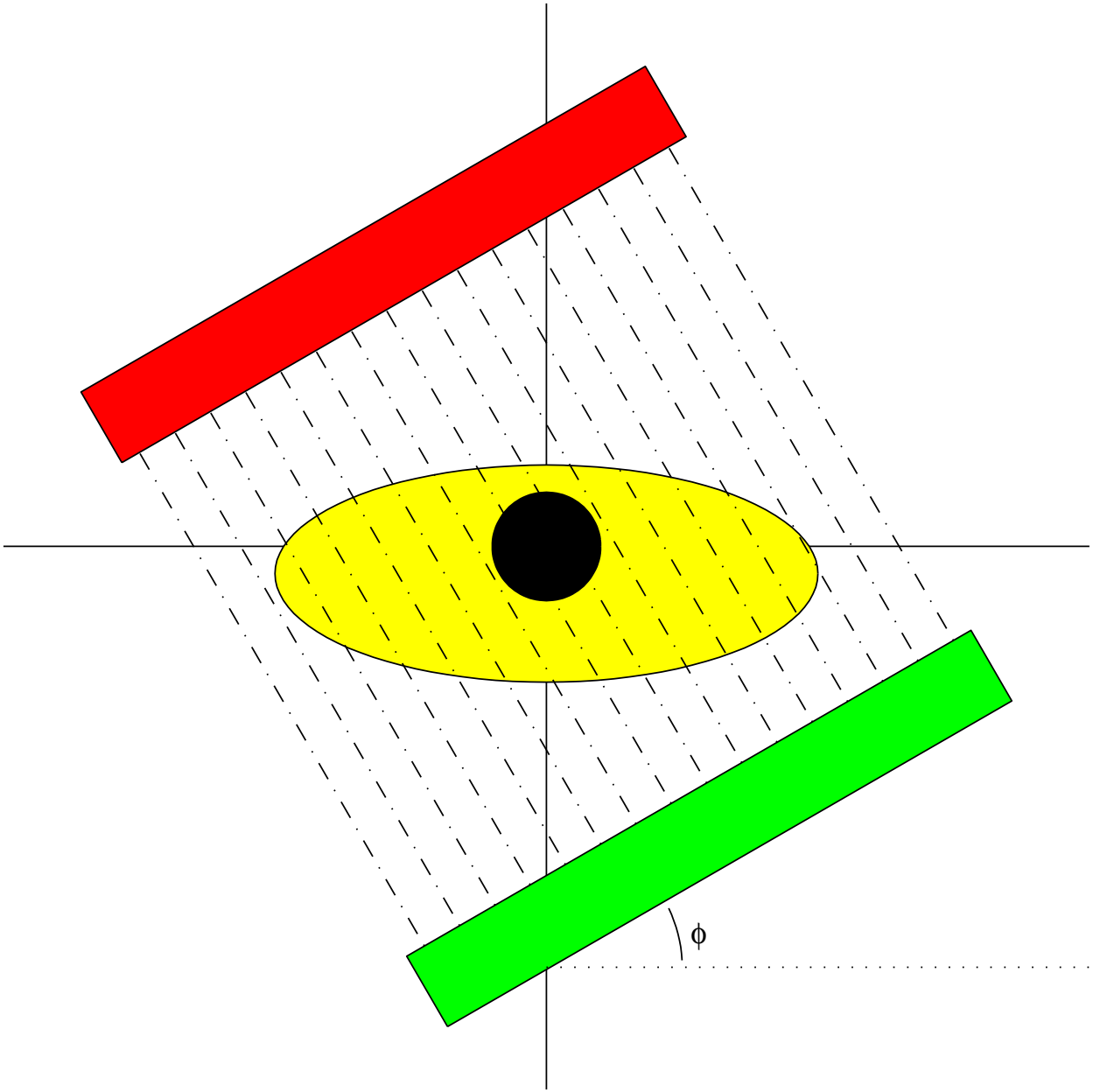
Computerised Tomography



Computerised Tomography

X-rays are transmitted from a straight line (the red beam in the picture) through an object, a slab of material (the yellow and black figure). The material partly 'absorbs' the x-rays. The intensity of the x-rays is measured at the detector (the green line parallel to the red line).

The detector is constructed to measure the intensity of those beams that pass straight through the object (scattered beams will not be detected).

The absorption depends on the kind of material and on the thickness of the slab of material.

If a x-ray with initial intensity I_0 travels through d cm of homogeneous material with absorption coefficient κ, then the measured intensity I equals

$$I = I_0\, e^{-\kappa d}.$$

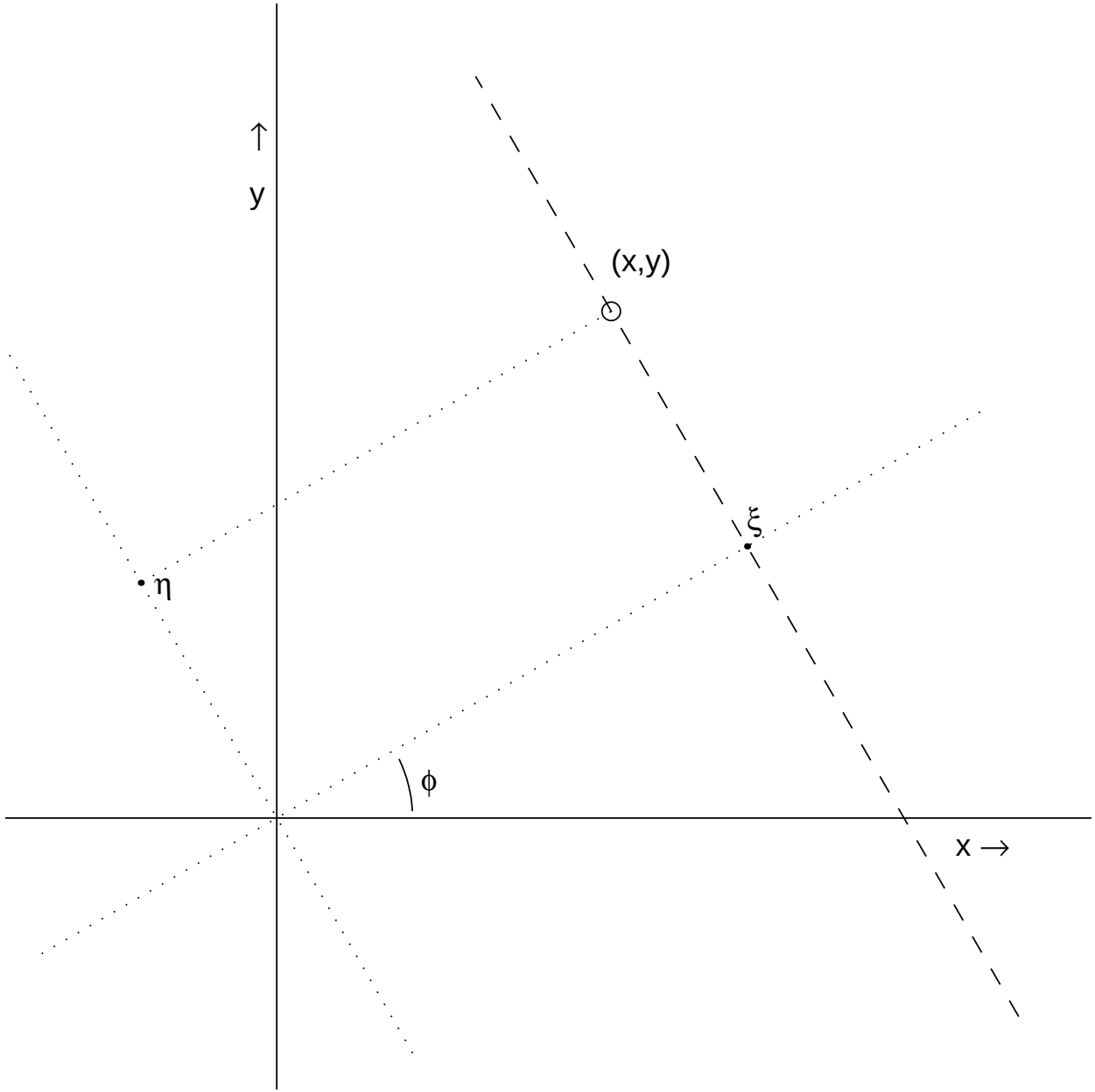
Computerised Tomography

Use Cartesian coordinates (x, y) to describe the scanner.

Suppose the absorption coefficient at point (x, y) of the object to be scanned is $f(x, y)$. The value of f at (x, y) depends on the (concentration of the) material at (x, y) of which the object is composed.

Consider an x-ray that travels along a line orthogonal to the detector.

Computerised Tomography



Computerised Tomography

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Suppose the absorption coefficient at point (x, y) of the object to be scanned is $f(x, y)$. The value of f at (x, y) depends on the (concentration of the) material at (x, y) of which the object is composed.

Consider an x-ray that travels along a line orthogonal to the detector: this is a line of points (x, y) with

$$x = x(\eta) = \xi \cos(\phi) - \eta \sin(\phi), \quad y = y(\eta) = \xi \sin(\phi) + \eta \cos(\phi)$$

with ξ fixed and ϕ the angle of the detector with x-axis (the dashed line in the picture).

Note. (ξ, η) is the point (x, y) in the plane with **rotated Cartesian coordinates**, rotated over an angle ϕ.

Computerised Tomography

Use Cartesian coordinates (x, y) to describe the scanner.

Suppose the absorption coefficient at point (x, y) of the object to be scanned is $f(x, y)$. The value of f at (x, y) depends on the (concentration of the) material at (x, y) of which the object is composed.

Consider an x-ray that travels along a line orthogonal to the detector: this is a line of points (x, y) with

$$x = x(\eta) = \xi \cos(\phi) - \eta \sin(\phi), \quad y = y(\eta) = \xi \sin(\phi) + \eta \cos(\phi)$$

with ξ fixed and ϕ the angle of the detector with x-axis (the dashed line in the picture).

At the detector at the end of this line the intensity is

$$I = I_0 \exp\left(-\int f(x(\eta), y(\eta))\, \mathrm{d}\eta\right).$$

Computerised Tomography

Use Cartesian coordinates (x, y) to describe the scanner.

Suppose the absorption coefficient at point (x, y) of the object to be scanned is $f(x, y)$. The value of f at (x, y) depends on the (concentration of the) material at (x, y) of which the object is composed.

Consider an x-ray that travels along a line orthogonal to the detector: this is a line of points (x, y) with

$$x = x(\eta) = \xi \cos(\phi) - \eta \sin(\phi), \quad y = y(\eta) = \xi \sin(\phi) + \eta \cos(\phi)$$

with ξ fixed and ϕ the angle of the detector with x-axis (the dashed line in the picture).

We therefore, can measure

$$p_\phi(\xi) \equiv \int f(x(\eta), y(\eta)) \, \mathrm{d}\eta.$$

Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int f(x(\eta), y(\eta))\, \mathrm{d}\eta \quad .$$

p_ϕ "is" the **projection** f onto the ξ-coordinate.

Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int \quad f(x(\eta), y(\eta))\, \mathrm{d}\eta \qquad .$$

We measure the intensity for all values of ξ in an interval (corresponding with the surface of the detector). η ranges between transmission device and detector. For mathematical simplicity, we assume that both η and ξ (as well as x and y) range between $-\infty$ and $+\infty$ (or, equivalently, if f is defined on a bounded domain, then we assume f to be extended to $\mathbb{R}^2$ with the value 0.

Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int_{-\infty}^{+\infty} f(x(\eta), y(\eta))\, \mathrm{d}\eta \quad (\xi \in \mathbb{R}).$$

Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

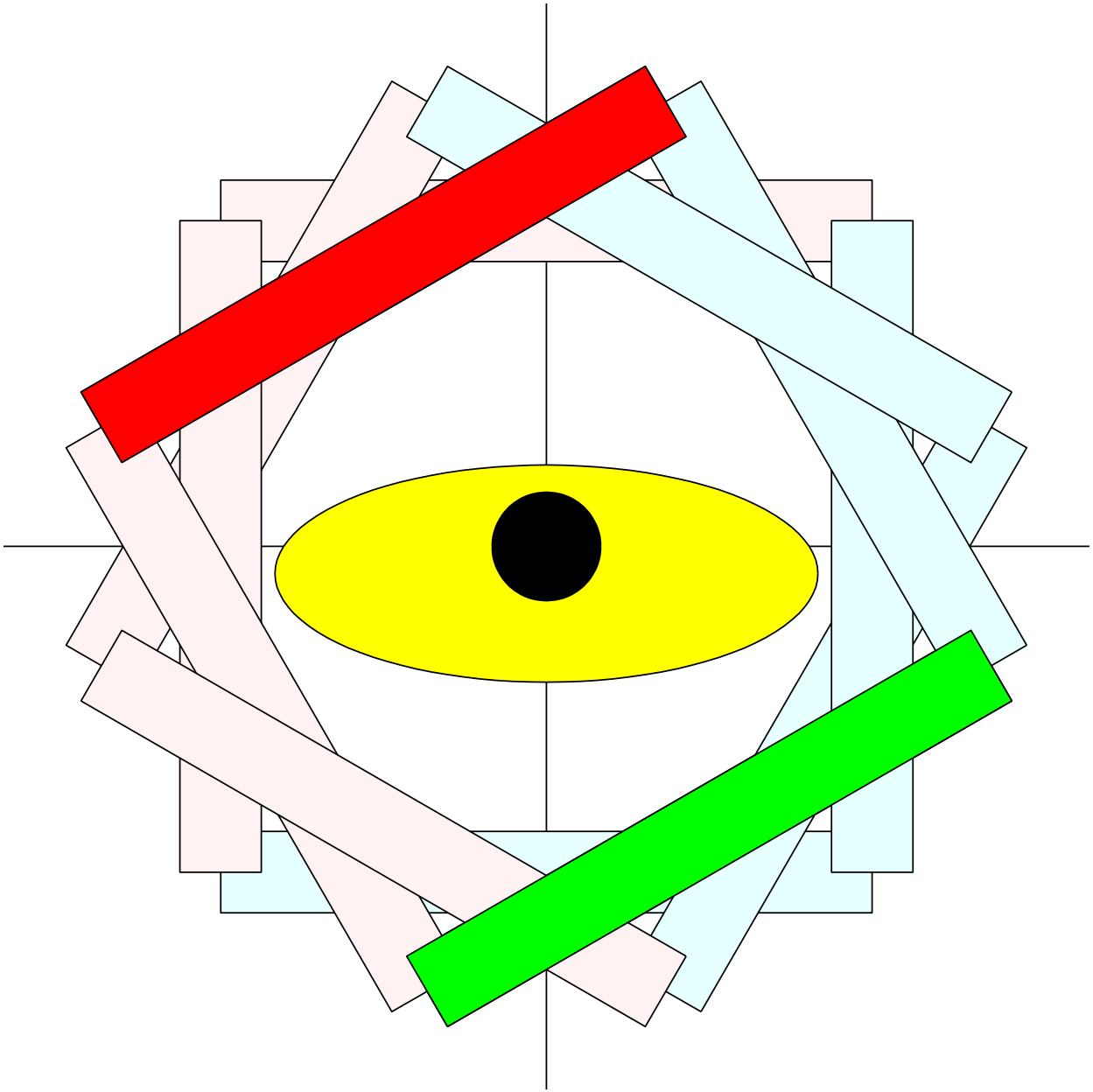
we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int \quad f(x(\eta), y(\eta))\, \mathrm{d}\eta \qquad .$$

We are interested in computing (reconstructing) f from our measurements, that is, from p_ϕ.

Note that p_ϕ does not determine f.

Computerised Tomography



Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int \quad f(x(\eta), y(\eta))\, \mathrm{d}\eta \qquad .$$

We are interested in computing (reconstructing) f from our measurements, that is, from p_ϕ.

Note that p_ϕ does not determine f.

Rotating the scanner, i.e., measuring p_ϕ for a range of ϕ (taking pictures from a number of directions), gives more information.

Note that $p_\phi(\xi) = p_{\pi+\phi}(-\xi)$.

Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int \quad f(x(\eta), y(\eta))\, \mathrm{d}\eta \qquad .$$

Assignment.

Given $p_\phi(\xi)$ for all $\xi \in \mathbb{R}$ and all $\phi \in [0, 2\pi)$, compute f.

Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int \quad f(x(\eta), y(\eta))\, \mathrm{d}\eta \qquad .$$

Assignment.
Given $p_\phi(\xi)$ for all $\xi \in \mathbb{R}$ and all $\phi \in [0, 2\pi)$, compute f.

With $p(\xi, \phi) \equiv p_\phi(\xi)$,

the map $f \rightsquigarrow p$ is the **Radon transformation** of f,

the graph of p as a 2-d picture is **the sinogram** of f.

Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int \quad f(x(\eta), y(\eta))\, \mathrm{d}\eta \qquad .$$

Assignment.

Given $p_\phi(\xi)$ for all $\xi \in \mathbb{R}$ and all $\phi \in [0, 2\pi)$, compute f.

Compute (reconstruct) f from its sinogram.

Computerised Tomography

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int \quad f(x(\eta), y(\eta))\, \mathrm{d}\eta \qquad .$$

Assignment.

Given $p_\phi(\xi)$ for all $\xi \in \mathbb{R}$ and all $\phi \in [0, 2\pi)$, compute f.

Practical observations.

• In practice, the x-rays are directed in a series of thin parallel beams homogeneously distributed: ξ is discretised.

• A 3-dimensional image can be obtained by piling the the 2-dimensional images of slices.

Back projection

If we express the point (x, y) is polar coordinates

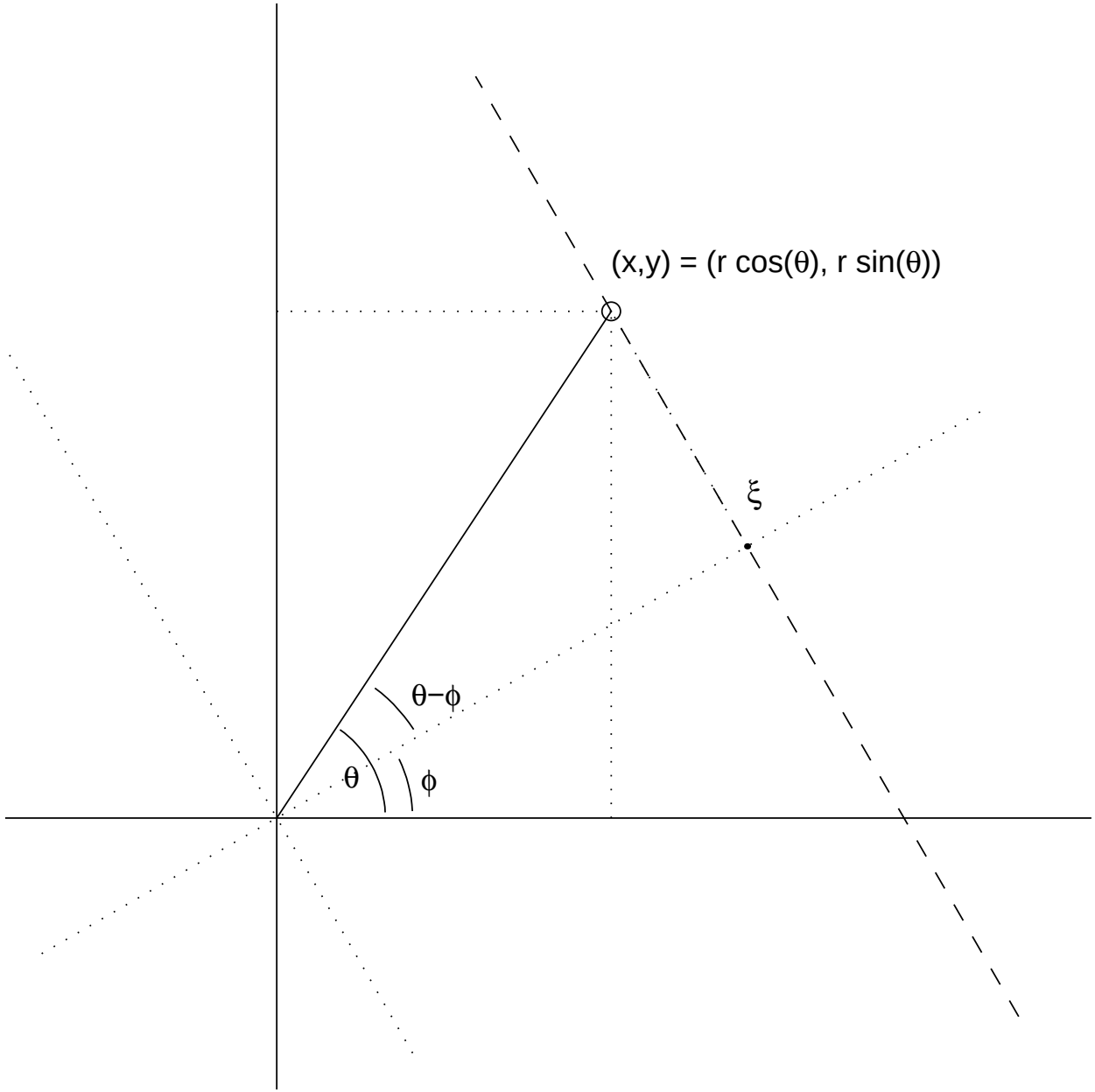
$$(x, y) = (r\cos(\theta), r\sin(\theta)),$$

then the ξ coordinate of (x, y) (i.e., the coordinate in rotated Cartesian coordinates) equals

$$\xi = r\cos(\theta - \phi).$$

(*see the picture on the next transparency*).

Computerised Tomography



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The transformation

$$\widetilde{f}(r\cos(\theta), r\sin(\theta)) \equiv \int_0^\pi p_\phi(r\cos(\theta - \phi))\,\mathrm{d}\phi$$

can be viewed as the **back-projection** (**BP**) of p_ϕ.

Observation. *The* **BP** *gives a blurred reconstruction of* f.

To obtain a sharp reconstruction,
we use Fourier transforms.

$$\widehat{f}(\omega_1,\omega_2)=\iint f(x,y)\,e^{-2\pi i(x\omega_1+y\omega_2)}\,\mathrm{d}x\,\mathrm{d}y.$$

Rotate the coordinates in both (x,y)-plane as well as in (ω_1,ω_2)-plane:

$$\begin{cases} x=\xi\,c_\phi-\eta\,s_\phi, \qquad y=\xi\,s_\phi+\eta\,c_\phi \\ \omega_1=\rho_1 c_\phi-\rho_2 s_\phi, \quad \omega_2=\rho_1 s_\phi+\rho_2 c_\phi. \end{cases}$$

Then

$$\begin{aligned}&\widehat{f}(\rho_1 c_\phi-\rho_2 s_\phi,\rho_1 s_\phi+\rho_2 c_\phi)\\ &\qquad=\iint f(\xi c_\phi-\eta s_\phi,\xi s_\phi+\eta c_\phi)\,e^{-2\pi i(\xi\rho_1+\eta\rho_2)}\,\mathrm{d}\eta\,\mathrm{d}\xi.\end{aligned}$$

In particular, if $\rho_2=0$ and putting $\rho\equiv\rho_1$

$$\begin{aligned}\widehat{f}(\rho c_\phi,\rho s_\phi) &= \textstyle\int\int f(\xi c_\phi-\eta s_\phi,\xi s_\phi+\eta c_\phi)\,e^{-2\pi i\xi\rho}\,\mathrm{d}\eta\,\mathrm{d}\xi\\ &= \textstyle\int\Big(\int f(\xi c_\phi-\eta s_\phi,\xi s_\phi+\eta c_\phi)\,\mathrm{d}\eta\Big)\,e^{-2\pi i\xi\rho}\,\mathrm{d}\xi\\ &= \textstyle\int p_\phi(\xi)\,e^{-2\pi i\xi\rho}\,\mathrm{d}\xi=\widehat{p}_\phi(\rho).\end{aligned}$$

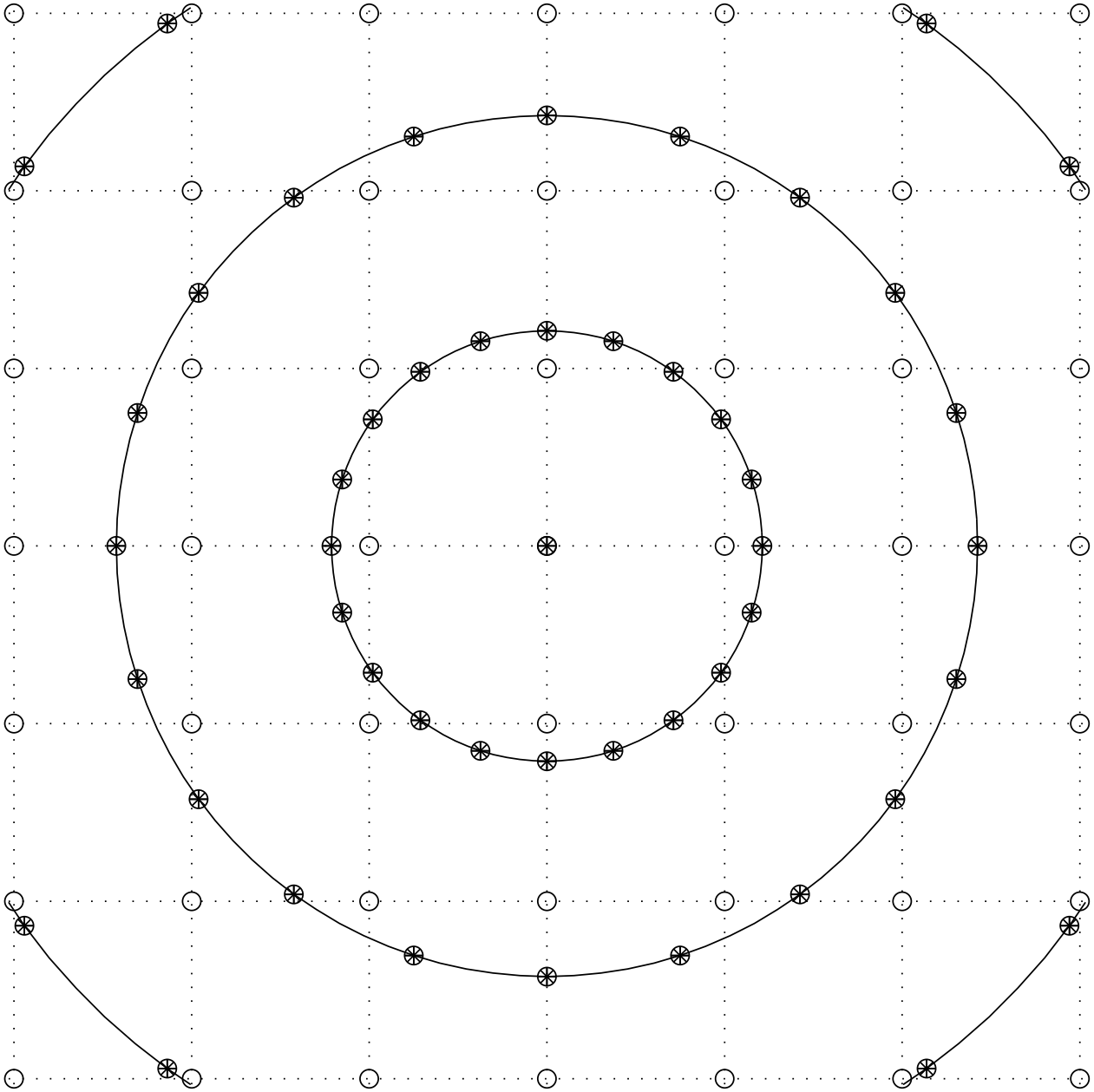
Theorem. $\widehat{f}(\rho c_\phi, \rho s_\phi) = \widehat{p}_\phi(\rho) \qquad (\rho \in \mathbb{R}, \phi \in [0, \pi))$.

Note. The point $(\rho c_\phi, \rho s_\phi)$ represents an arbitrary point in (ω_1, ω_2)-plane in **polar coordinates**.

In particular, $\widehat{f}$ is known in all frequencies and the Fourier back transform yields f.

In practice, we have to deal with discretised versions. Unfortunately, homogeneous discretization in polar coordinates does not match well homogeneous discretization of Cartesian coordinates (see picture next transparency).

Computerised Tomography



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Note. The point $(\rho c_\phi, \rho s_\phi)$ represents an arbitrary point in (ω_1, ω_2)-plane in **polar coordinates**.

We therefore express the Fourier back transform

$$f(x, y) = \iint \widehat{f}(\omega_1, \omega_2)\, e^{2\pi i(x\omega_1 + y\omega_2)}\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2.$$

into polar coordinates:

$$\begin{cases} (x, y) = (rc_\theta, rs_\theta) \\ (\omega_1, \omega_2) = (\rho c_\phi, \rho s_\phi) \end{cases}$$

Then

$$x\omega_1 + y\omega_2 = r\rho \cos(\theta - \phi), \qquad \mathrm{d}x\, \mathrm{d}y = |\rho|\, \mathrm{d}\rho\, \mathrm{d}\phi$$

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$$\begin{cases} (x, y) = (rc_\theta, rs_\theta) \\ (\omega_1, \omega_2) = (\rho c_\phi, \rho s_\phi) \end{cases}$$

Then

$$\begin{aligned} f(rc_\theta, rs_\theta) &= \int_0^\pi \int_{-\infty}^{+\infty} \widehat{f}(\rho c_\phi, \rho s_\phi)\, e^{2\pi i \rho (rc_{\theta-\phi})} |\rho|\, \mathrm{d}\rho\, \mathrm{d}\phi \\ &= \int_0^\pi \int_{-\infty}^{+\infty} \widehat{p}_\phi(\rho)\, e^{2\pi i \rho (rc_{\theta-\phi})} |\rho|\, \mathrm{d}\rho\, \mathrm{d}\phi \end{aligned}$$

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Theorem. With $\widetilde{p}_\phi(\xi) \equiv \int |\rho|\, \widehat{p}_\phi(\rho)\, e^{2\pi i \rho \xi}\, \mathrm{d}\rho$, we have that

$$f(rc_\theta, rs_\theta) = \int_0^\pi \widetilde{p}_\phi(rc_{\theta-\phi})\, \mathrm{d}\phi$$

With

$$\begin{cases} (x, y) = (rc_\theta, rs_\theta) \\ (\omega_1, \omega_2) = (\rho c_\phi, \rho s_\phi) \end{cases}$$

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Interpretation. The multiplication of $\widehat{p}_\phi(\rho)$ by $|\rho|$ act as a **filter**, damping low frequency components ($\rho \approx 0$) and amplifying high frequency ones.

f is obtained as a **filtered back-projection**, i.e., the **BP** of the filtered Fourier transform of the Radon transformed p_ϕ.

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Recall that the **BP** without filtering (i.e., **BP** of p_ϕ, rather than of $\widetilde{p}_\phi$) leads to a blurred version of f. This can be viewed as an over estimation of low frequency components. The filtering by $|\rho|$ seems to correct this.

With

$$\begin{cases} (x, y) = (rc_\theta, rs_\theta) \\ (\omega_1, \omega_2) = (\rho c_\phi, \rho s_\phi) \end{cases}$$

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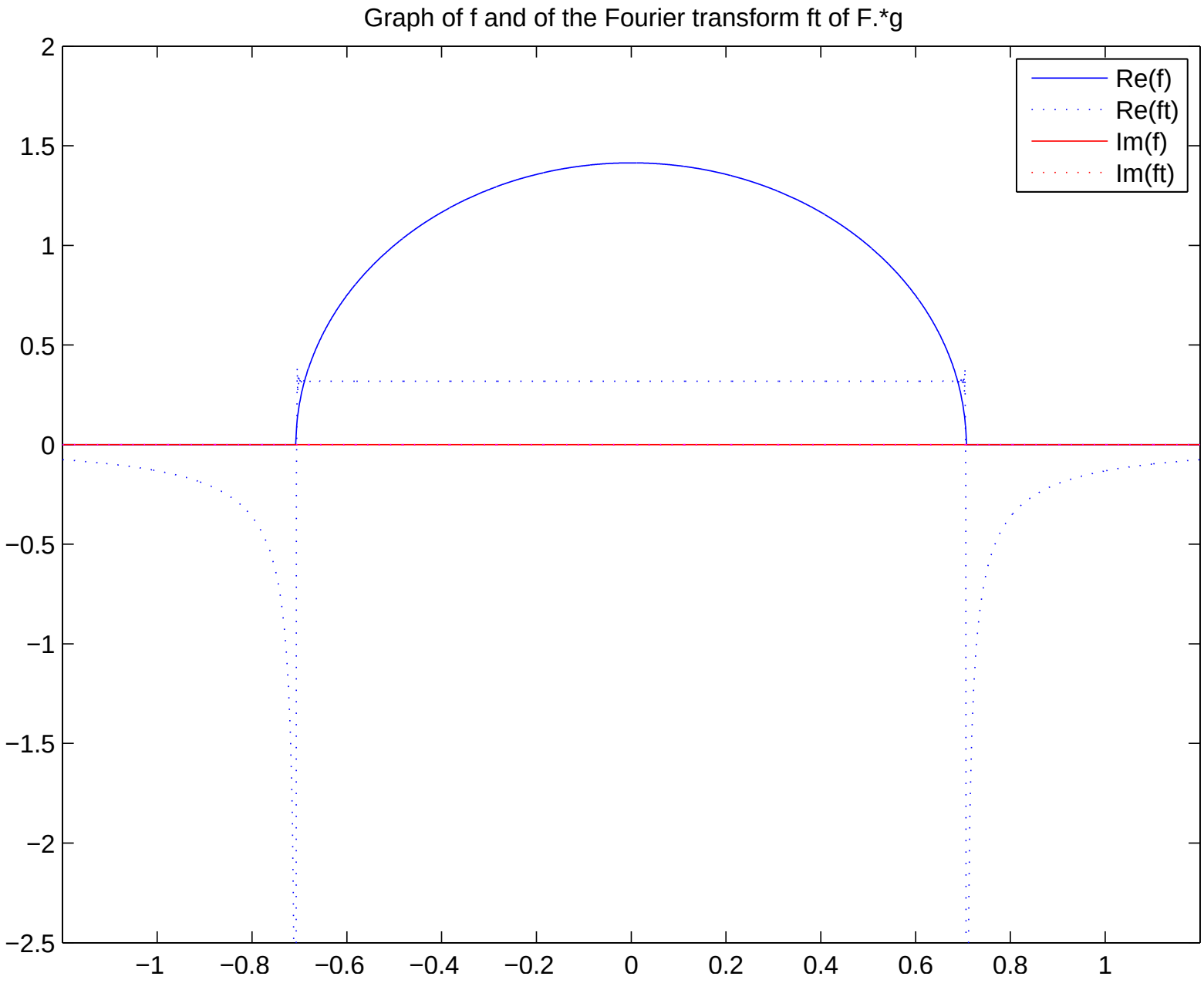
On the next picture, we see,
for $f = 1$ on the unit disk and 0 elswhere,

- p_ϕ *(denoted by* `f`*, the solid graph) and*
- $\widetilde{p}_\phi$ *(denoted by* `ft`*, the dotted graph)*

Computerised Tomography



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Interpretation. The multiplication of $\widehat{p}_\phi(\rho)$ by $|\rho|$ act as a **filter**, damping low frequency components ($\rho \approx 0$) and amplifying high frequency ones.

Noise. The formulae indicate that $\widehat{p}_\phi(\rho)$ decays 'rapidly' for $|\rho| \to \infty$.
On average, noise is equally large in all frequency components.
Therefore, the filter $|\rho|$ amplifies (high frequent) noise.
In practice, a version of $|\rho|$ that levels off for larger values of $|\rho|$ is used.

Examples. For some $\kappa > 0$, $\quad \min(|\rho|, \kappa)$, $\quad \kappa \arctan(|\rho|/\kappa)$, etc..

Pixels

If f is discretised, as a step function, then

$$f = \sum_{i,j} f_{i,j}\, \chi_{i,j}$$

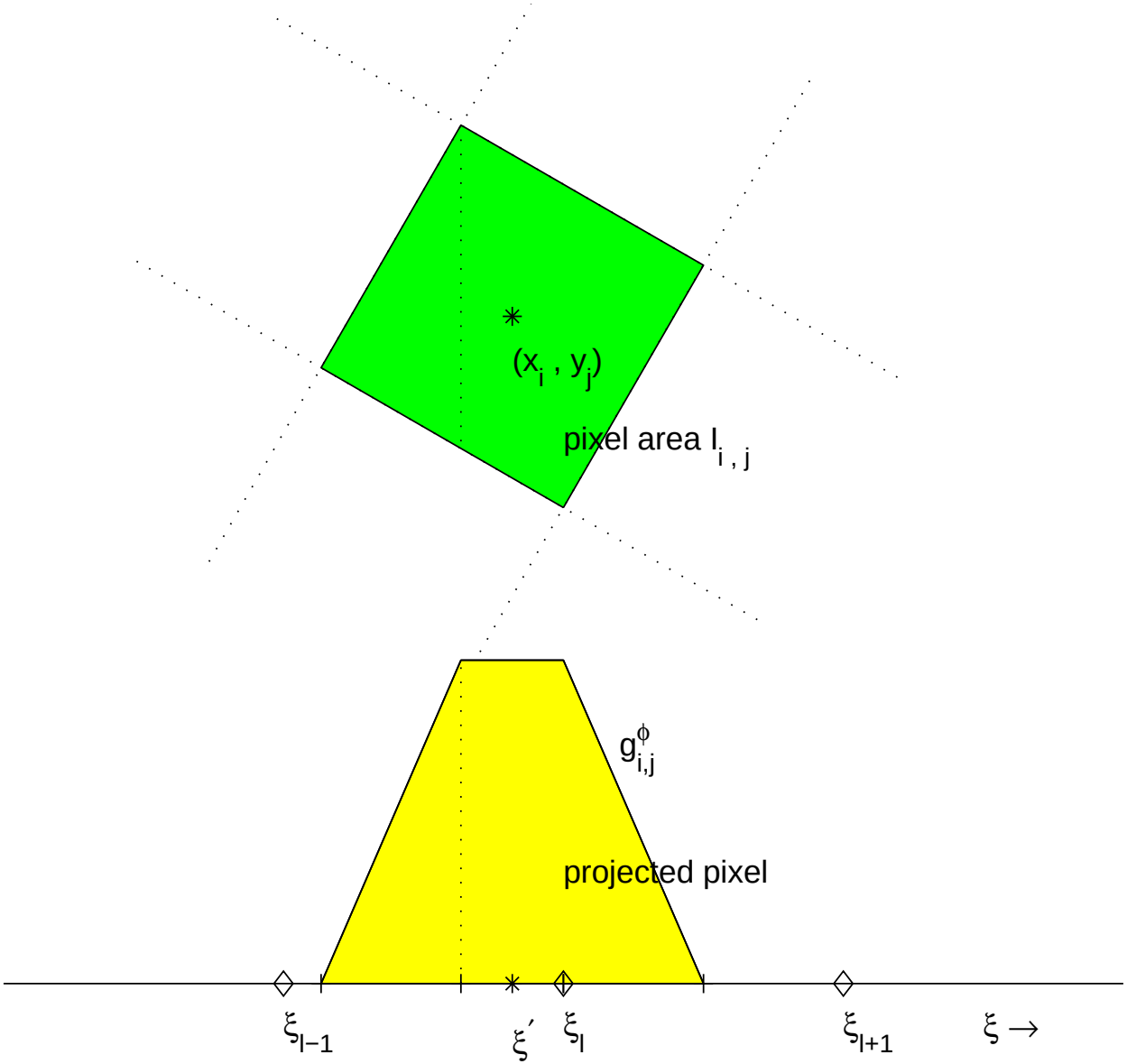
where $f_{i,j}$ is the average value of f on the (i,j)th pixel. The (i,j)th pixel S_{ij} is the square

$$S_{ij} \equiv [x_i - \tfrac{1}{2}h, x_i + \tfrac{1}{2}h] \times [y_j - \tfrac{1}{2}h, y_j + \tfrac{1}{2}h]$$

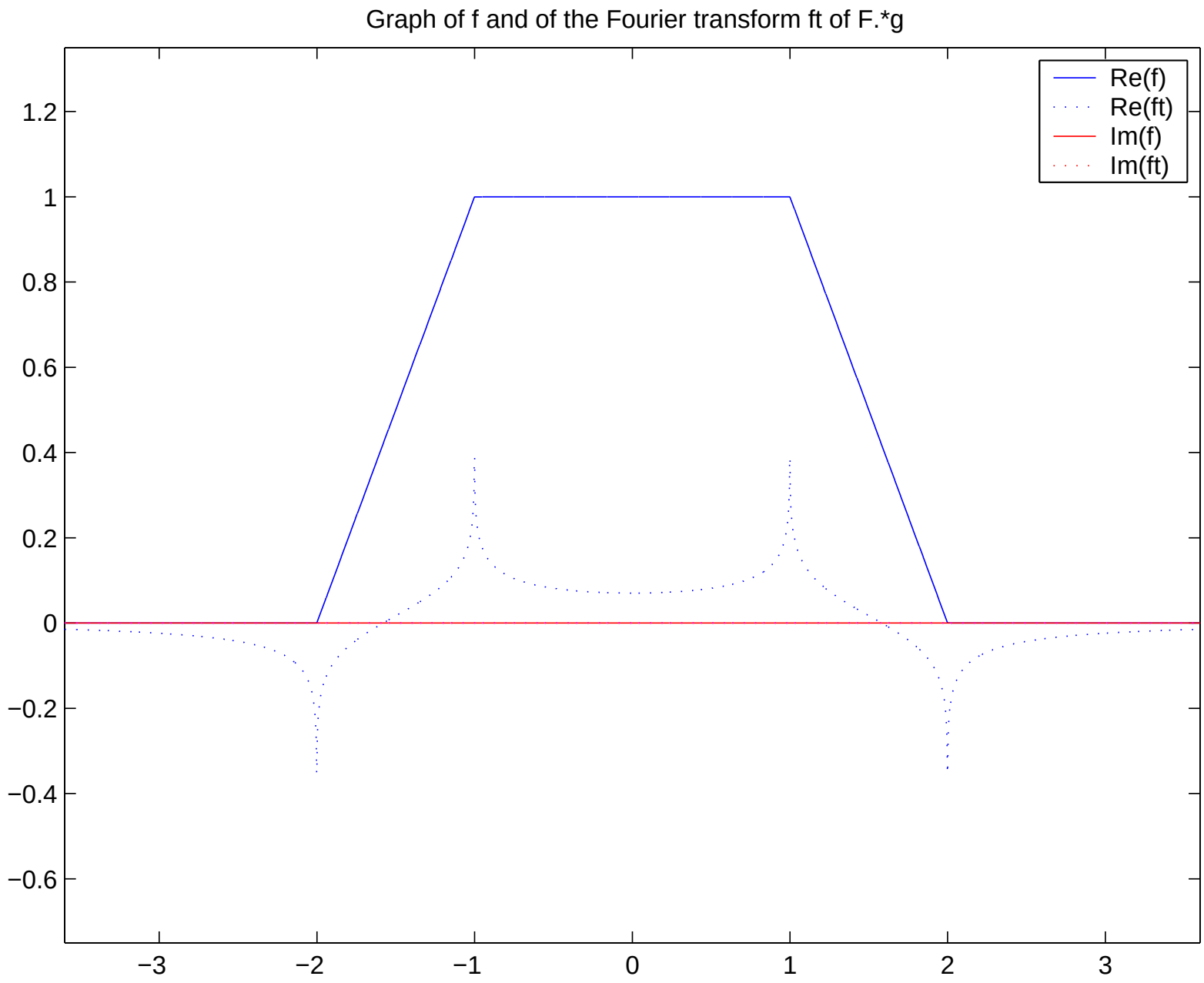
with $x_i = ih$, $y_j = jh$ $(i,j \in \mathbb{Z})$ and h the size of the pixels. χ_{ij} takes the value 1 at S_{ij} and the value 0 elsewhere.

Since all operations are linear, it suffices to study the results of the Radon transformation on $\chi_{0,0}$ and the filtered back-projection.

Computerised Tomography



Computerised Tomography



CT and Fourier transforms

Theorem. With $\widetilde{p}_\phi(\xi) \equiv \int |\rho|\, \widehat{p}_\phi(\rho)\, e^{2\pi i \rho \xi}\, \mathrm{d}\rho$, we have that

$$f(rc_\theta, rs_\theta) = \int_0^\pi \widetilde{p}_\phi(rc_{\theta-\phi})\, \mathrm{d}\phi$$

Summary.

The statement in the theorem involves
1) a 1-dimensional Fourier transform (FT) (to make $\widehat{p}_\phi$),
2) a filter operation in frequency space,
3) a 1-d inverse FT and
4) BP.

The proof exploits 2-d FT, switching between
Cartesian coordinates,
rotated Cartesian coordinates, and
polar coordinates.

Practical issues

Computation requires discretization (the scanners emits the x-rays in a series of thin parallel beams, each beam of distance $\Delta\xi$ to the next beam, that is, ξ is discretised).

The discretization leads to artifacts, as the Gibb's phenomenon (wiggles).

The discretization of ϕ has to be related to the discretization of ξ: a small $\Delta\phi$ (as compared to $\Delta\xi$) may be a waist of scanning time, a large $\Delta\phi$ may introduce additional non-negligible errors.

Computerised Tomography