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Fourier Transforms
Wavelets
Theory and Applications

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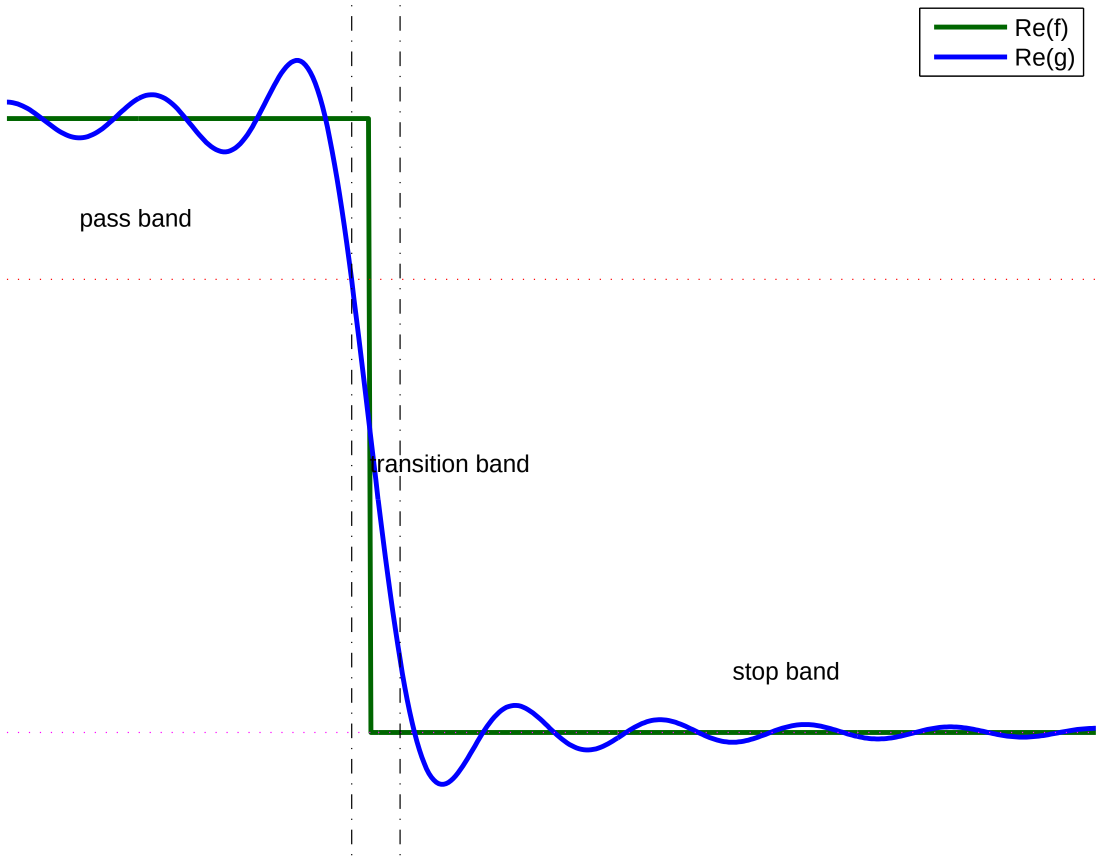
Filters



Program

- Filters
- Finite Impulse Response Filters
- Windows
- Signals of finite duration and bounded bandwidth?
- Infinite Impulse Response Filters
- Analog filters (hardware)
- Digital filters (software)

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Filters

Let $H \in L^\infty(\mathbb{R})$.

The map $f \rightsquigarrow g$ with g s.t. $\widehat{g} = \widehat{f}\,H$ is the **ideal H-filter**.
Here, f is a **signal**, i.e., a function in $L^2(\mathbb{R})$.

Example.

$$H(\omega) = \begin{cases} 1 & \big|\, |\omega| - |\Omega| \,\big| < \varepsilon \\ 0 & \text{else} \end{cases}$$

Note that $H = \Pi_{\Omega+\varepsilon} - \Pi_{\Omega-\varepsilon}$.
We will mainly focus on the ideal Π_Ω-filter.

Filters

Let $H \in L^\infty(\mathbb{R})$.

The map $f \rightsquigarrow g$ with g s.t. $\widehat{g} = \widehat{f}\, H$ is the **ideal H-filter**. Here, f is a **signal**, i.e., a function in $L^2(\mathbb{R})$.

H is the **transfer f.** or **(frequency) response function**, f is the **input**, g is the **output** of the filter.

Write $H(\omega) = |H(\omega)|\, e^{-i\phi(\omega)}$.

$|H(\omega)|$ is the **gain** at frequency ω.

Filters and delays

$$H(\omega) = |H(\omega)|\, e^{-i\phi(\omega)}.$$

What is the effect of the 'complex part' of the filter?

• Suppose $H(\omega) = e^{-ic\omega}$, i.e., $\phi(\omega) = c\omega$.

Then, $g(t) = (\widehat{f}\, H)\hat{}(-t) = \int \widehat{f}(\omega)e^{2\pi it\omega - ic\omega}\, \mathrm{d}\omega = f(t - \frac{c}{2\pi})$:

Conclusion. The phase shift leads to a **time delay**.

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What if $\frac{\phi(\omega)}{\omega}$ *is not constant?*

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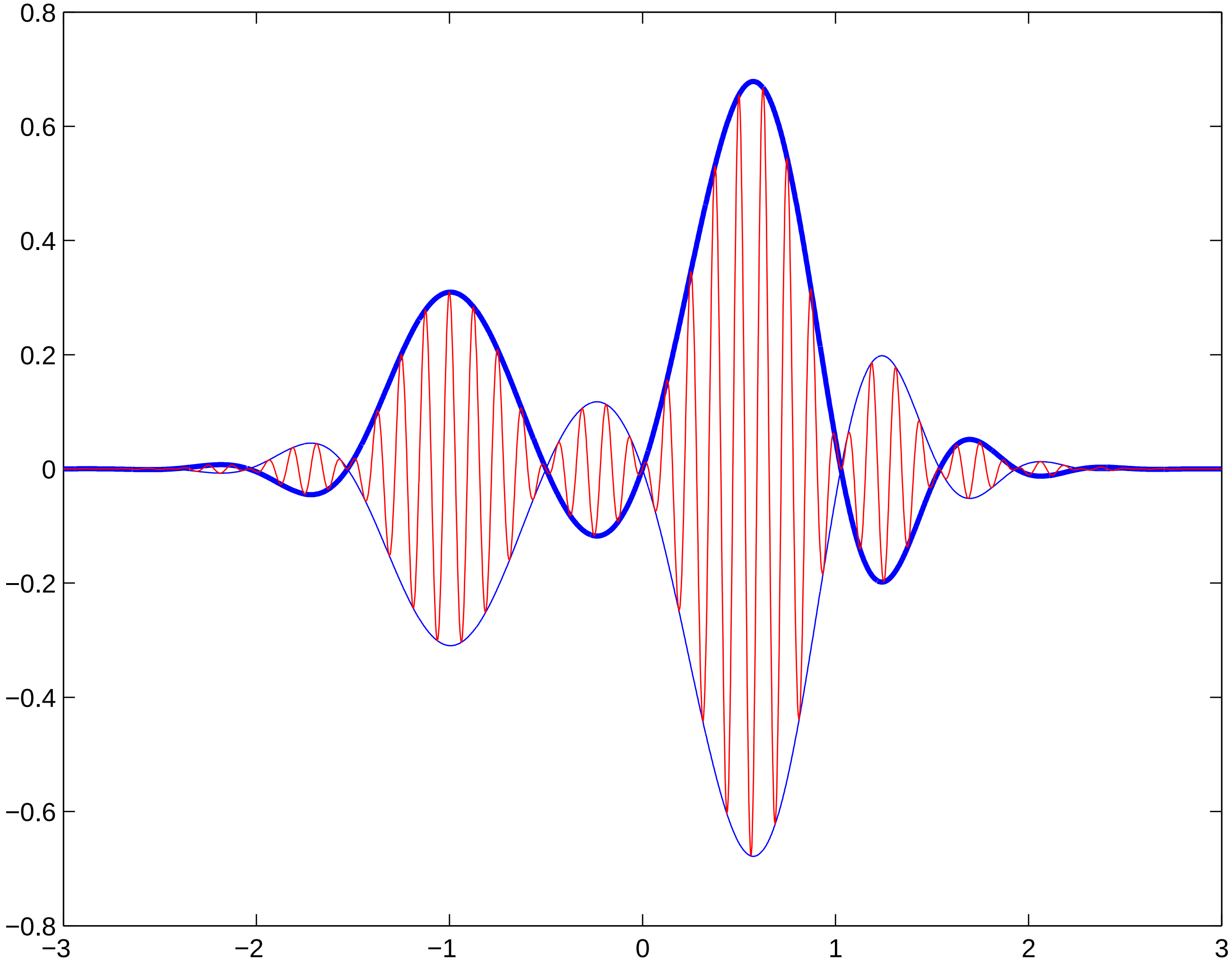
- **Ex.** $f(t) = f_0(t) e^{2\pi i t\Omega}$ with $\widehat{f}_0$ concentrated in $[-\varepsilon, +\varepsilon]$.

f is a **wave packet**, f_0 is the **envelop**.



$$e^{2\pi i(k(\omega)x-\omega t)}$$

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$$H(\omega) \approx |H(\omega)|\, e^{-i(\phi(\Omega)+\phi'(\Omega)(\omega-\Omega))} \qquad \text{for} \qquad \omega \approx \Omega$$

$$\begin{aligned} g(t) &= (\widehat{f}H)\widehat{\ }(-t) \approx \int \widehat{f}(\omega)\, |H(\Omega)|\, e^{i(\phi(\Omega)-\phi'(\Omega)\Omega)}\, e^{2\pi i t\omega - i\phi'(\Omega)\omega}\, \mathrm{d}\omega \\ &= |H(\Omega)|\, e^{i(\phi(\Omega)-\phi'(\Omega)\Omega)}\, f(t - \tfrac{\phi'(\Omega)}{2\pi}) \\ &= |H(\Omega)|\, f_0(t - \tfrac{\phi'(\Omega)}{2\pi})\, e^{2\pi i\Omega(t - \frac{\phi(\Omega)}{2\pi\Omega})} \end{aligned}$$

Filters and delays

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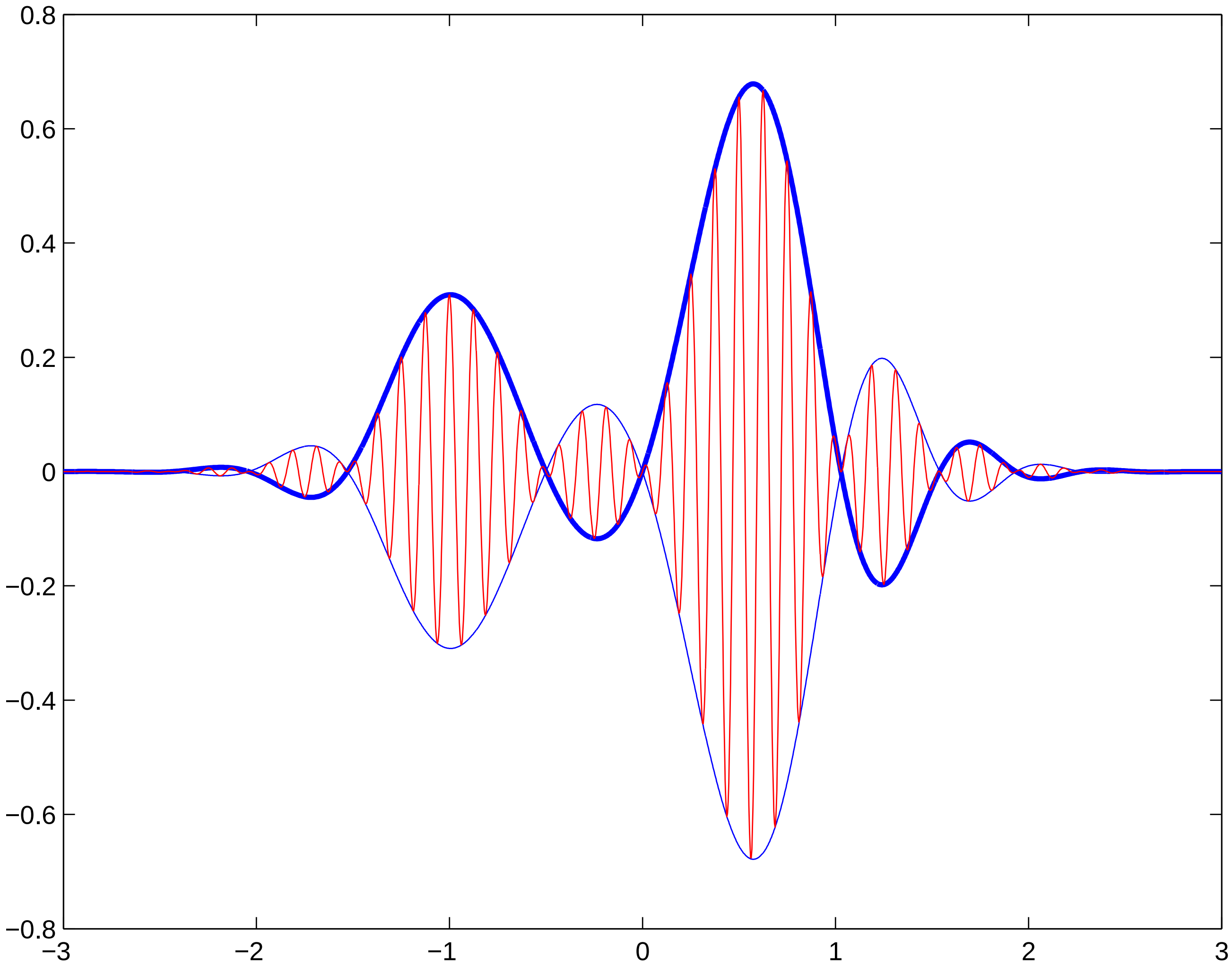
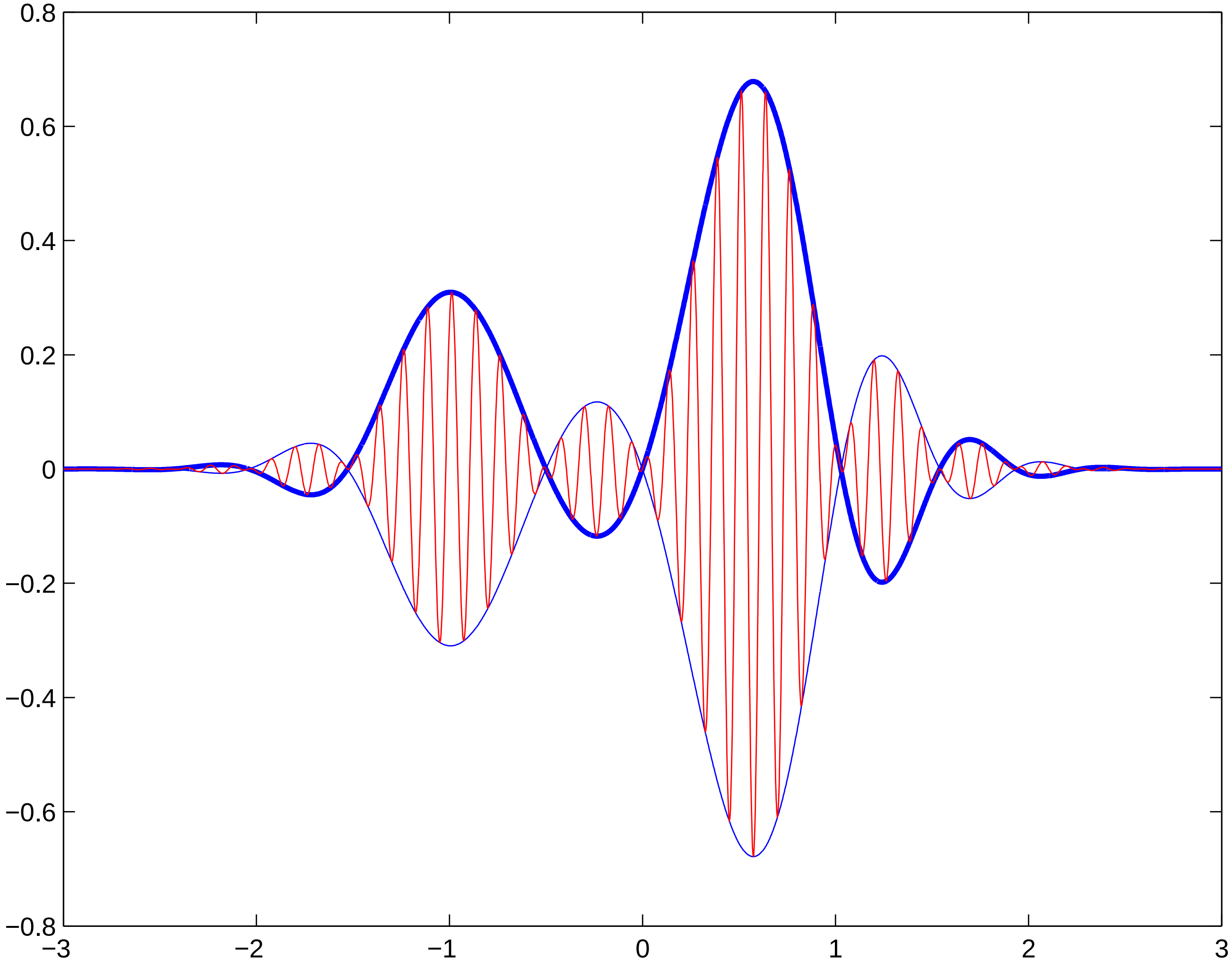
$$g(t) = |H(\Omega)|\, f_0\left(t - \frac{\phi'(\Omega)}{2\pi}\right) e^{2\pi i\Omega(t - \frac{\phi(\Omega)}{2\pi\Omega})}$$

$\frac{\phi'(\Omega)}{2\pi}$ is the **group delay** and $\frac{\phi(\Omega)}{2\pi\Omega}$ the **time delay**



$$e^{2\pi i(k(\omega)x-\omega t)}$$



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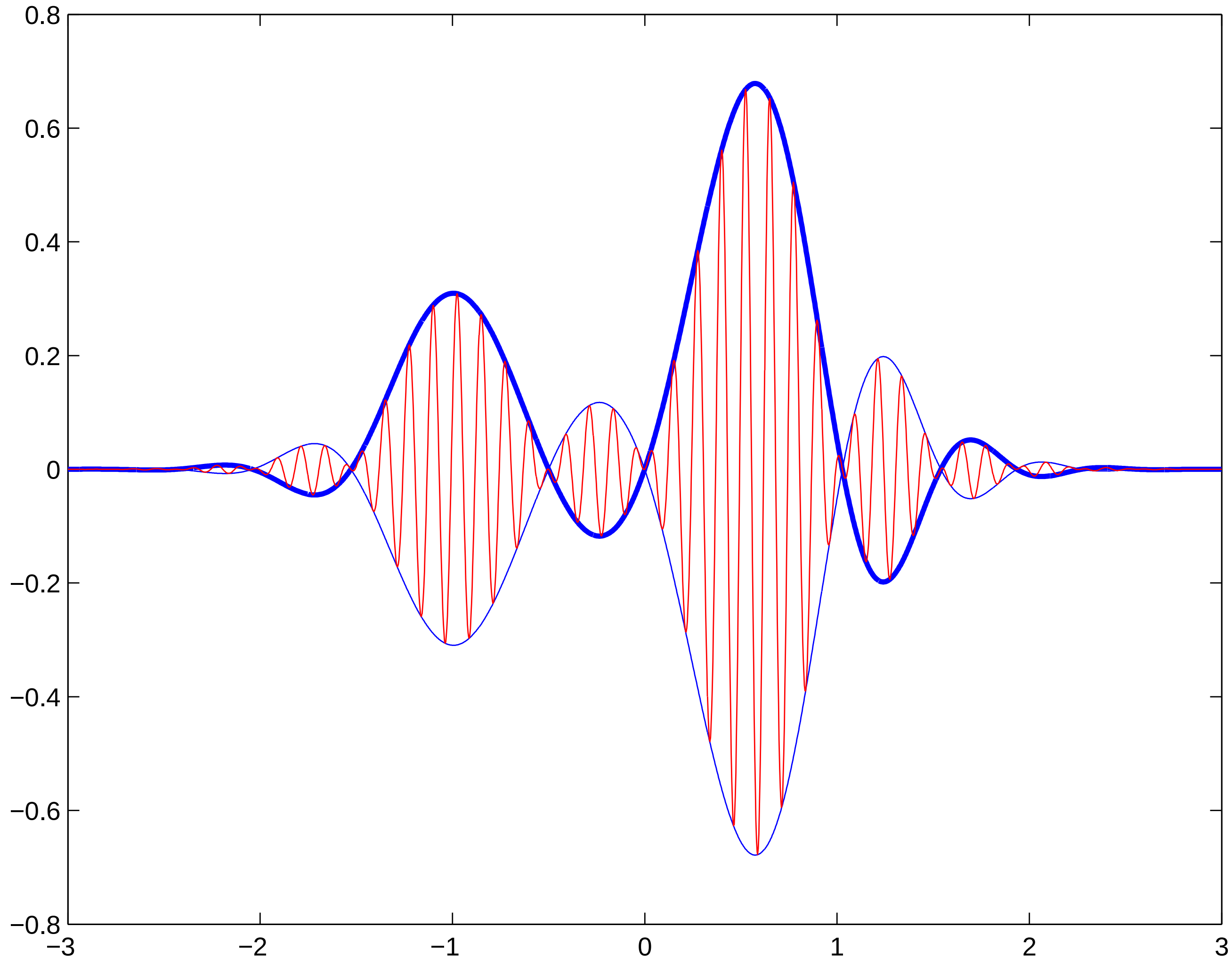

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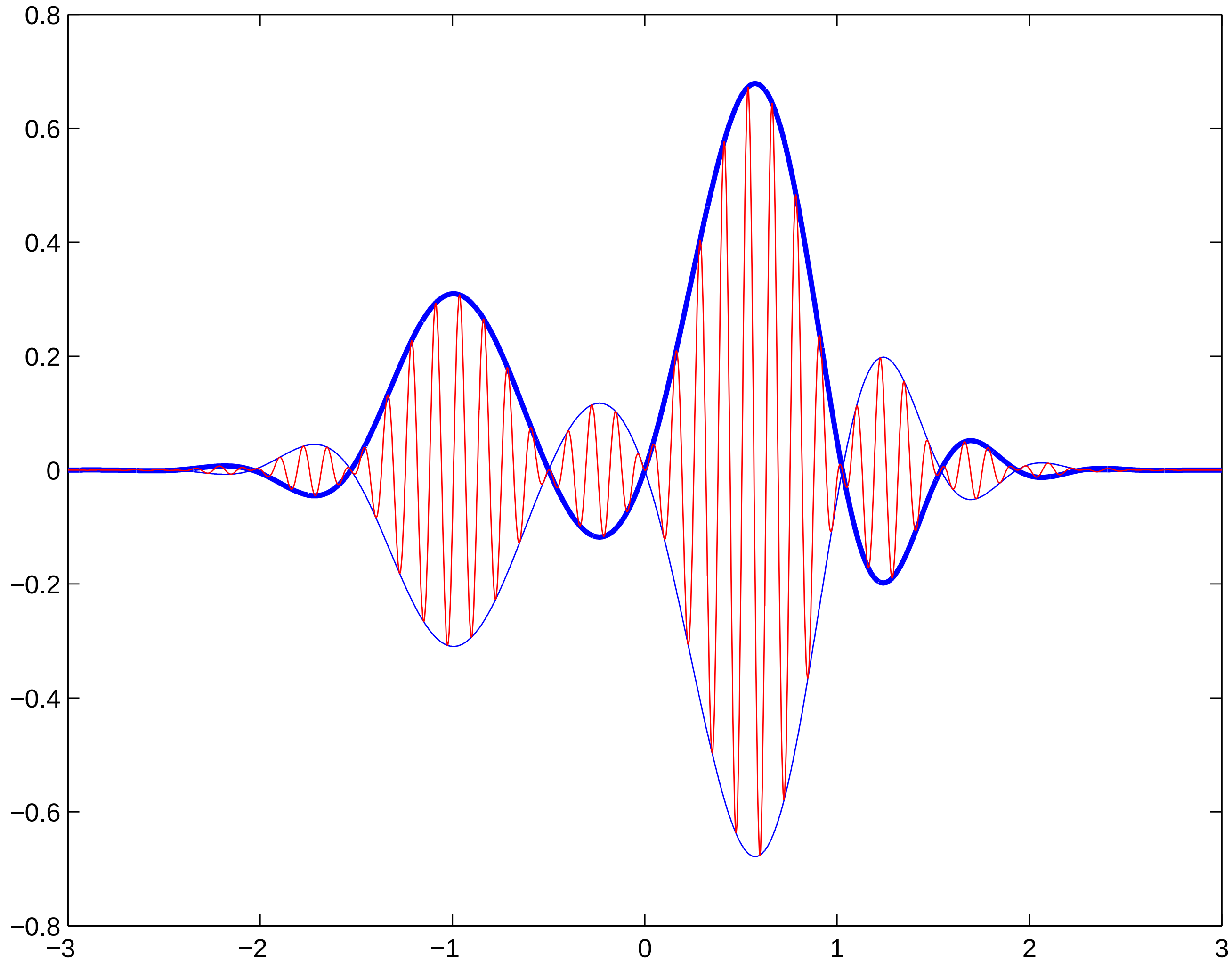

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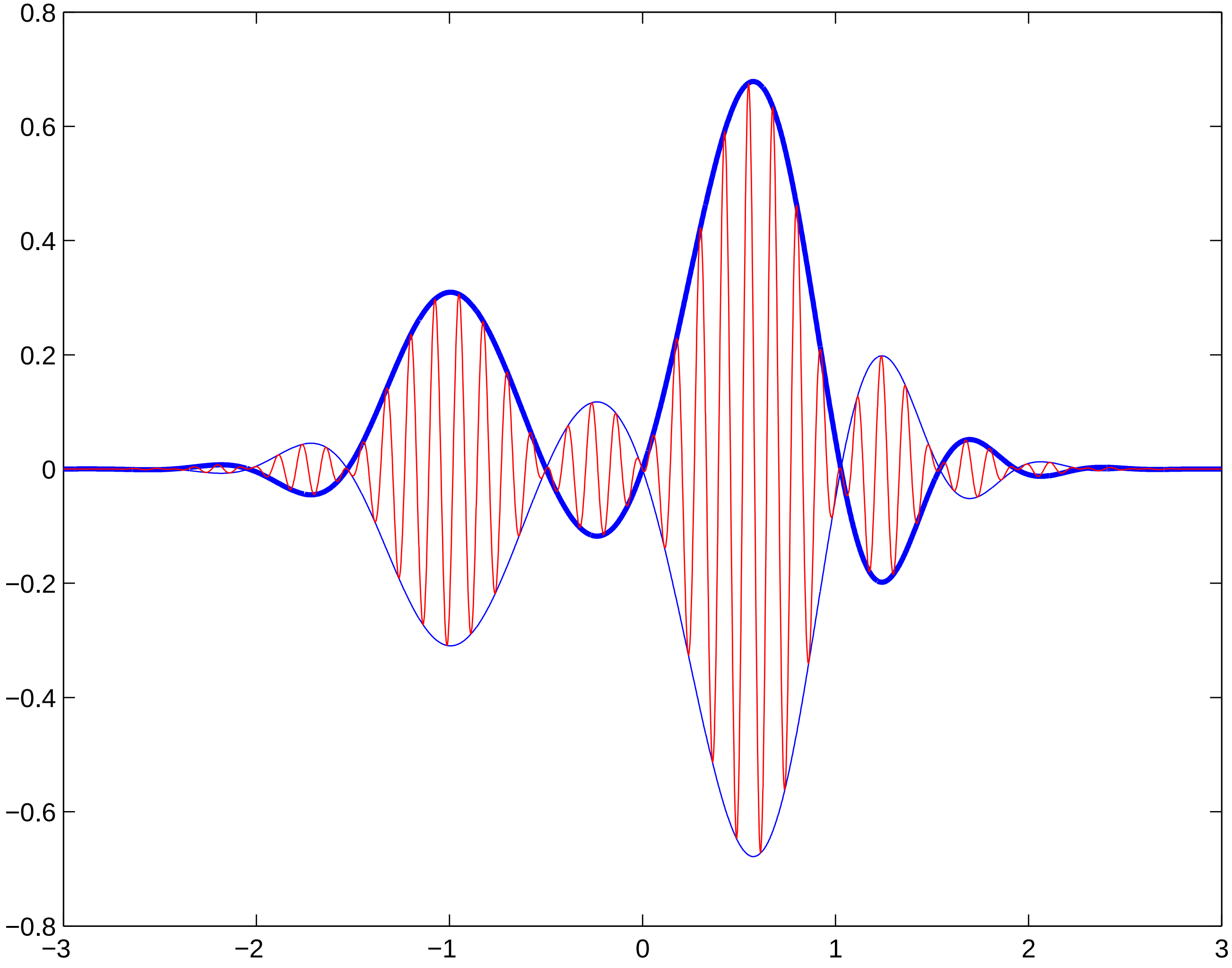

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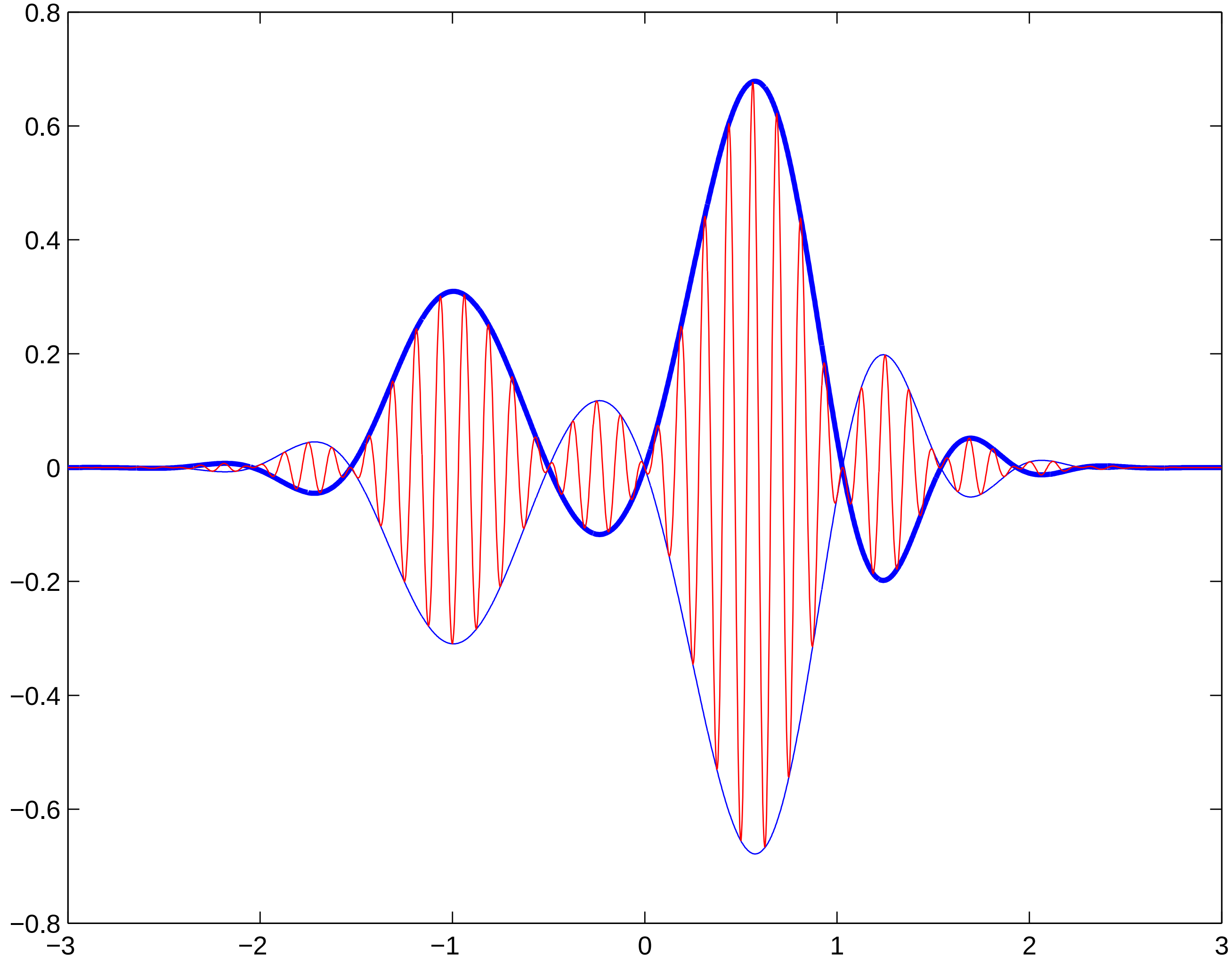

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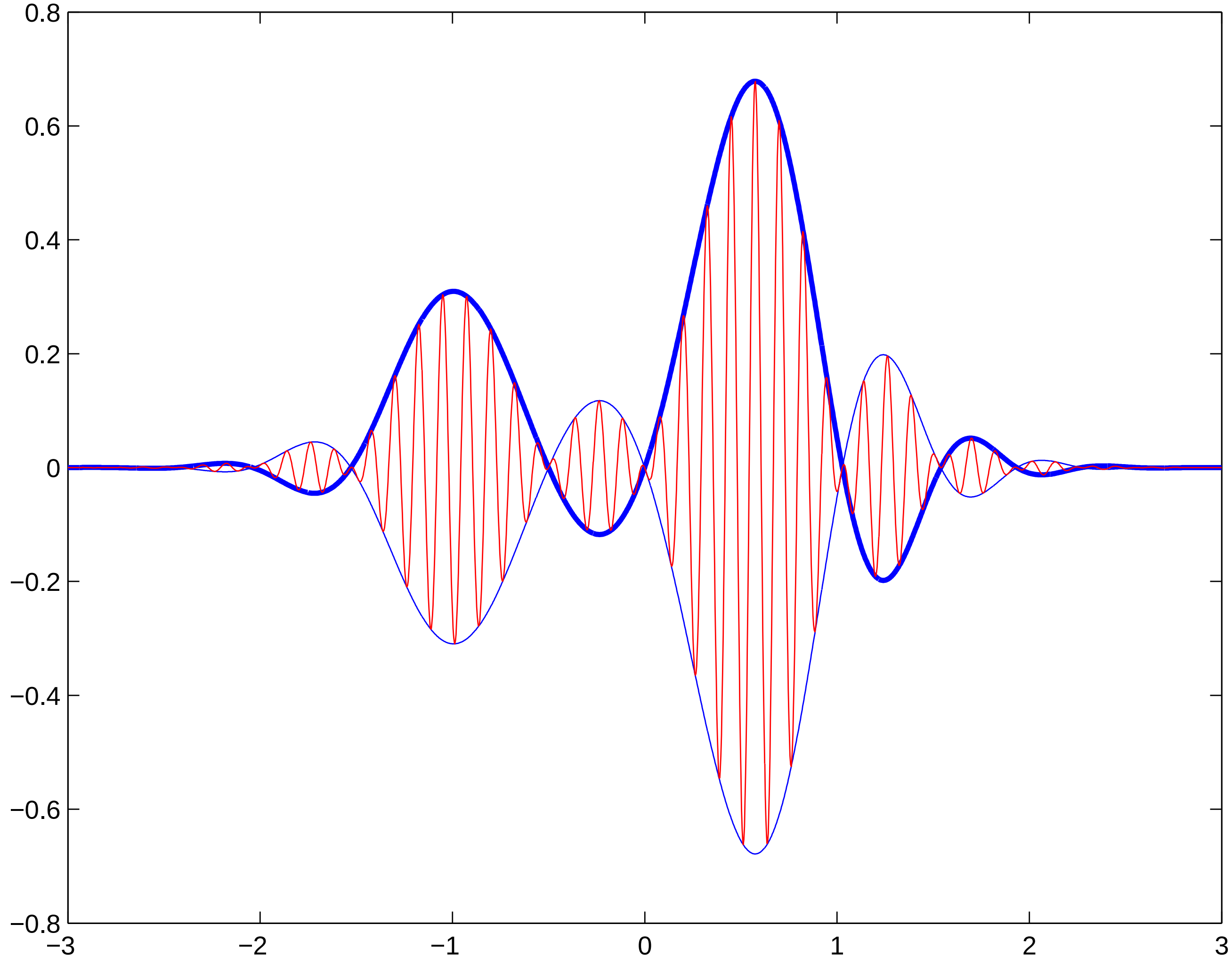

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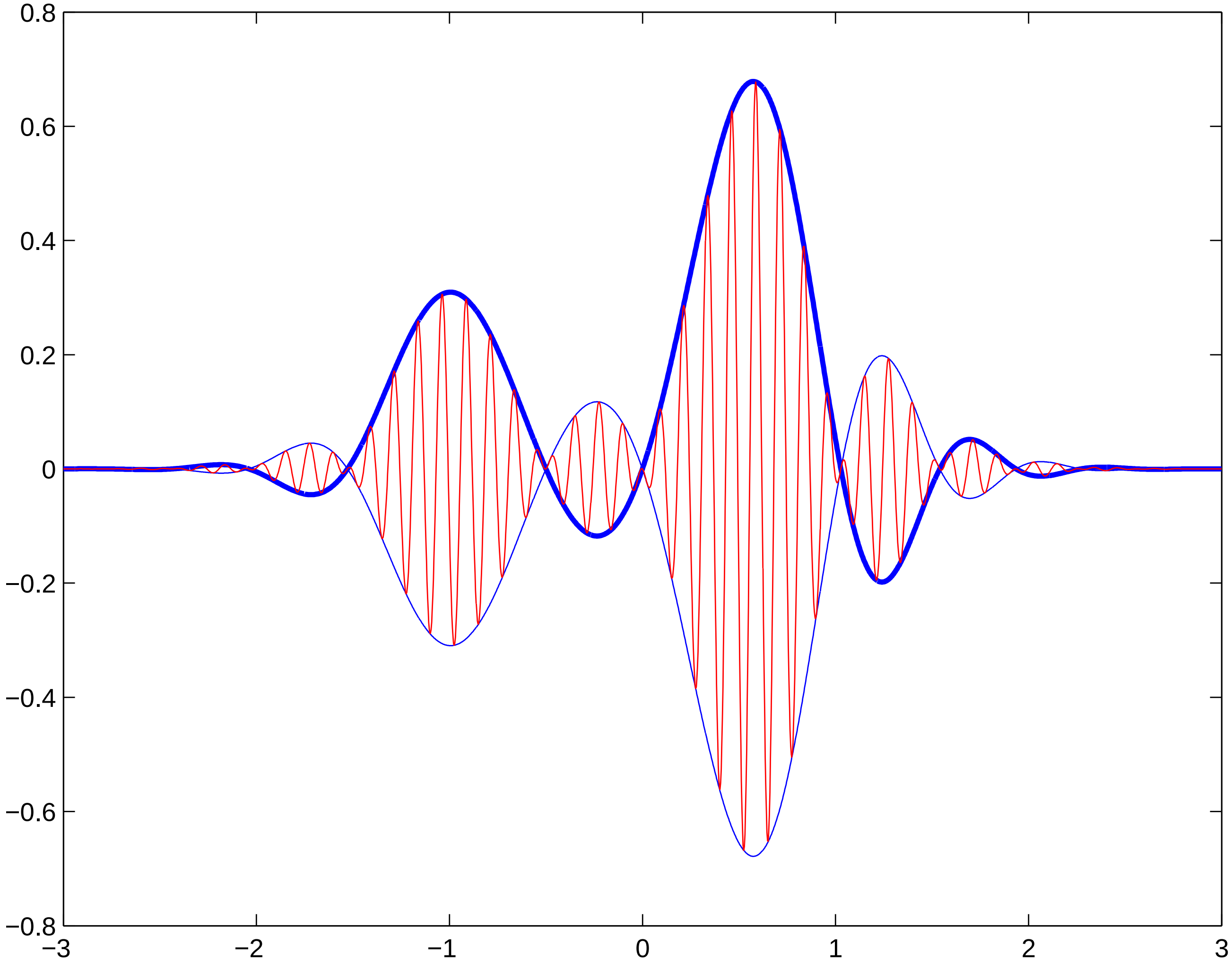

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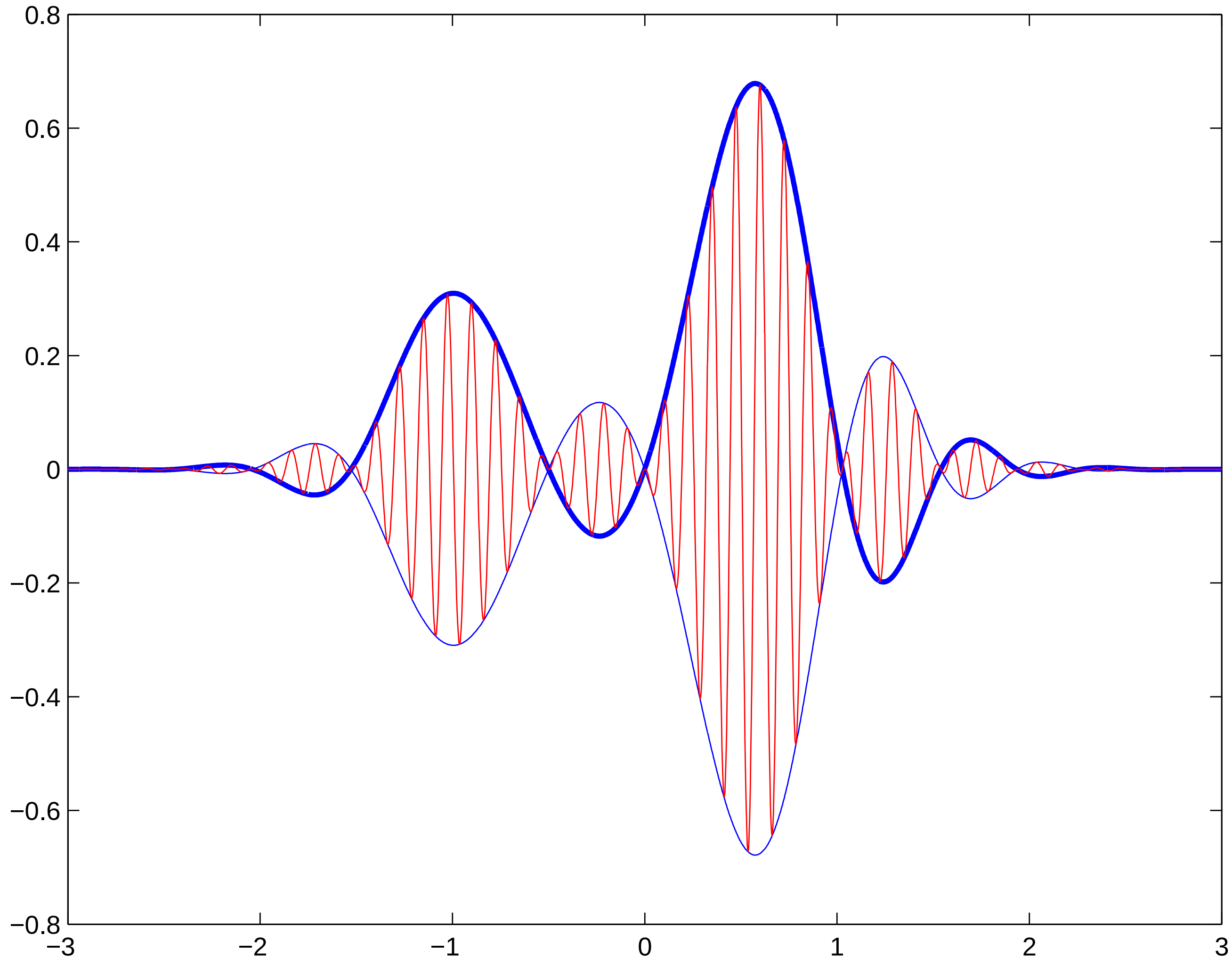

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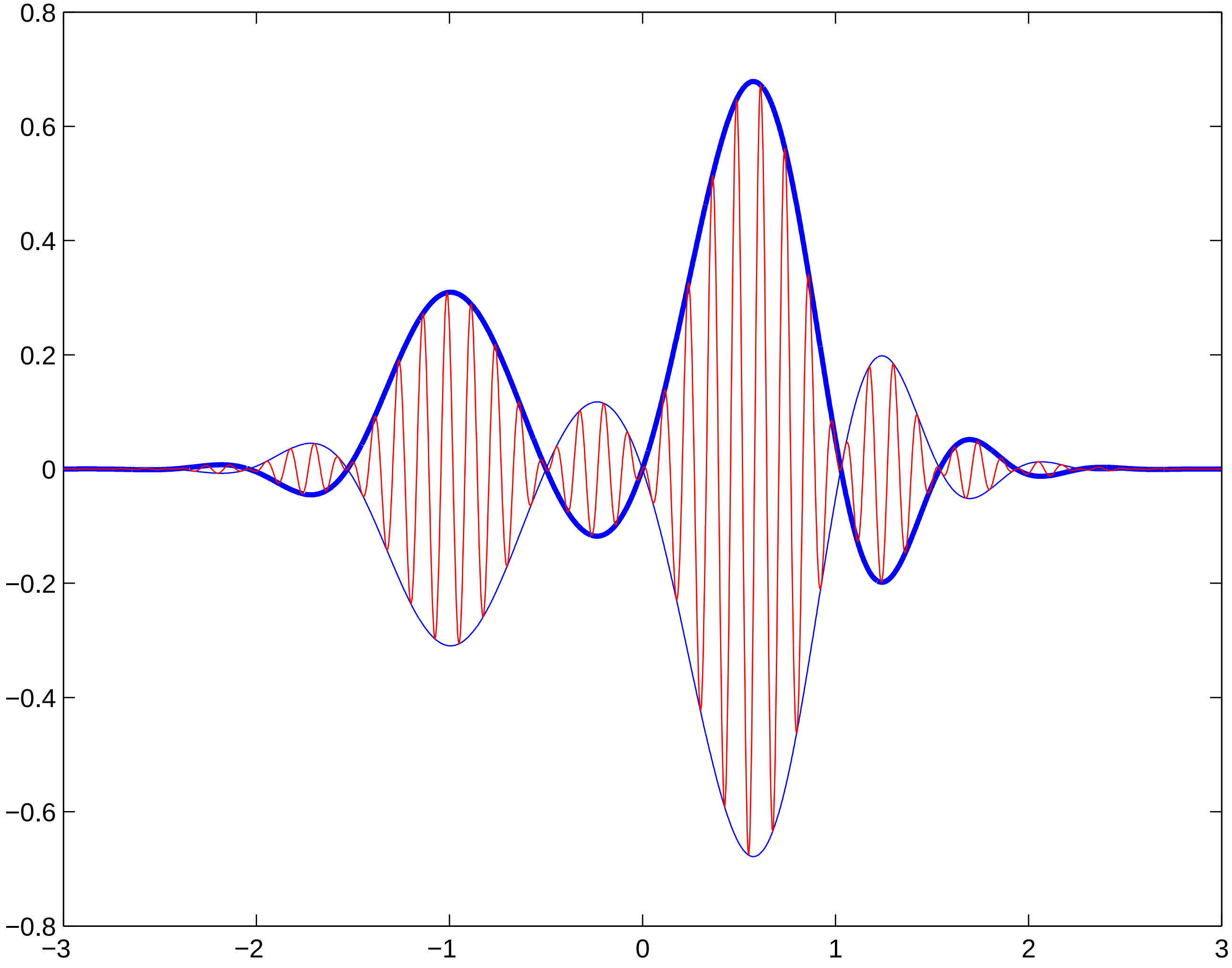

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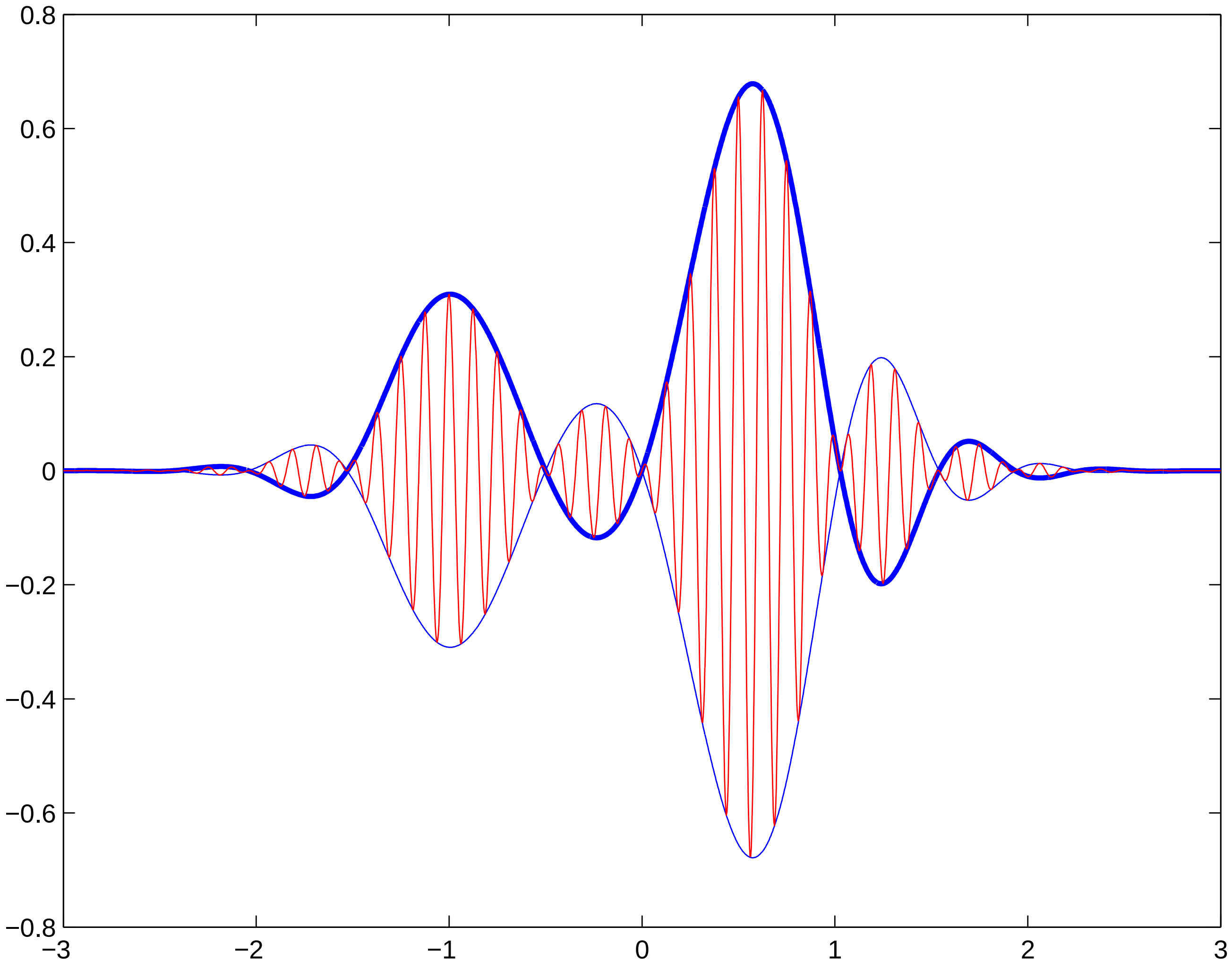

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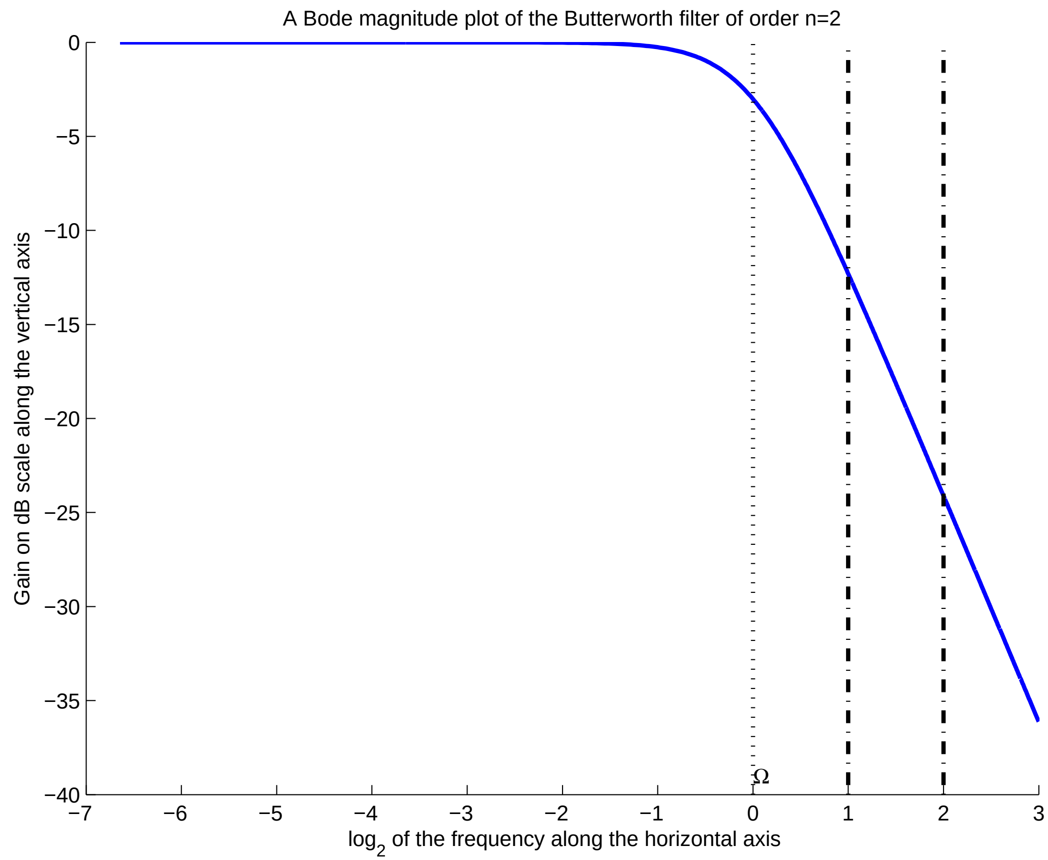
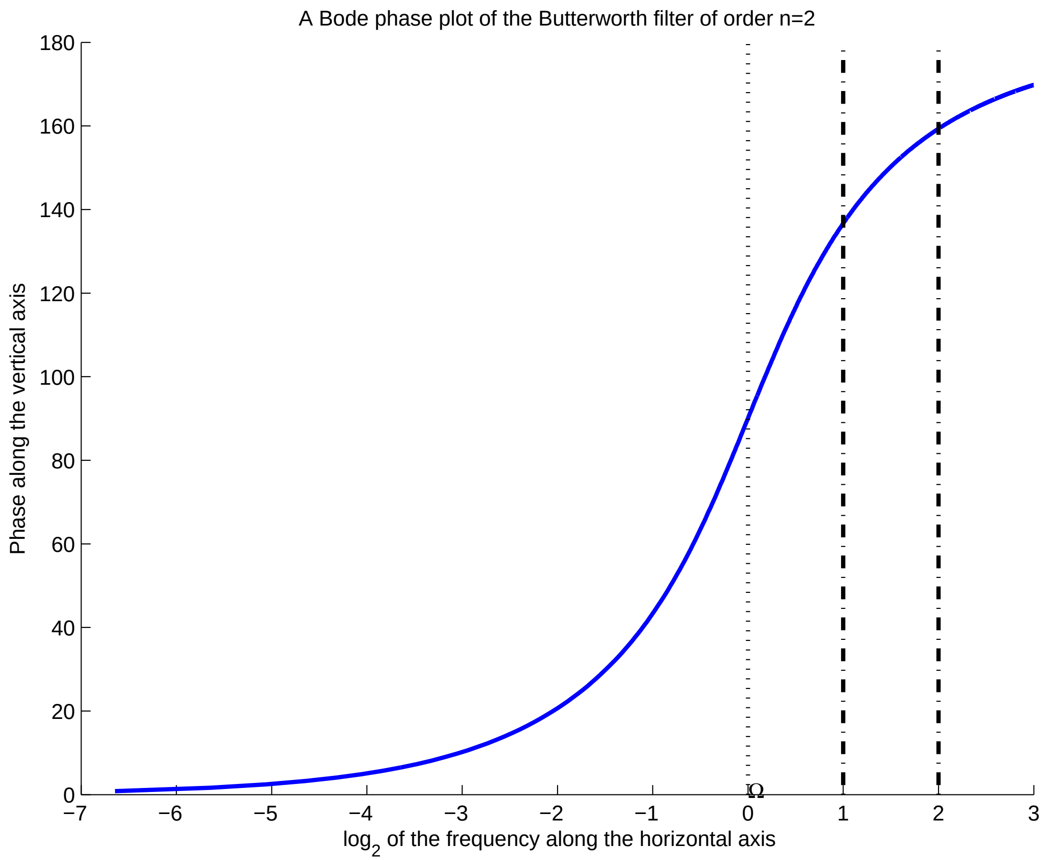
Filters



A gain plot ($|H(\omega)|$ against ω) is called a **Bode plot** of the filter. **deciBel** scale ($20 \log_{10} |H(\omega)|$) is used on the vertical axis.

This the Bode plot of the Butterworth filter of degree 2 (see later).

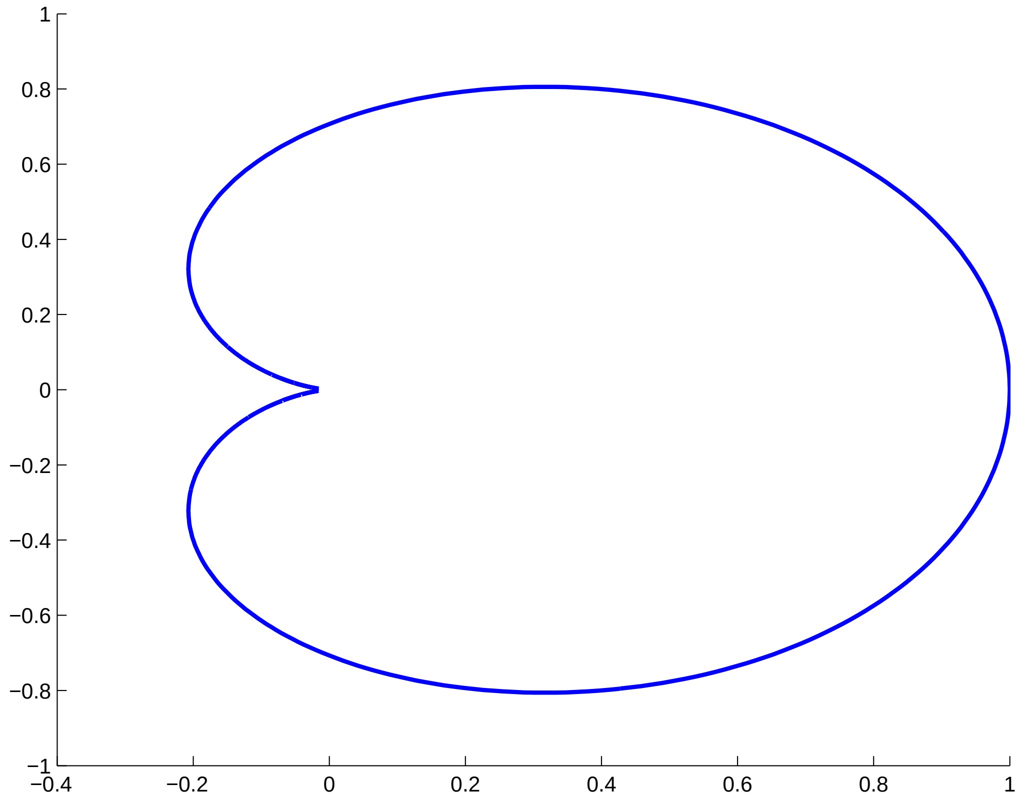
Filters



The **Bode phase plot** is the plot of $\phi(\omega)$ versus ω.

Here ϕ is such that $H(\omega) = |H(\omega)|\, e^{-i\phi(\omega)}$.

Filters



The curve $\omega \rightsquigarrow H(\omega)$ (imaginary part versus real part) in the complex plane is the **Nyquist plot**.

If $H \in L^2(\mathbb{R})$, then

$$(\widehat{f}\, H)\widehat{\ }(-t) = f * h, \quad \text{where} \quad \widehat{h} = H.$$

Filtering in frequency domain

$\rightsquigarrow$ convolution in time domain.

Convolution can be viewed as weighted averaging.

h is the **(im)pulse response function**.

h is the representation of the filter in time domain,
H is the representation in frequency domain.:

With $f_\delta \equiv \frac{1}{2\delta}\Pi_\delta$ we have $f_\delta * h \to h$
f_δ is a **pulse**. Its response to the filter H approximates h.

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Example. $H = \Pi_\Omega$. Then

$$\widehat{\Pi}_\Omega(t) = \frac{\sin(2\pi t\Omega)}{\pi t} = 2\Omega \mathrm{sinc}(2t\Omega).$$

Filters in time domain

Recall $$f * h(t) = \int f(s)h(t-s)\,\mathrm{d}s.$$

It is not practical if 'future' function values $f(s)$, i.e., for $s > t$, are required for the computation of $f * h$.

If h is **causal**, i.e., $h(t) = 0$ for all $t < 0$, then

$$f * h(t) = \int_{-\infty}^{t} f(s)h(t-s)\,\mathrm{d}s$$

and only 'old' function values $f(s)$ with $s \leq t$ are required.

Note. If $h(t) = 0$ for all $t < -s$ for some $s > 0$, and $h(-s) \neq 0$, then h is not causal. However, h_s is causal and

$$(f_s) * h = f * (h_s) = (f * h)_s$$

We call h causal if, for some s, $h(t) = 0$ for all $t < -s$.

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Example. $\widehat{\Pi}_\Omega$ is not causal.

Take $T > 0$ large. $\widehat{\Pi}_\Omega \, \Pi_T$ is **not** causal (except for a delay).

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A **Finite Impulse Response** filter is a causal filter that, in time domain, has a bounded support, i.e.,

there is a $T > 0$ such that $h(t) = 0$ if $t > T$:

for short, h has a "bounded" or "finite" time domain.

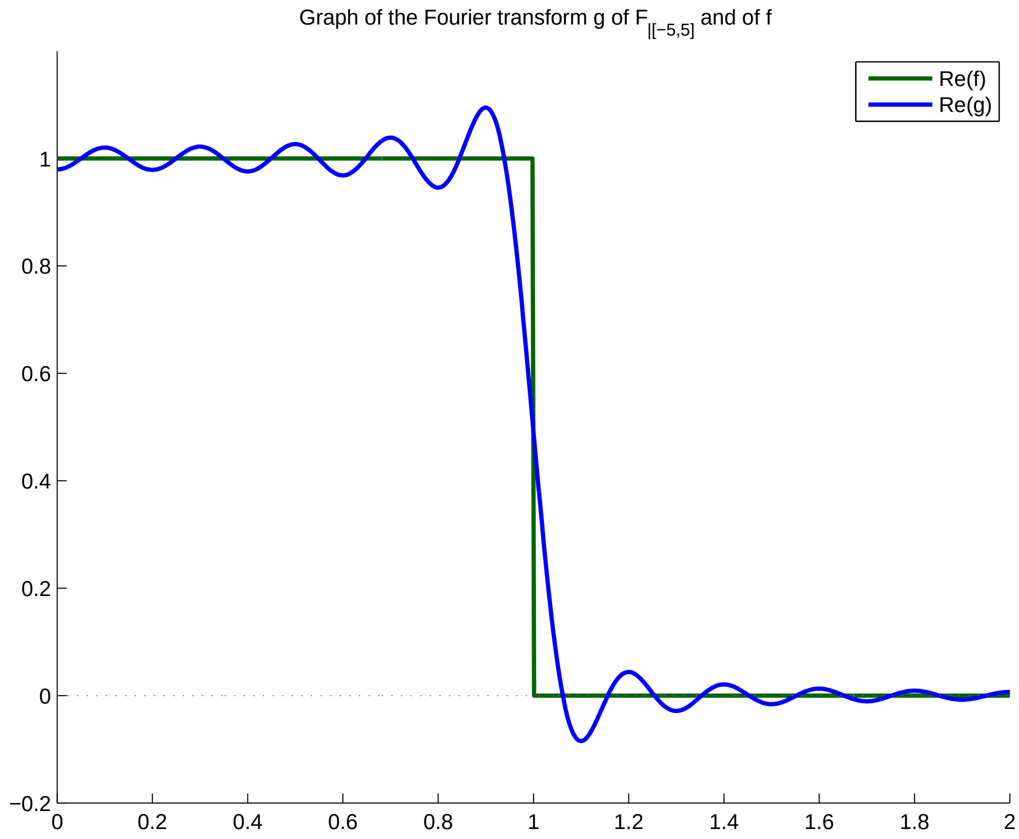
The desired filter H (in frequency domain), can be approximated by the filter $H * \widehat{\Pi}_T$ which is bounded time domain.

How large is the approximation error?

The Gibbs' phenomenon

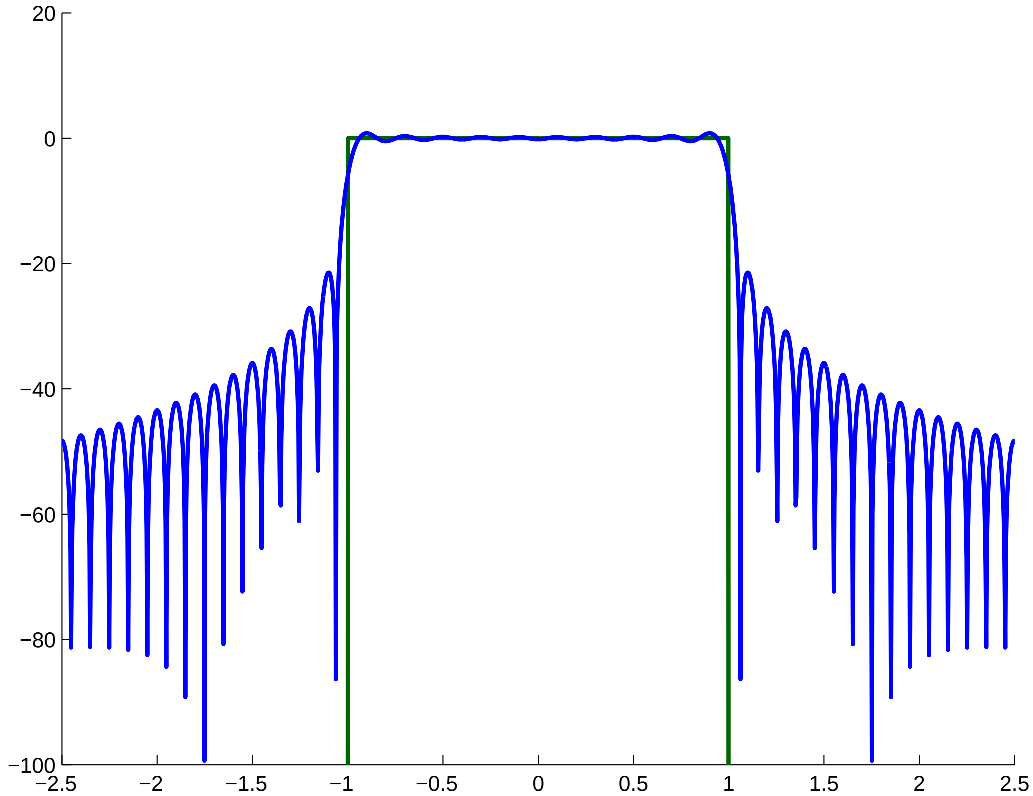
How close is $(\widehat{\Pi}_\Omega \, \Pi_T)\hat{} = \Pi_\Omega * \widehat{\Pi}_T$ to Π_Ω?

The Gibbs' phenomenon



$H = \sqcap_{\Omega} \qquad [\Omega = 1, T = 5]$

The Gibbs' phenomenon



$H = \Pi_\Omega \qquad [\Omega = 1, T = 5]$

The Gibbs' phenomenon

$$\Pi_\Omega * \widehat{\Pi}_T(\omega) = \int_{-\Omega}^{\Omega} \widehat{\Pi}_T(\omega - \rho)\,\mathrm{d}\rho = \int_{\omega-\Omega}^{\omega+\Omega} \widehat{\Pi}_T(\rho)\,\mathrm{d}\rho$$

$$= \int_{\omega-\Omega}^{\omega+\Omega} \frac{\sin(2\pi T\rho)}{\pi\rho}\,\mathrm{d}\rho = U_T(\omega+\Omega) - U_T(\omega-\Omega),$$

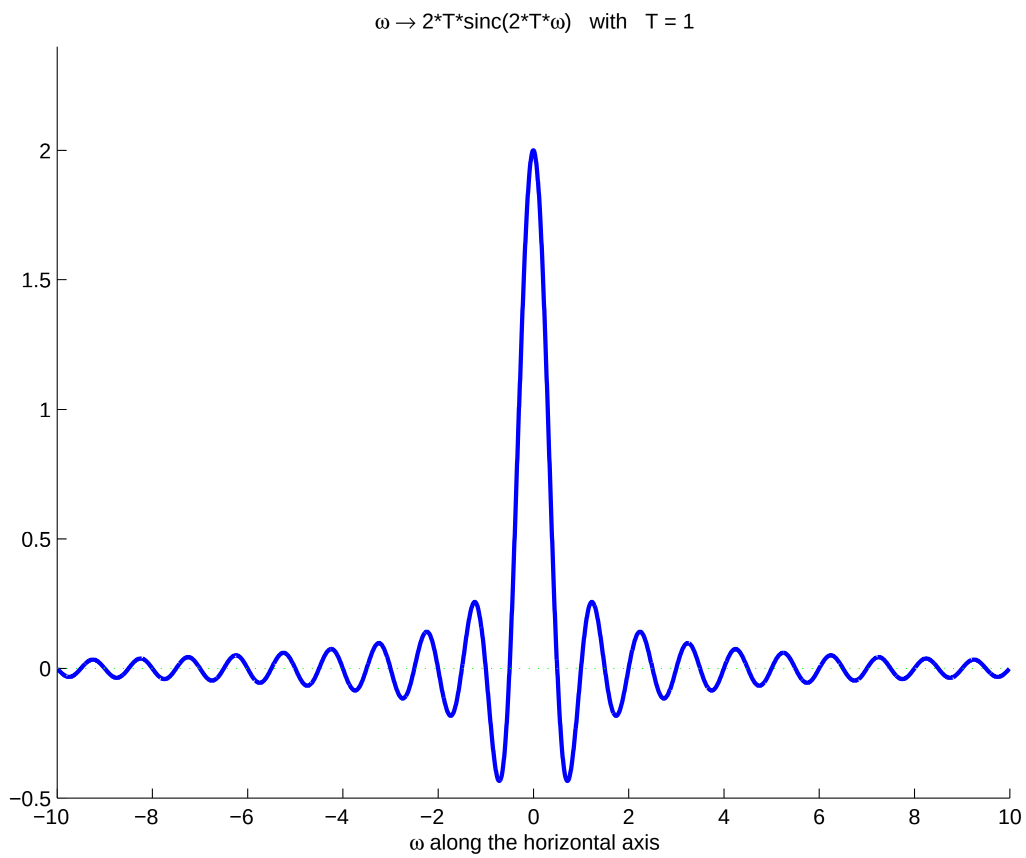
where

$$U_T(\omega) \equiv \int_{-\infty}^{\omega} \frac{\sin(2\pi T\rho)}{\pi\rho}\,\mathrm{d}\rho$$

$$= \int_{-\infty}^{T\omega} \frac{\sin(2\pi\sigma)}{\pi\sigma}\,\mathrm{d}\sigma = U_1(T\omega)$$

Conclusion. T rescales the ω-axis.
It does not affect the height of the ripples.

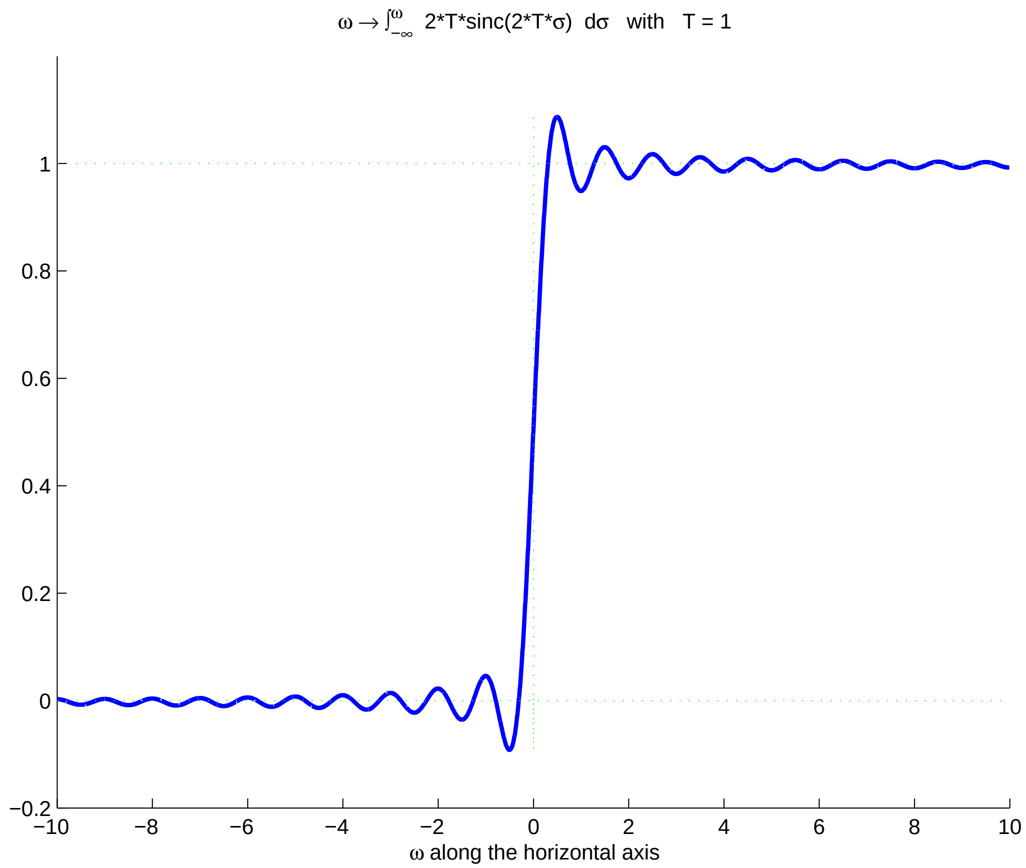
The Gibbs' phenomenon



The Sinc function $\rho \rightsquigarrow \dfrac{\sin(2\pi T\rho)}{\pi\rho} = 2T\mathrm{sinc}(2\pi T\rho)$.

In the picture here: $T = 1$.

The Gibbs' phenomenon

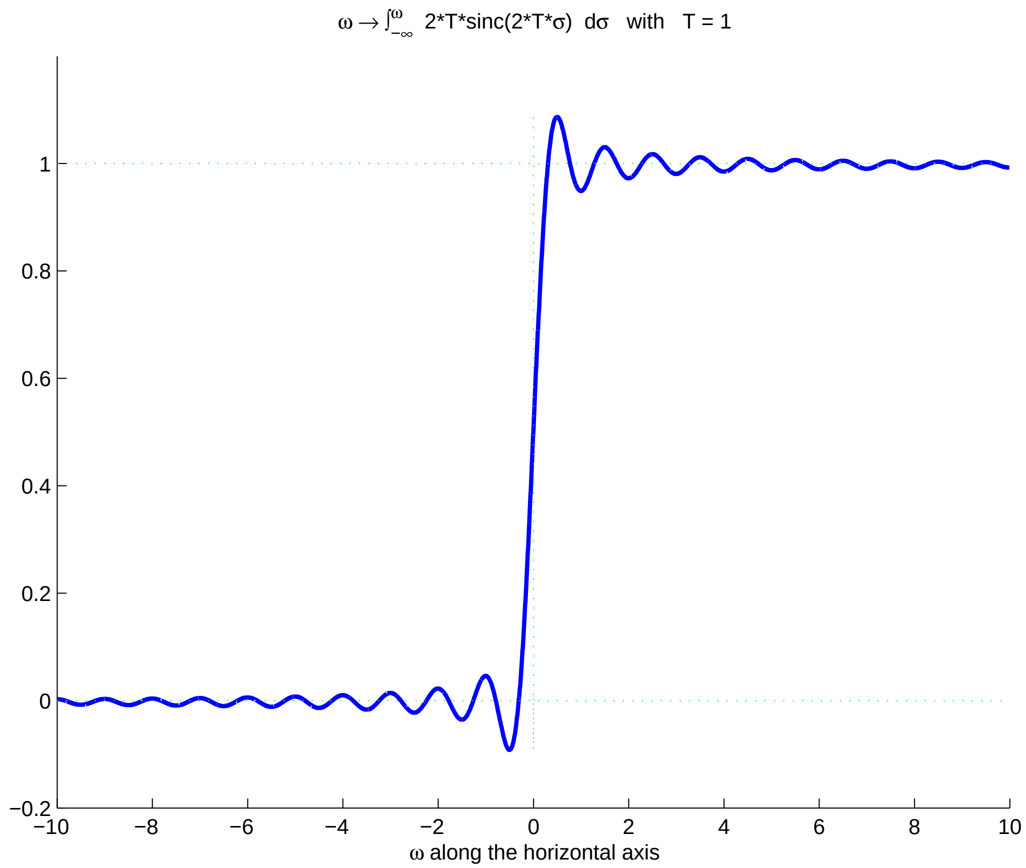


And its primitive U_T, i.e., $U_T'(\rho) = 2T\,\mathrm{sinc}(2\pi T\rho)$.

Note that

$$\int 2T\mathrm{sinc}(2\pi T\rho)\,\mathrm{d}\rho = \int 2T\mathrm{sinc}(2\pi T\rho)e^{2\pi i 0\rho}\,\mathrm{d}\rho = \Pi_T(0) = 1.$$

The Gibbs' phenomenon

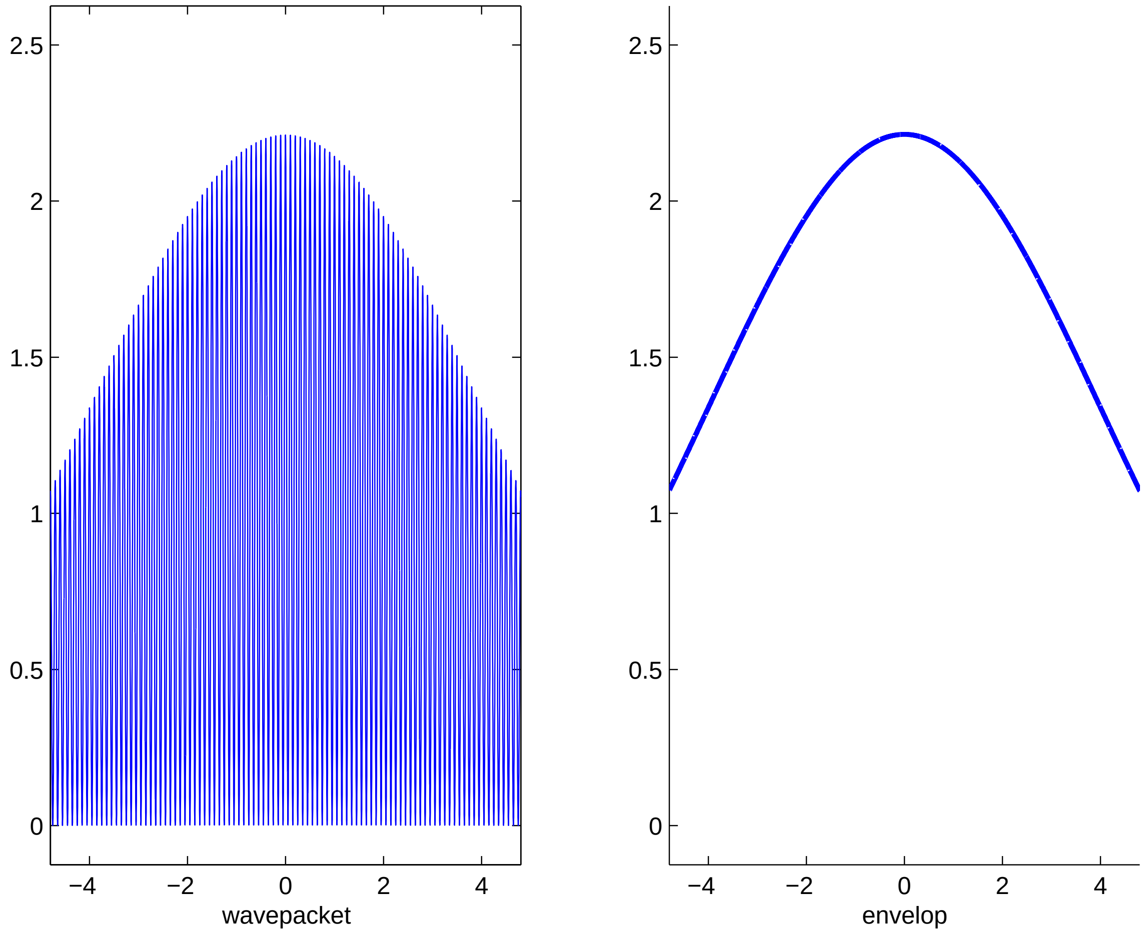


And its primitive U_T, i.e., $U'_T(\rho) = 2T\,\mathrm{sinc}(2\pi T\rho)$.

Note that

$$U_T(\omega) = 1 - U_T(-\omega) \qquad (\omega > 0).$$

Filters

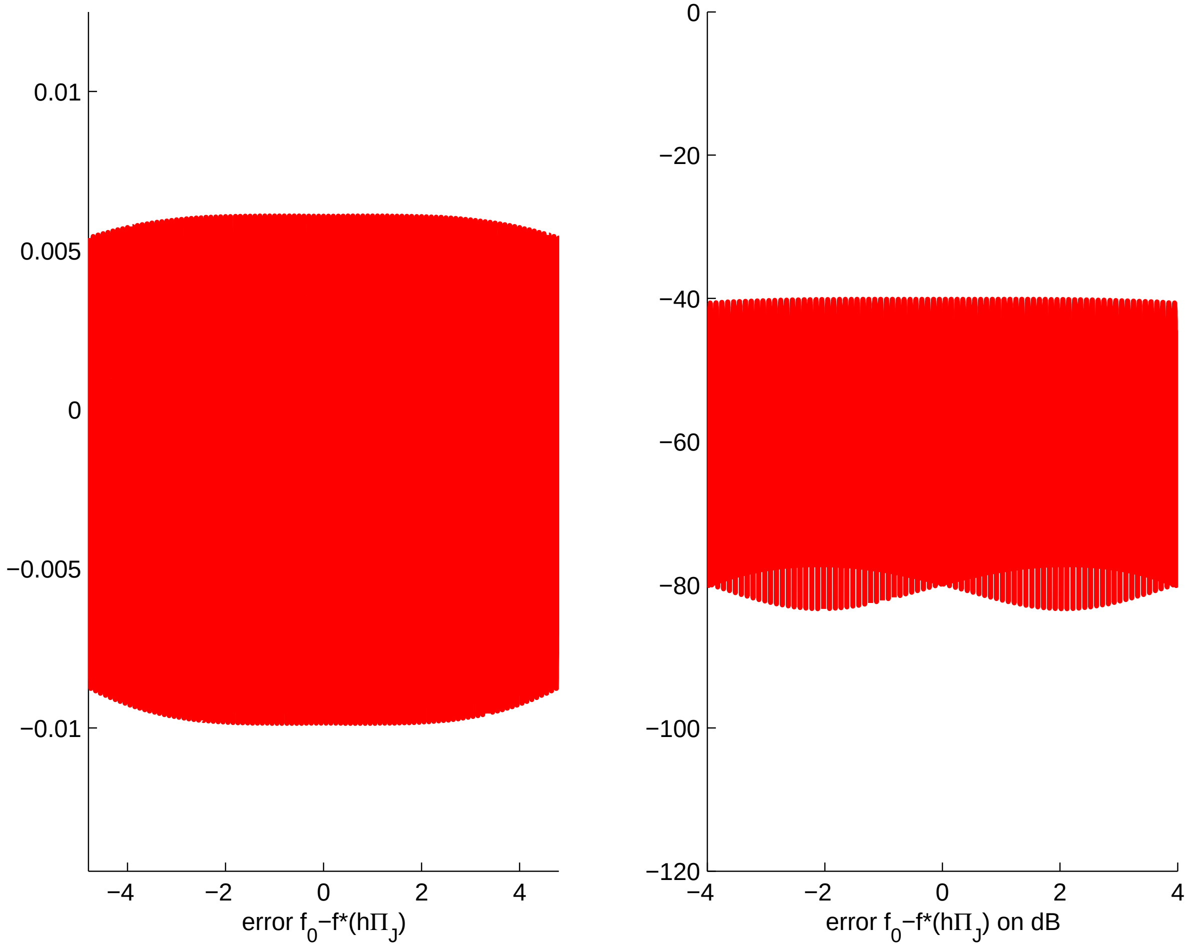


$f(t) = 2\, f_0(t)\, \cos^2(2\pi\nu t)$ (in the left picture) with

$f_0(t) = 3.5\sqrt{\alpha}\, \exp(-\pi\alpha^2 t^2)$ (in the right picture)

and $\alpha = 0.1, \quad \nu = 5$

Filters



With $\Omega = 9$, we have that $f * \widehat{\Pi}_\Omega = f_0$.

The pictures show the error $f_0 - \widetilde{f}_0$ with $\widetilde{f}_0 \equiv f * (\widehat{\Pi}_\Omega \Pi_T)$ and $T = 4$ (the right picture in dB-scale).

Bat detectors

Bats use infra sound acoustic waves for navigation (radar).

How to make this sounds audible?

Bat detectors

Bats use infra sound acoustic waves for navigation (radar).

These waves can be described by

$$f(t) = f_0(t) \cos(2\pi\nu t) \qquad (t \in \mathbb{R}),$$

where ν is high (ultra sound) and the frequencies of f_0 are **concentrated around** ω_0 in the low frequency range, i.e.,

$\omega_0 > 0$ is low and there is a (small) $\delta > 0$

such that $\dfrac{|f_0(\omega)|}{|f_0(\omega_0)|}$ negligible if $\big|\,|\omega| - |\omega_0|\,\big| > \delta$.

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Note.
The frequencies of f are concentrated around $\nu+\omega_0$ in the ultra sound frequency range.

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Property.

$$f_0(t)\cos(2\pi\nu t)\cos(2\pi\nu t) = \tfrac{1}{2}\,f_0(t)\,[\cos(4\pi\nu t) + 1]$$

Our ear will filter out the high frequencies 2ν:

we will hear $\frac{1}{2}f_0$

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Since

$\cos(2\pi\nu t)\cos(2\pi\widetilde{\nu} t) = \frac{1}{2}[\cos(2\pi(\nu+\widetilde{\nu})t) + \cos(2\pi(\nu-\widetilde{\nu})t)]$ multiplication with a wave with frequency $\approx$ the frequency ν of the carrier wave $\cos(2\pi\nu t)$ also makes the bat waves audible.

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The approach with the transformation $H \rightsquigarrow H * \widehat{\Pi}_T$ to make the filter causal (& finite) is called a **window method**.

Note. If $H \in L^2(\mathbb{R})$, then $H * \widehat{\Pi}_T$ is continuous (Why?).

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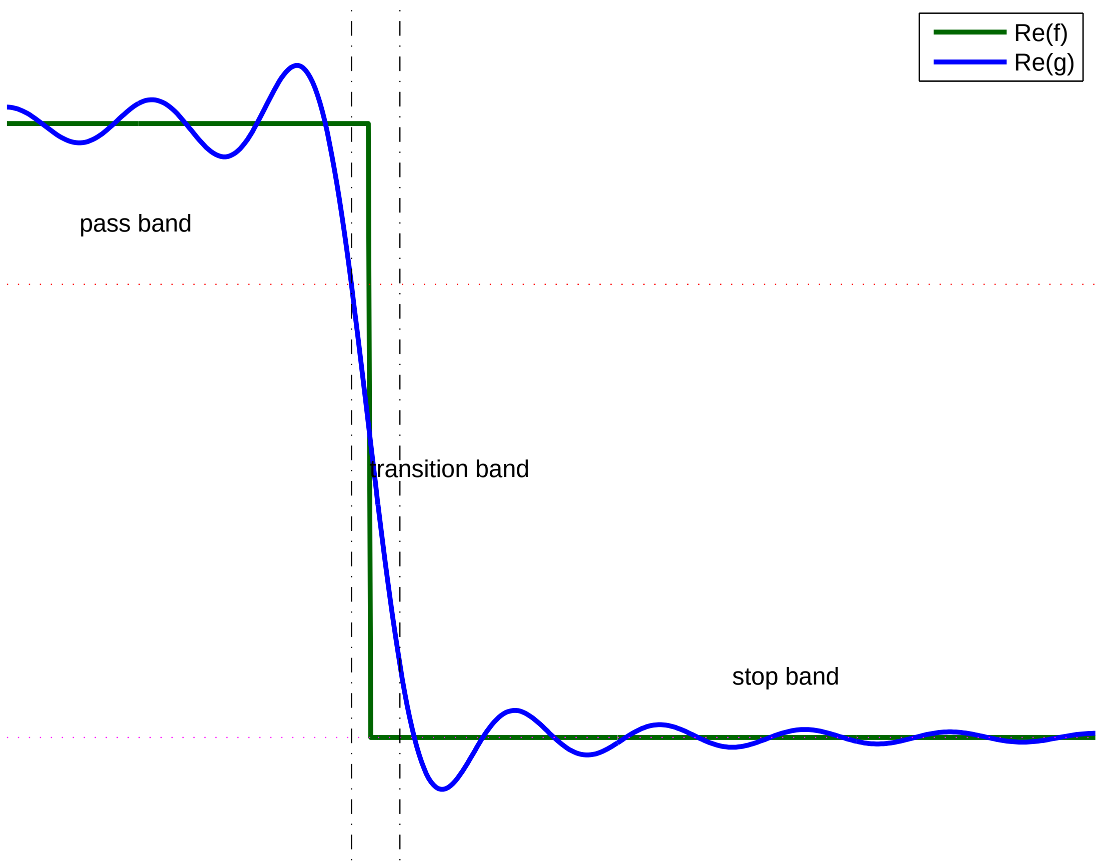
Note. If $H \in L^2(\mathbb{R})$, then $H * \widehat{\sqcap}_T$ is continuous (Why?).

It is **impossible** to form a step function with a filter h that has a bounded time domain:
$h \sqcap_T$ has bounded domain $\Rightarrow H * \widehat{\sqcap}_T$ is analytic.

Filters



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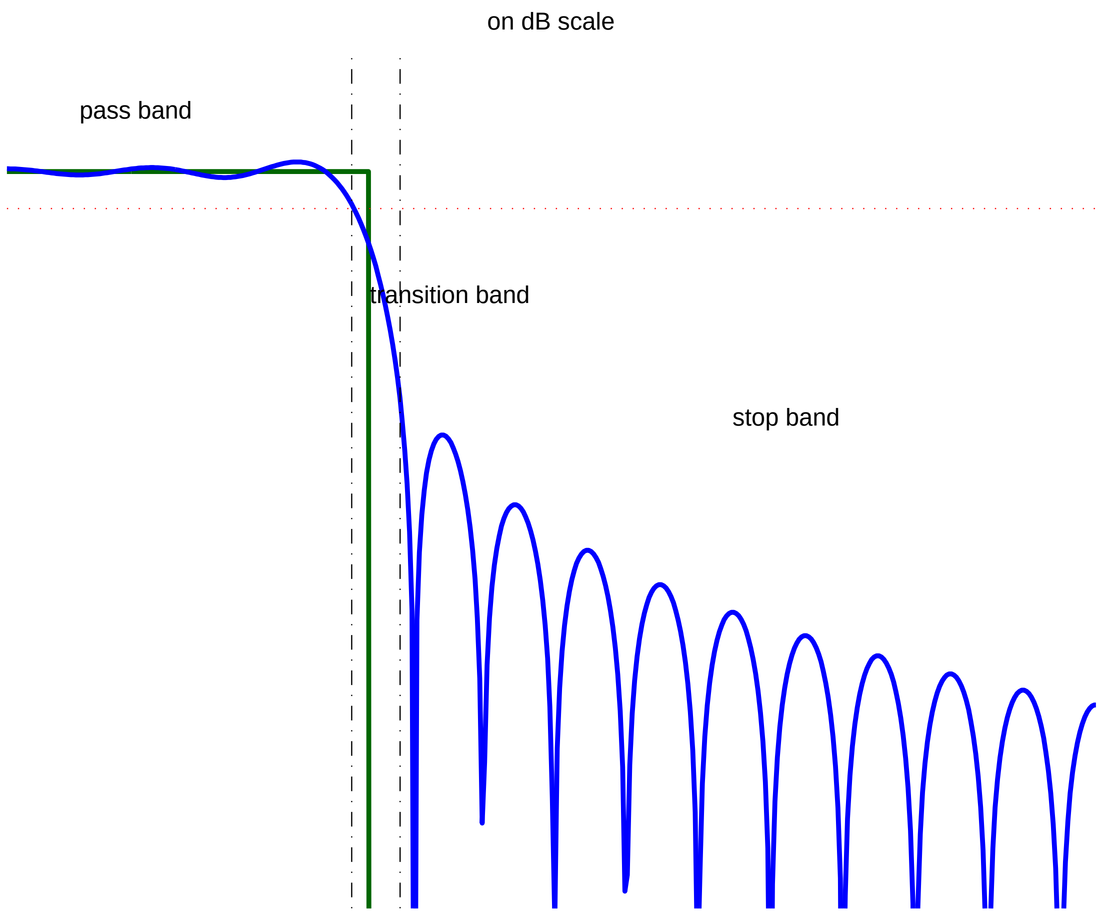

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To "damp" the "overshoot" (10%) effect of Gibbs' phenomenon in the stop band: take a continuous approximation of H (rather than H, as $H = \Pi_\Omega$, itself):

- **+** better stop properties in the stop band
- **+** less leakage in the pass band
- **−** wider transition band

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H smooth(er) $\Rightarrow$ $h = \widehat{H}$ decreases more rapidly at ∞

$\Rightarrow$ $h\Pi_T$ is a more accurate approx. of $\widehat{H}$ (than $\widehat{\Pi}_\Omega \Pi_T$ of $\widehat{\Pi}_\Omega$)

$\Rightarrow$ $H * \Pi_T$ is more accurate (uniform convergence for $T \to \infty$)

The approach with the transformation $H \rightsquigarrow H * \widehat{\Pi}_T$ to make the filter causal (& finite) is called a **window method**.

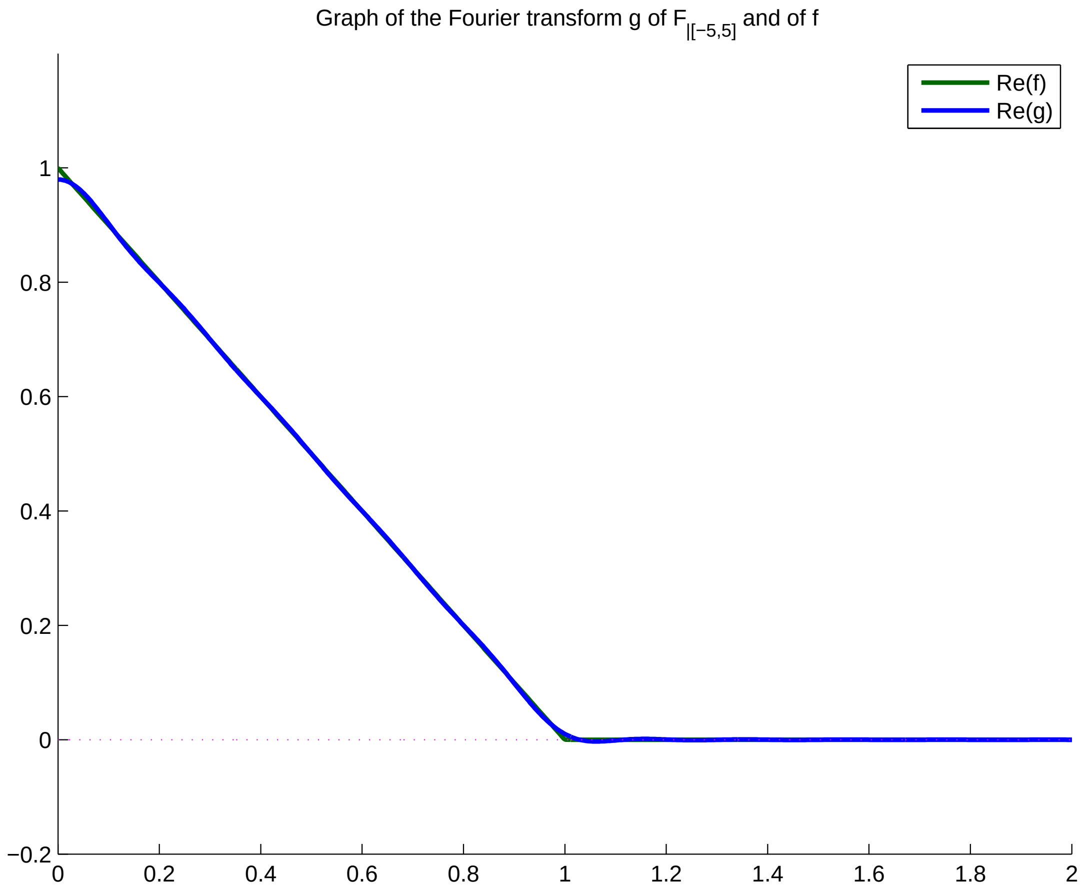
Note. If $H \in L^2(\mathbb{R})$, then $H * \widehat{\Pi}_T$ is analytic.

To "damp" the "overshoot" (10%) effect of Gibbs' phenomenon in the stop band: take a continuous approximation of H (rather than H, as $H = \Pi_\Omega$, itself):

- **+** better stop properties in the stop band
- **+** less leakage in the pass band
- **−** wider transition band

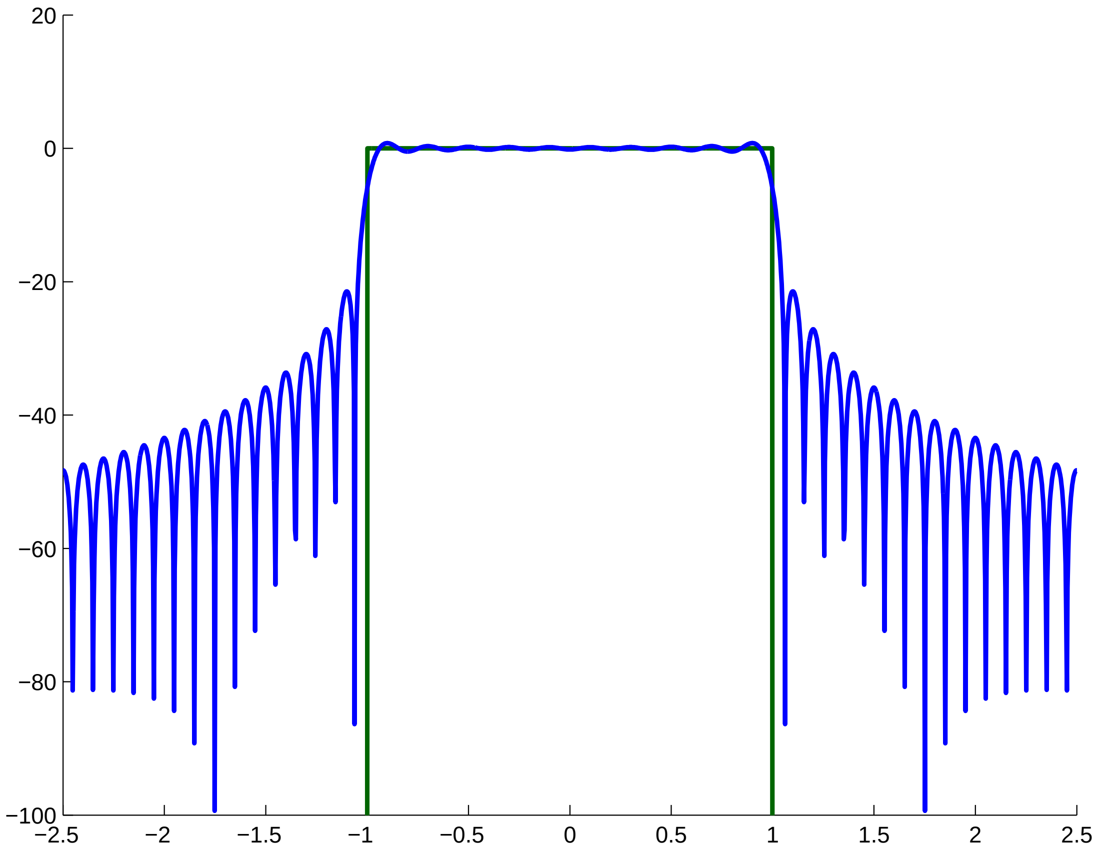
(Smooth) approximations of Π_Ω are also called **windows** (in frequency domain).

Smooth Windows



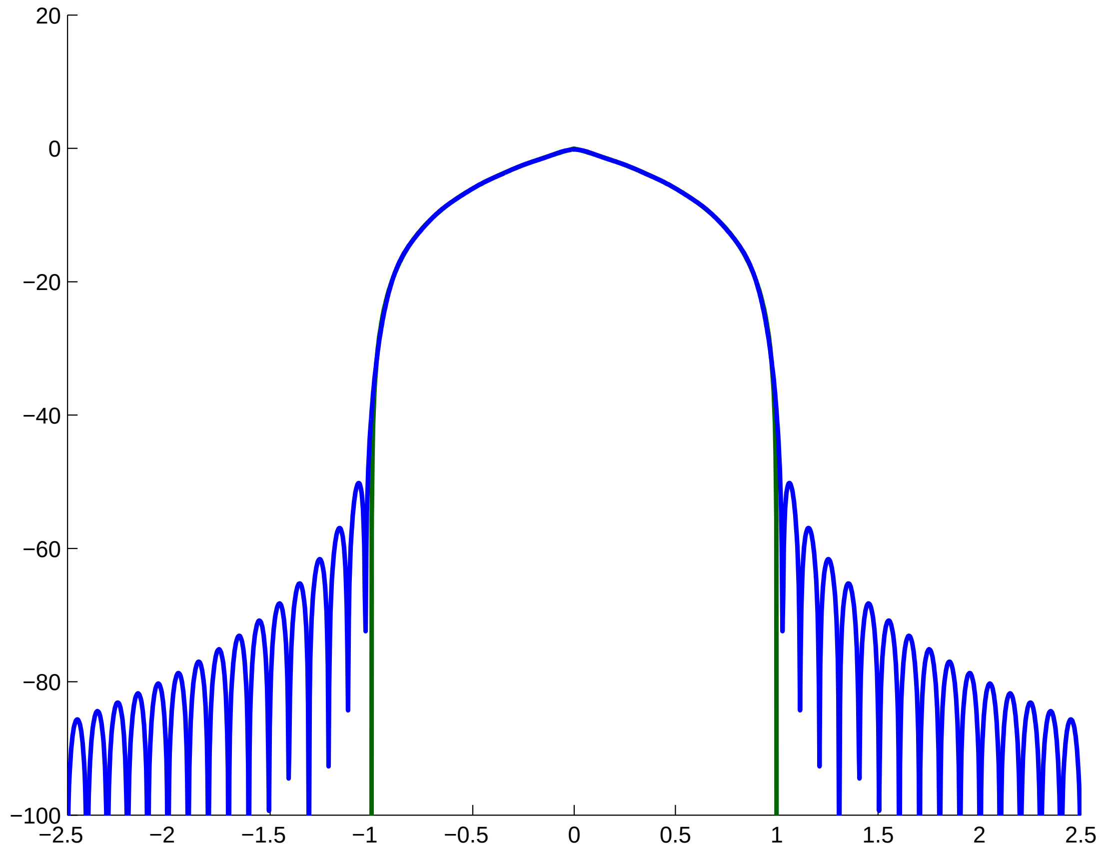
Barlett window: $H(\omega) \equiv (1 - \frac{\omega}{\Omega})\Pi_{\Omega}(\omega) \qquad [\Omega = 1, T = 5]$

Smooth Windows



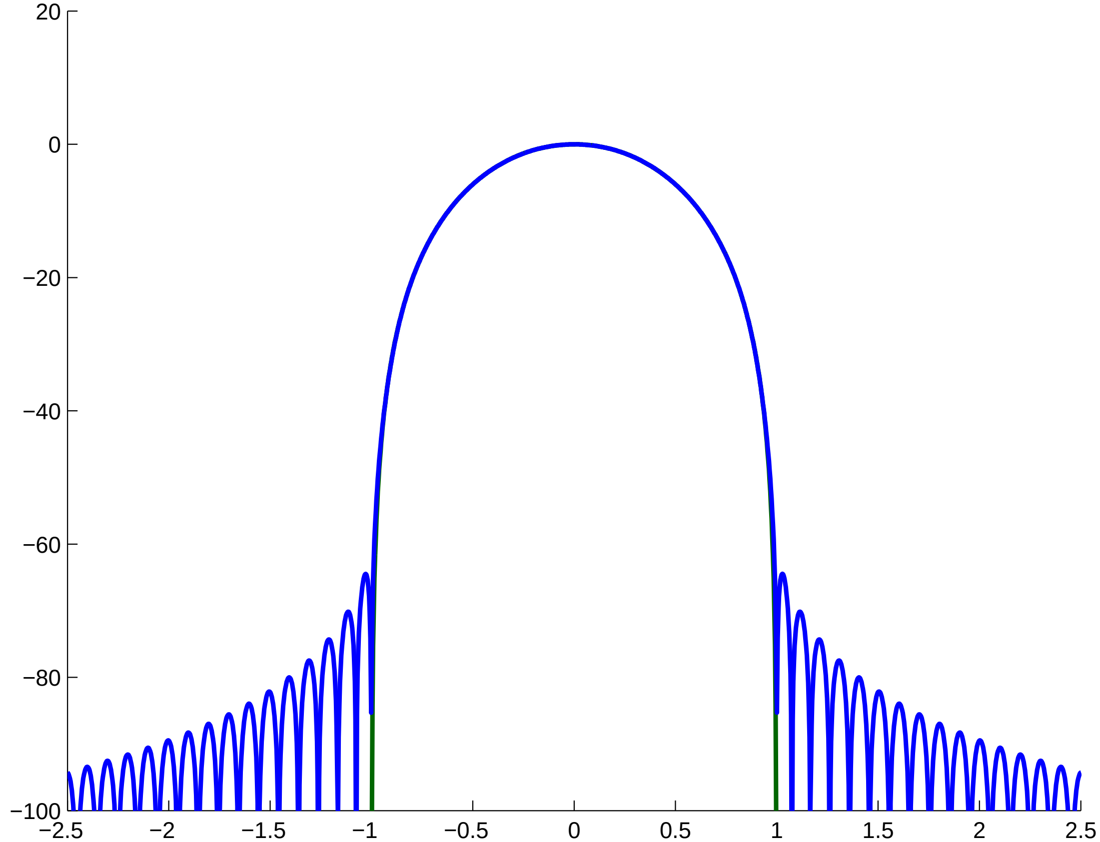
$H = \Pi_\Omega \qquad [\Omega = 1, T = 5]$

Smooth Windows



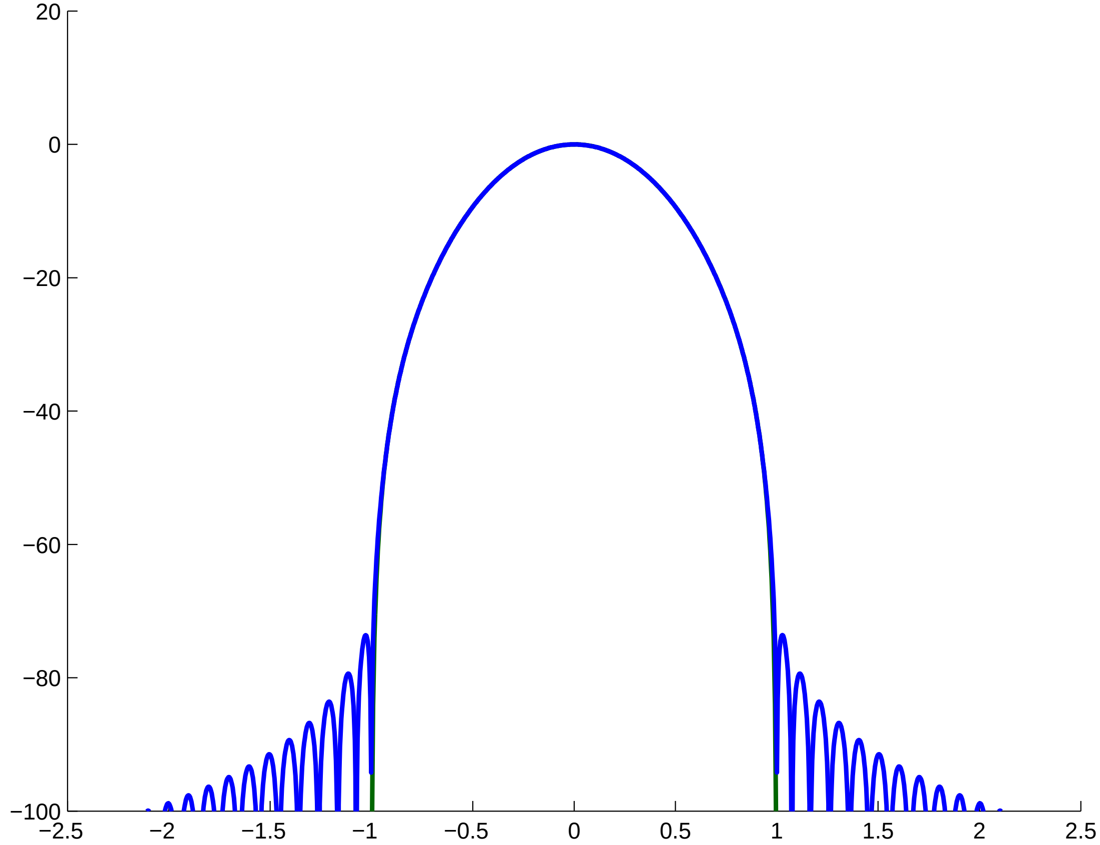
Barlett window: $H(\omega) \equiv (1 - \frac{\omega}{\Omega})\Pi_\Omega(\omega)$, on dB scale

Smooth Windows



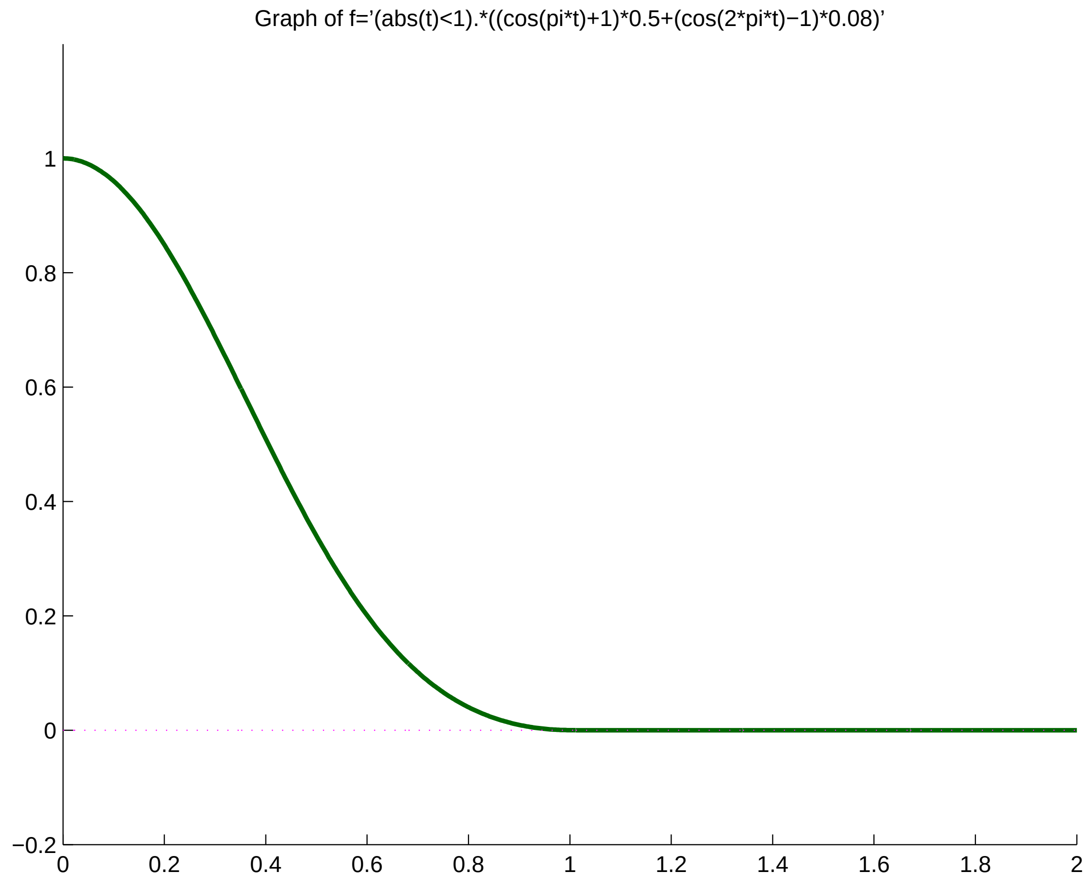
Hann window: $H(\omega) \equiv 0.5(\cos(\pi\frac{\omega}{\Omega})+1)\Pi_{\Omega}(\omega)$ on dB scale

Smooth Windows



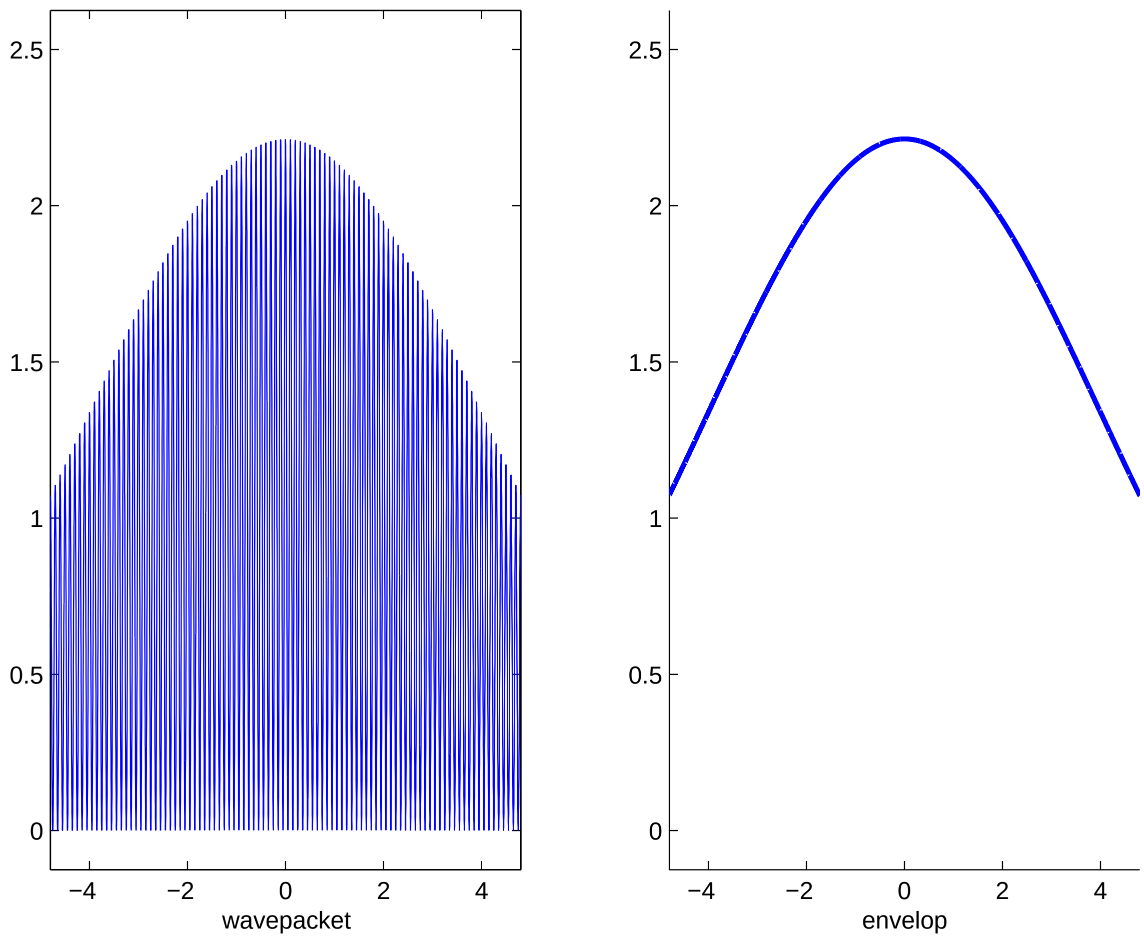
Blackman window: Hann $+ \; 0.08(\cos(2\pi\frac{\omega}{\Omega}) - 1)\Pi_\Omega(\omega)$

Smooth Windows



Blackman window: Hann + $0.08(\cos(2\pi\frac{\omega}{\Omega}) - 1)\Pi_{\Omega}(\omega)$

Filters

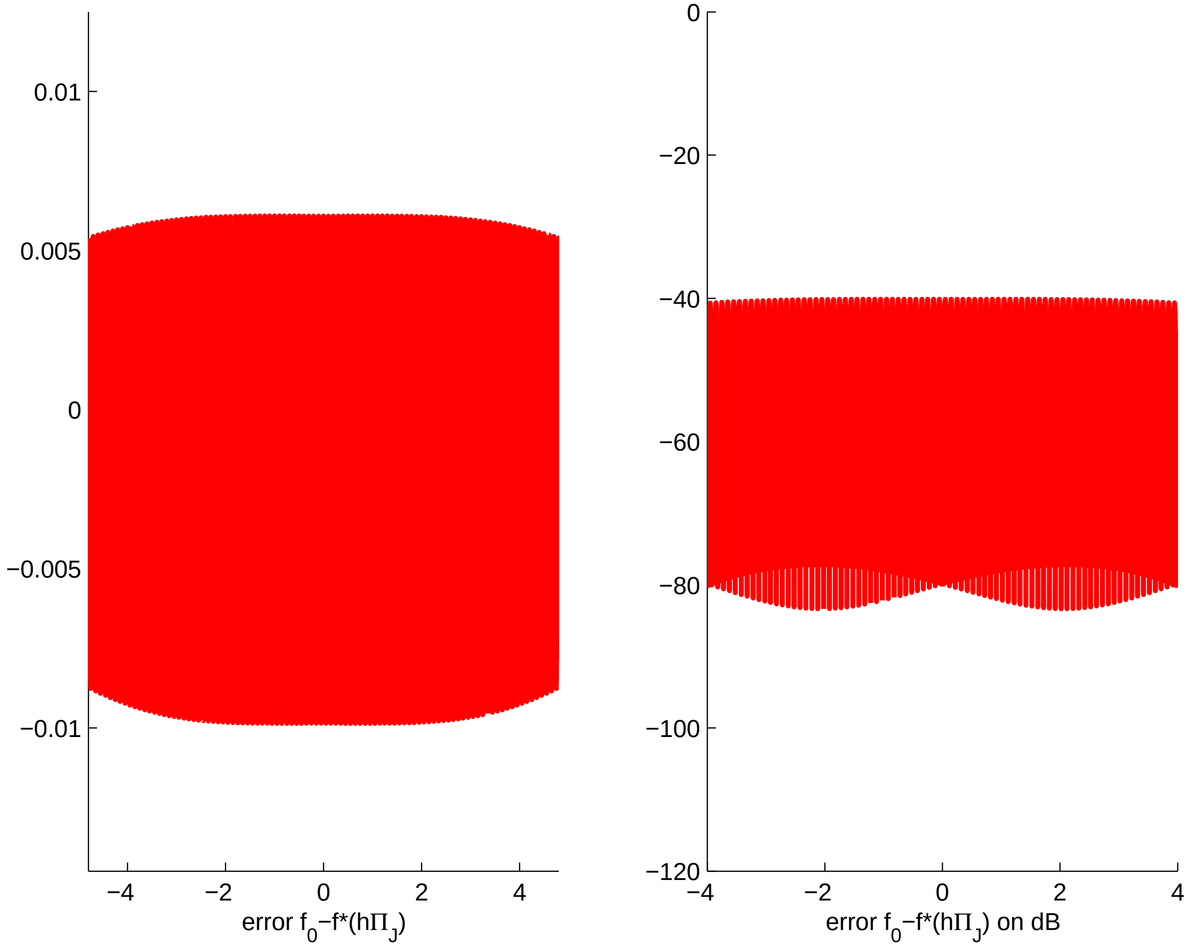


$f(t) = 2\, f_0(t)\, \cos^2(2\pi\nu t)$ (in the left picture) with

$f_0(t) = 3.5\sqrt{\alpha}\, \exp(-\pi\alpha^2 t^2)$ (in the right picture)

and $\alpha = 0.1, \quad \nu = 5$

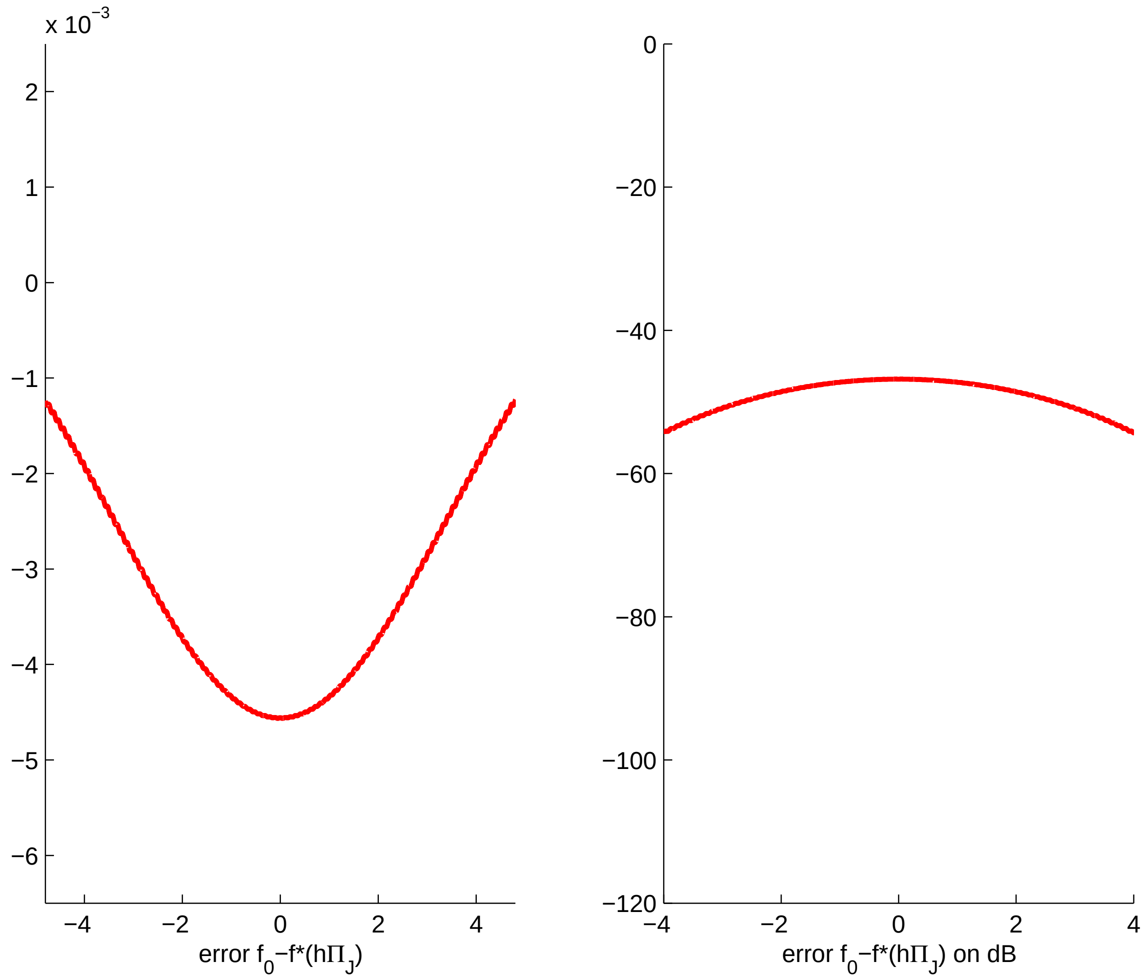
Filters



With $\Omega = 9$, we have that $f * \widehat{\Pi}_{\Omega} = f_0$.

The pictures show the error $f_0 - \widetilde{f}_0$ with $\widetilde{f}_0 \equiv f * (\widehat{\Pi}_{\Omega}\Pi_T)$ and $T = 4$ (the right picture in dB-scale).

Filters

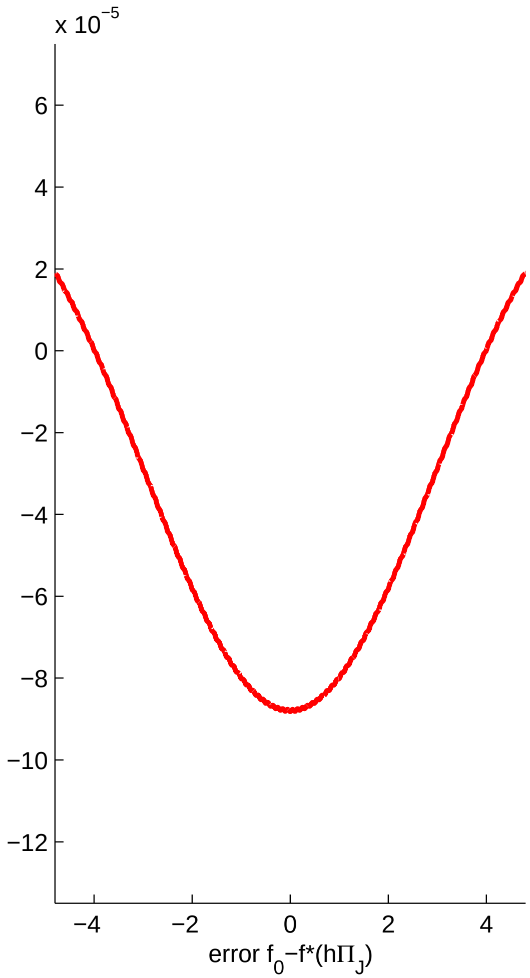


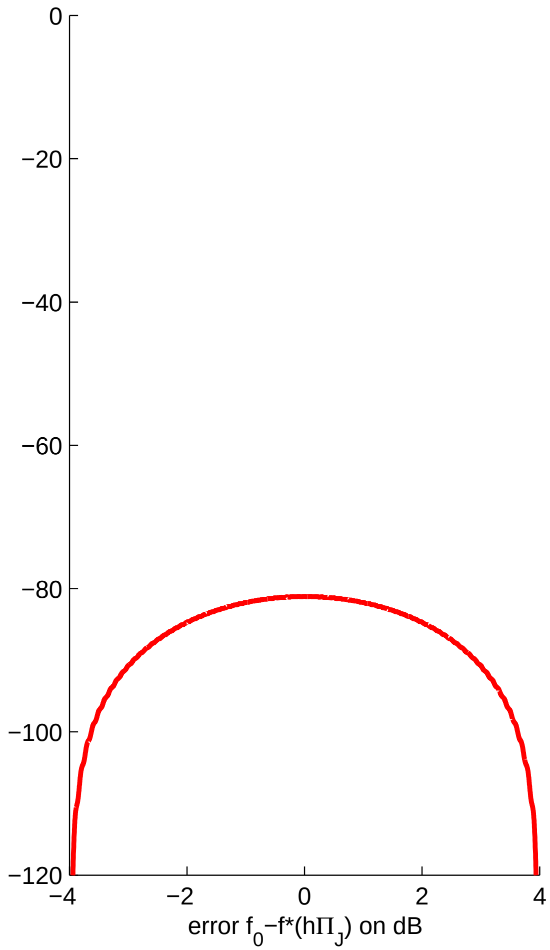
With $\Omega = 9$, we have that $f * \widehat{\Pi}_\Omega = f_0$.

The pictures show the error $f_0 - \widetilde{f}_0$ with $\widetilde{f}_0 \equiv f * (\widehat{H}\Pi_T)$

H Barlett's window (in the right picture at dB scale).

Filters





With $\Omega = 9$, we have that $f * \widehat{\Pi}_\Omega = f_0$.

The pictures show the error $f_0 - \widetilde{f}_0$ with $\widetilde{f}_0 \equiv f * (\widehat{H}\Pi_T)$

H Blackman's window (in the right picture at dB scale).

The approach with the transformation $H \rightsquigarrow H * \widehat{\Pi}_T$ to make the filter causal (& finite) is called a **window method**.

Note. The filters that we considered so far are real and symmetric (both in time as well as in frequency domain) and the windowing approach did not change this. In particular, these filters will not lead to group or time delays.

Program

- Filters
- Finite Impulse Response Filters
- Windows
- Signals of finite duration and bounded bandwidth?
- Infinite Impulse Response Filters
- Analog filters (hardware)
- Digital filters (software)

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\Pi_\Omega = \widehat{f}\}.$

$Df \equiv D_T(f) \equiv f\Pi_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$

D restricts to the bounded time domain $[-T, +T]$ and
B filters to the bounded frequency domain $[-\Omega, \Omega]$.

There are no functions f for which $BDf = f$ (Why not?).

$$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\,\Pi_\Omega = \widehat{f}\}.$$

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B filters to the bounded frequency domain $[-\Omega, \Omega]$.

There are no functions f for which $BDf = f$ (Why not?).

Find the functions ψ for which $BD\psi = \lambda\psi$:

ψ is an eigenfunctions of BD with eigenvalue λ.

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$Df \equiv D_T(f) \equiv f\Pi_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega$ $(f \in L^2(\mathbb{R}))$

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There are no functions f for which $BDf = f$ (Why not?).

Find the functions ψ for which $BD\psi = \lambda\psi$:

ψ is an eigenfunctions of BD with eigenvalue λ.

Note. $B\psi = \psi \in \mathcal{B}$. Hence,

$$\lambda\|\psi\|_2^2 = (BD\psi, \psi) = (D\psi, \psi) = \|D\psi\|_2^2 > 0.$$

By restricting a signal in $\mathcal{B}$ to $[-T, T]$, energy gets lost:

$$1 - \frac{\|D\psi\|_2^2}{\|\psi\|_2^2} = 1 - \lambda.$$

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\Pi_\Omega = \widehat{f}\}$.

$Df \equiv D_T(f) \equiv f\Pi_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega$ $(f \in L^2(\mathbb{R}))$

D restricts to the bounded time domain $[-T, +T]$ and
B filters to the bounded frequency domain $[-\Omega, \Omega]$.

There are no functions f for which $BDf = f$ (Why not?).

Find the functions ψ for which $BD\psi = \lambda\psi$:

ψ is an eigenfunctions of BD with eigenvalue λ.

Or, equivalently, $\widetilde{\psi}(= D\psi)$ for which $DB\widetilde{\psi} = \lambda\widetilde{\psi}$.

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\Pi_\Omega = \widehat{f}\}.$

$Df \equiv D_T(f) \equiv f\Pi_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$

$$BD\psi(t) = 2\Omega \int_{-T}^{T} \mathrm{sinc}(2\Omega(t-s))\,\psi(s)\,\mathrm{d}s = \lambda\psi(t)$$

Put $c \equiv 2\Omega T$ and $\phi(x) \equiv \psi(Tx)$. Then (with $s = Tx$)

$$c \int_{-1}^{1} \mathrm{sinc}(c(y-x))\,\phi(x)\,\mathrm{d}x = \lambda\phi(y)$$

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\sqcap_\Omega = \widehat{f}\}$.

$Df \equiv D_T(f) \equiv f\sqcap_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\sqcap}_\Omega \quad (f \in L^2(\mathbb{R}))$

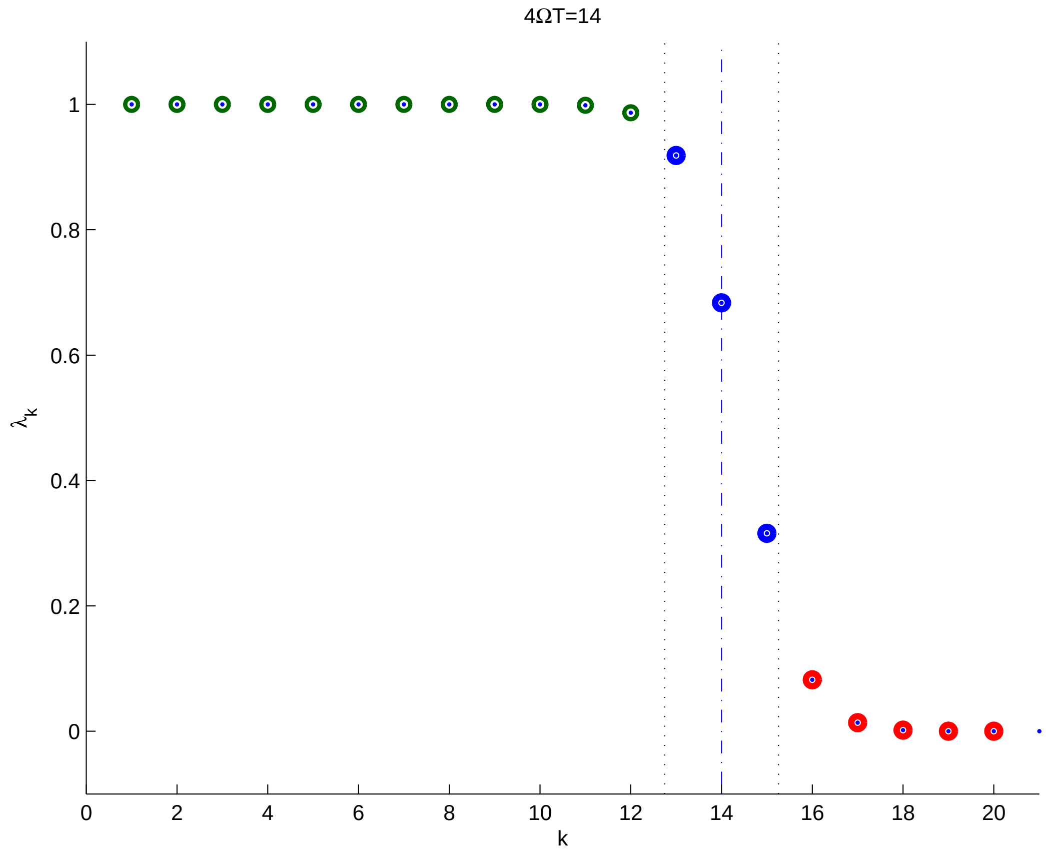
$$BD\psi(t) = 2\Omega \int_{-T}^{T} \mathrm{sinc}(2\Omega(t-s))\, \psi(s)\, \mathrm{d}s = \lambda\psi(t)$$

Put $c \equiv 2\Omega T$ and $\phi(x) \equiv \psi(Tx)$. Then (with $s = Tx$)

$$c \int_{-1}^{1} \mathrm{sinc}(c(y-x))\, \phi(x)\, \mathrm{d}x = \lambda\phi(y)$$

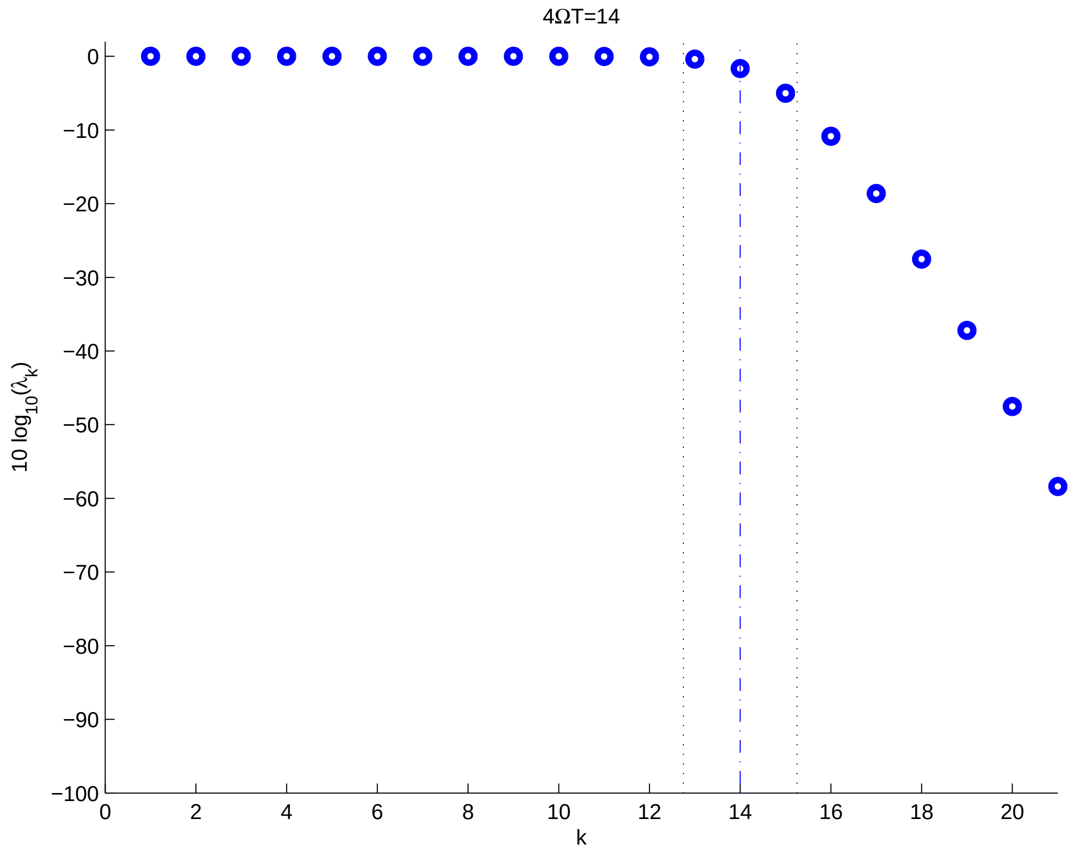
Except for a scaling of the t, the eigenfunctions depend on ΩT only (not on the individual values of T or Ω)

Eigenvalues



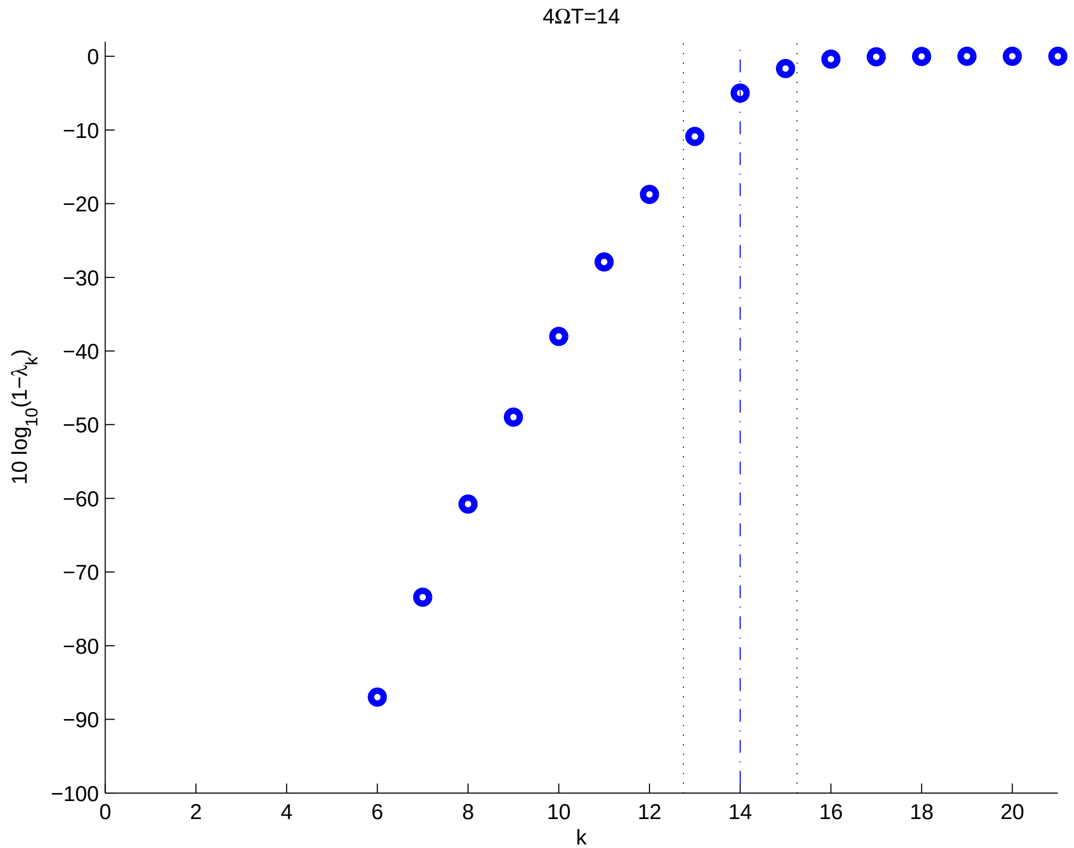
$4\Omega T = 14, \qquad k \rightsquigarrow \lambda_k, \qquad \ldots$ at $\pm\ln(\Omega T)$ from $4\Omega T$ $(-.)$

Eigenvalues



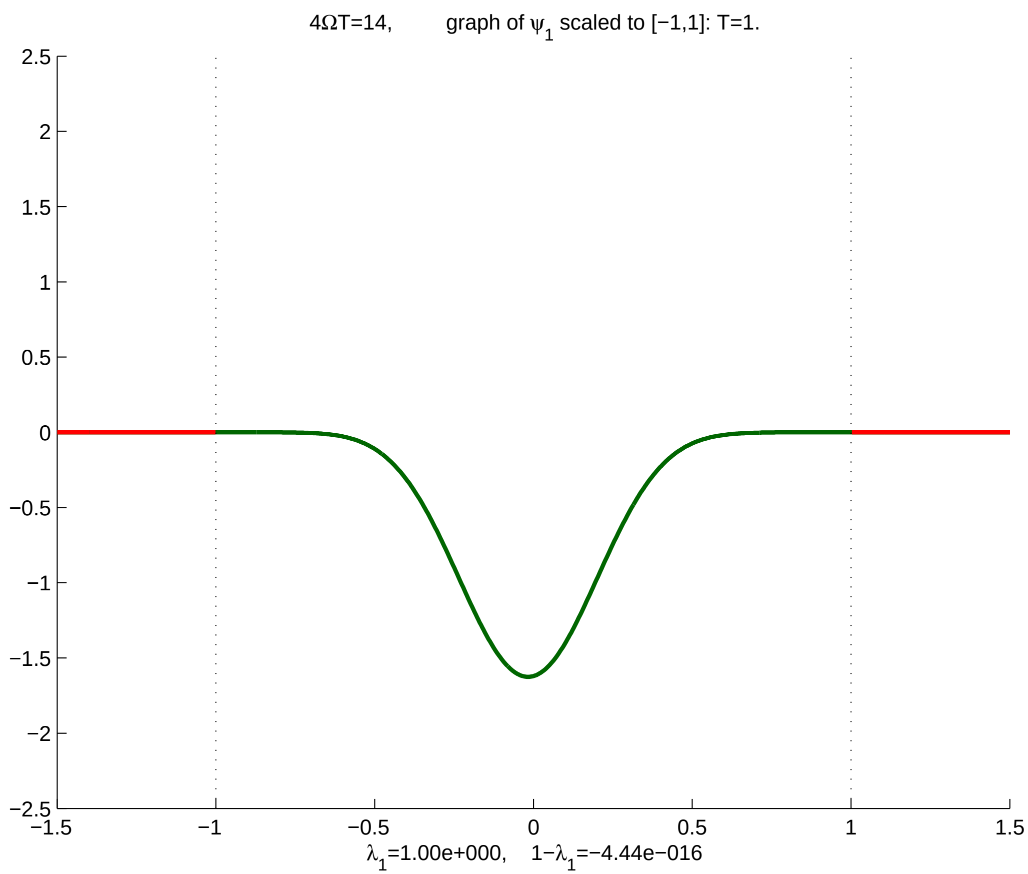
$$4\Omega T = 14, \qquad k \rightsquigarrow 10\log_{10}\lambda_k = 20\log_{10}\frac{\|D\psi_k\|_2}{\|\psi\|_2}$$

Eigenvalues



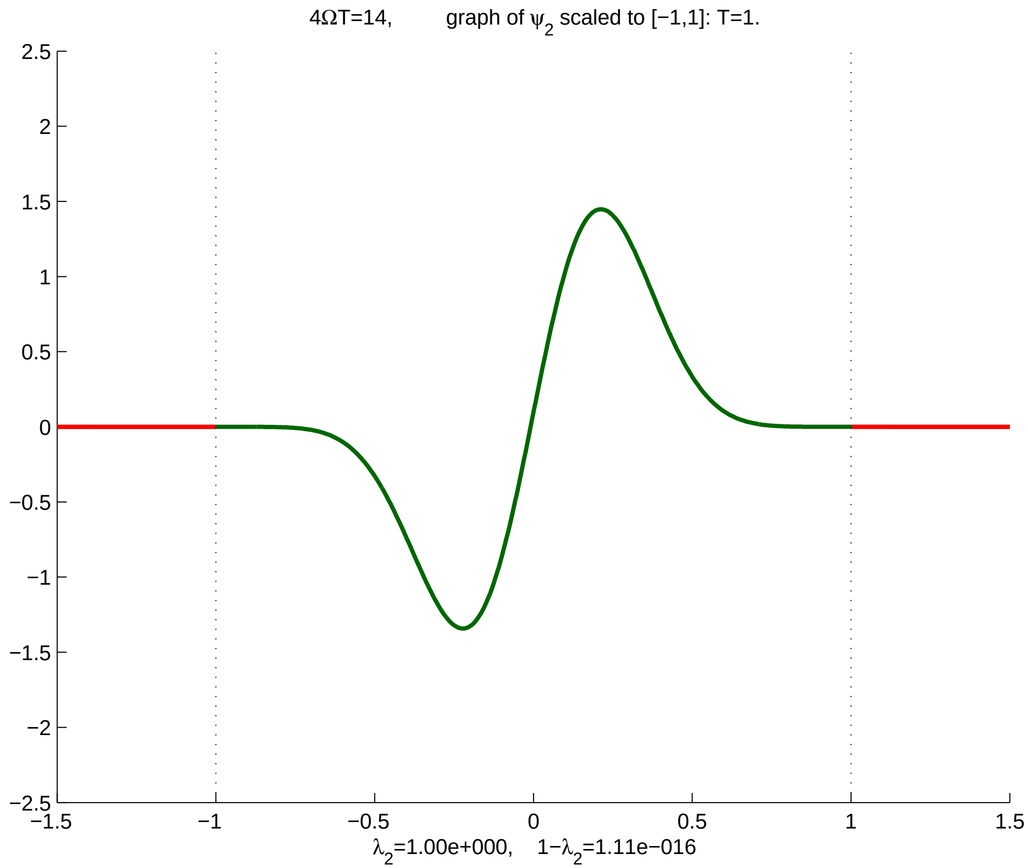
$$4\Omega T = 14, \qquad k \rightsquigarrow 10\log_{10}(1-\lambda_k)$$

Eigenfunctions



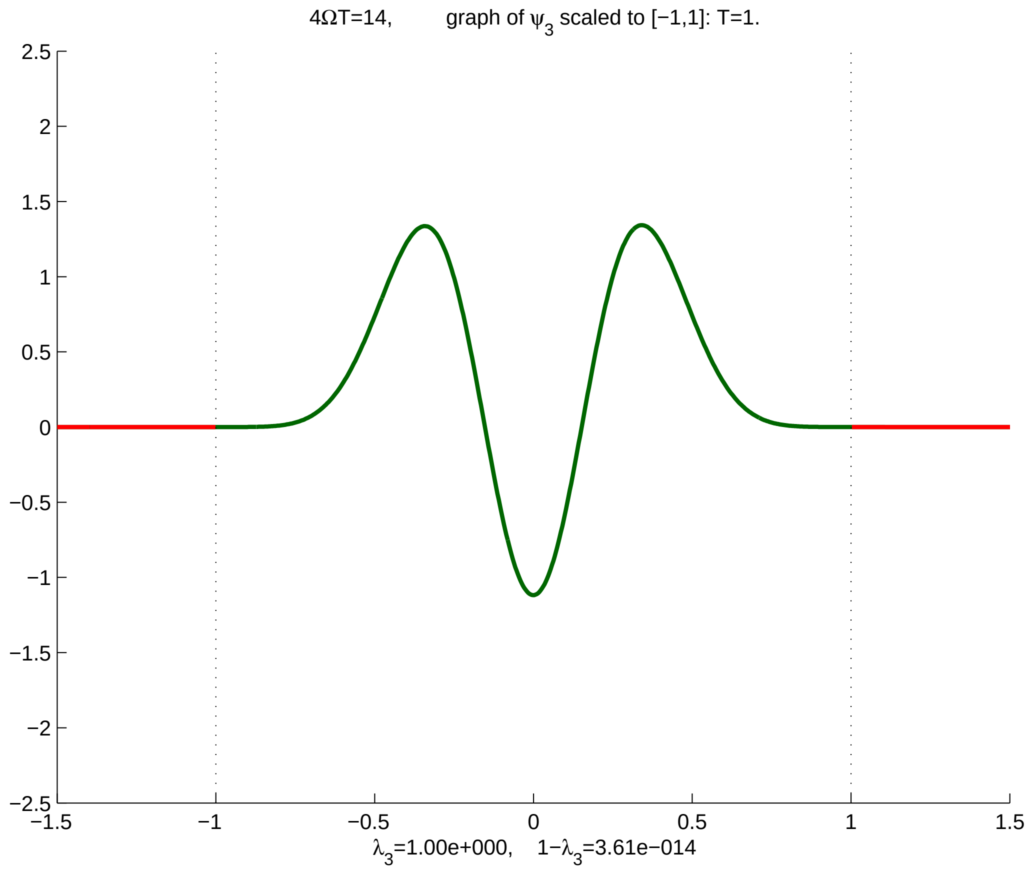
$4\Omega T = 14, \quad \phi_1$

Eigenfunctions



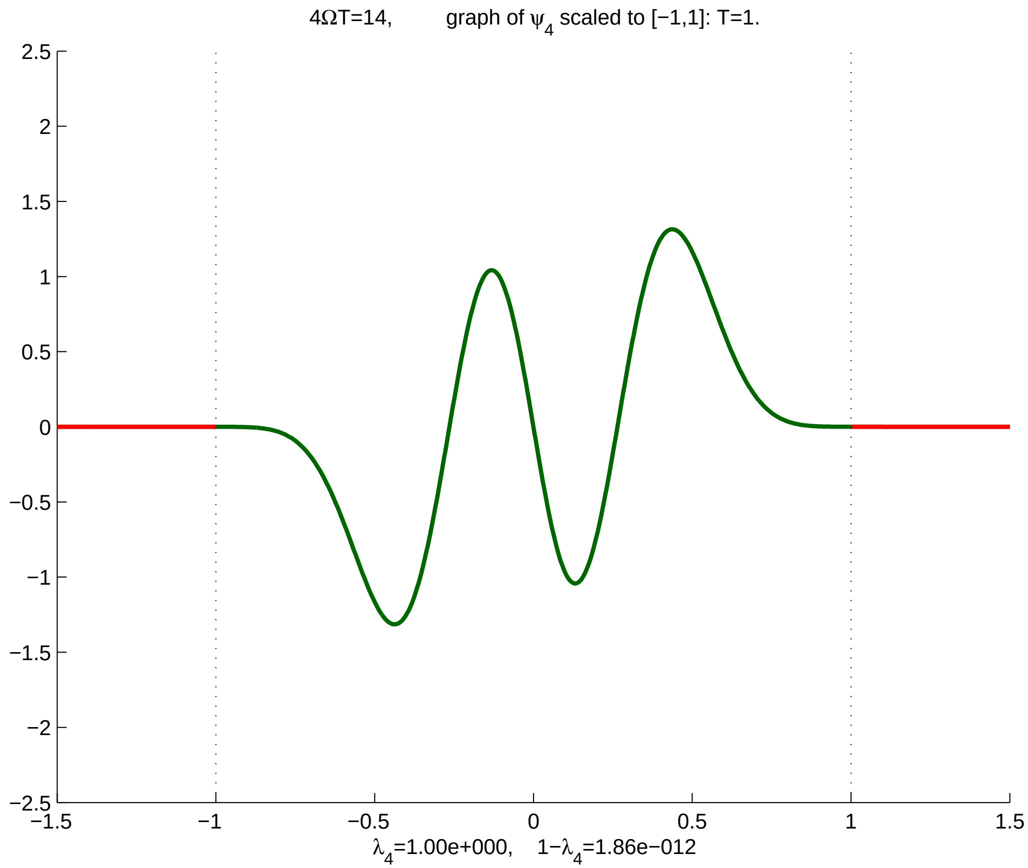
$4\Omega T = 14, \quad \phi_2$

Eigenfunctions



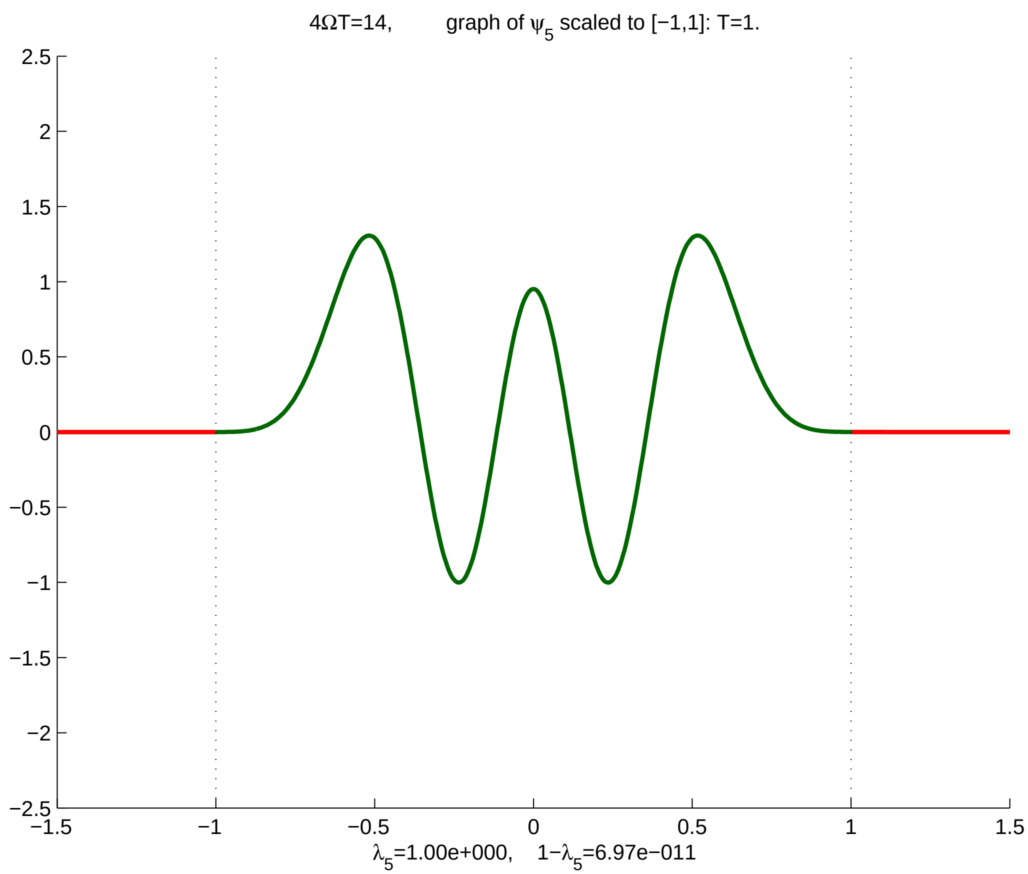
$4\Omega T = 14, \quad \phi_3$

Eigenfunctions



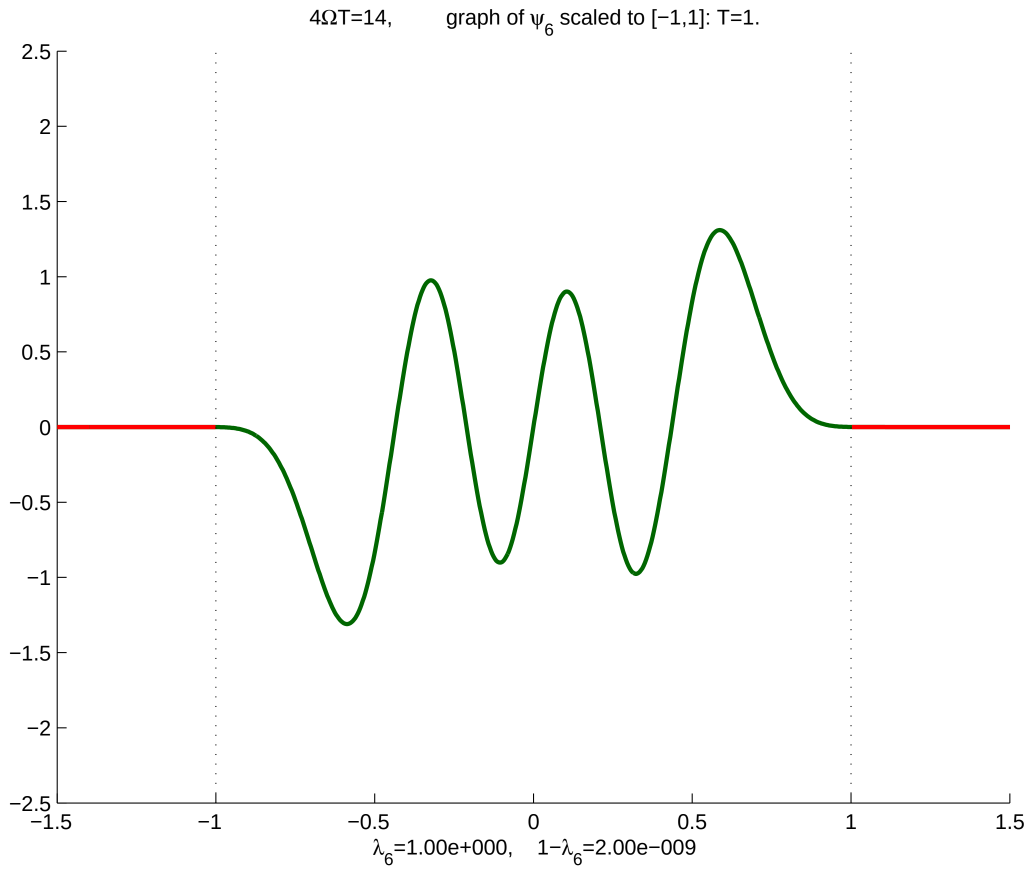
$4\Omega T = 14, \qquad \phi_4$

Eigenfunctions



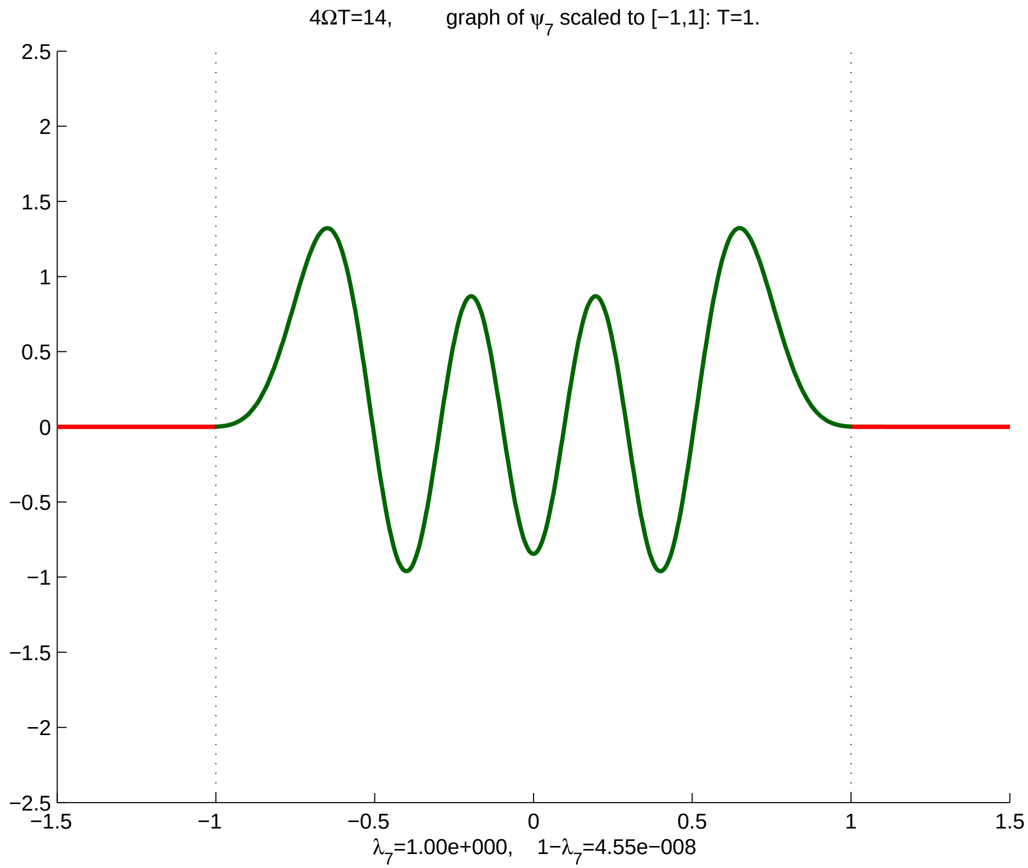
$4\Omega T = 14, \qquad \phi_5$

Eigenfunctions



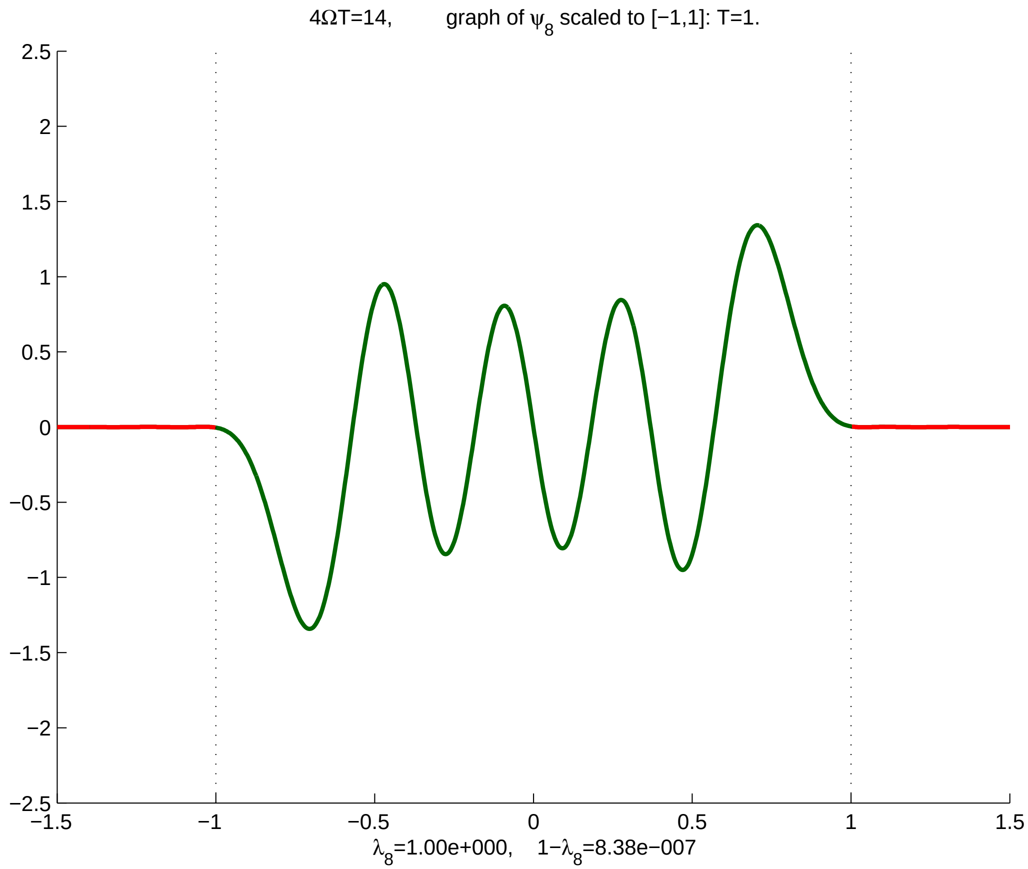
$4\Omega T = 14, \qquad \phi_6$

Eigenfunctions



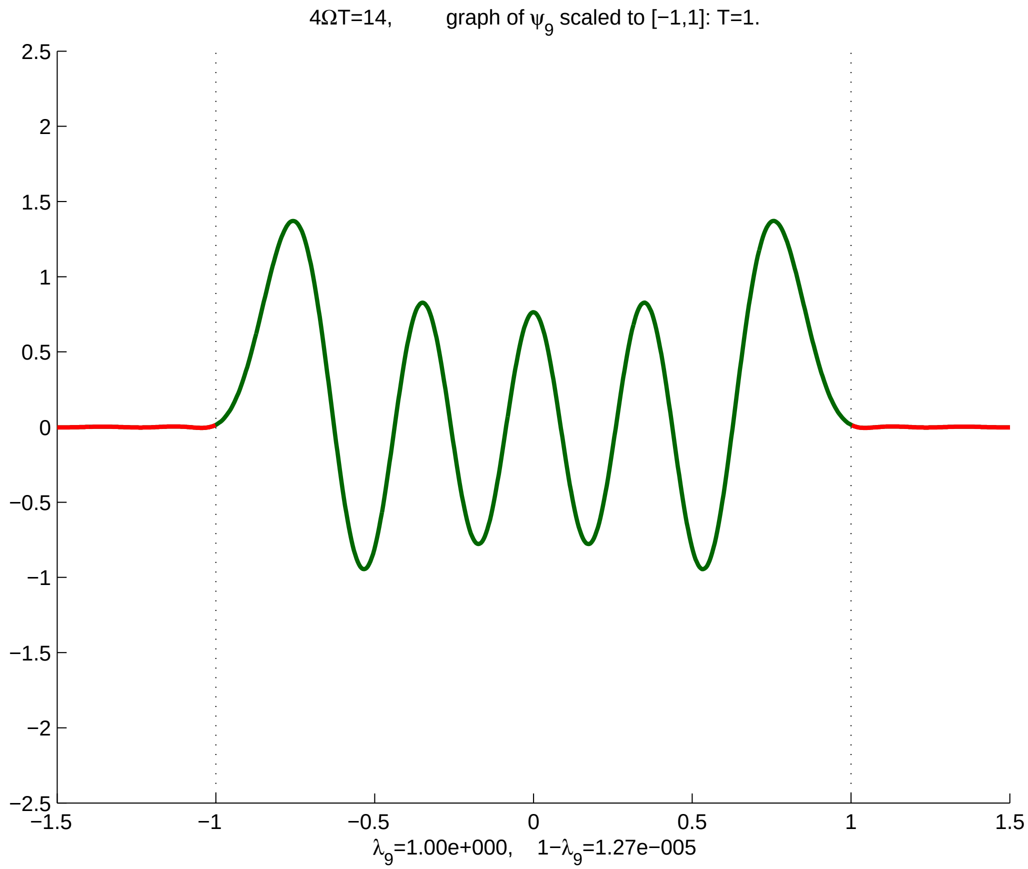
$4\Omega T = 14, \quad \phi_7$

Eigenfunctions



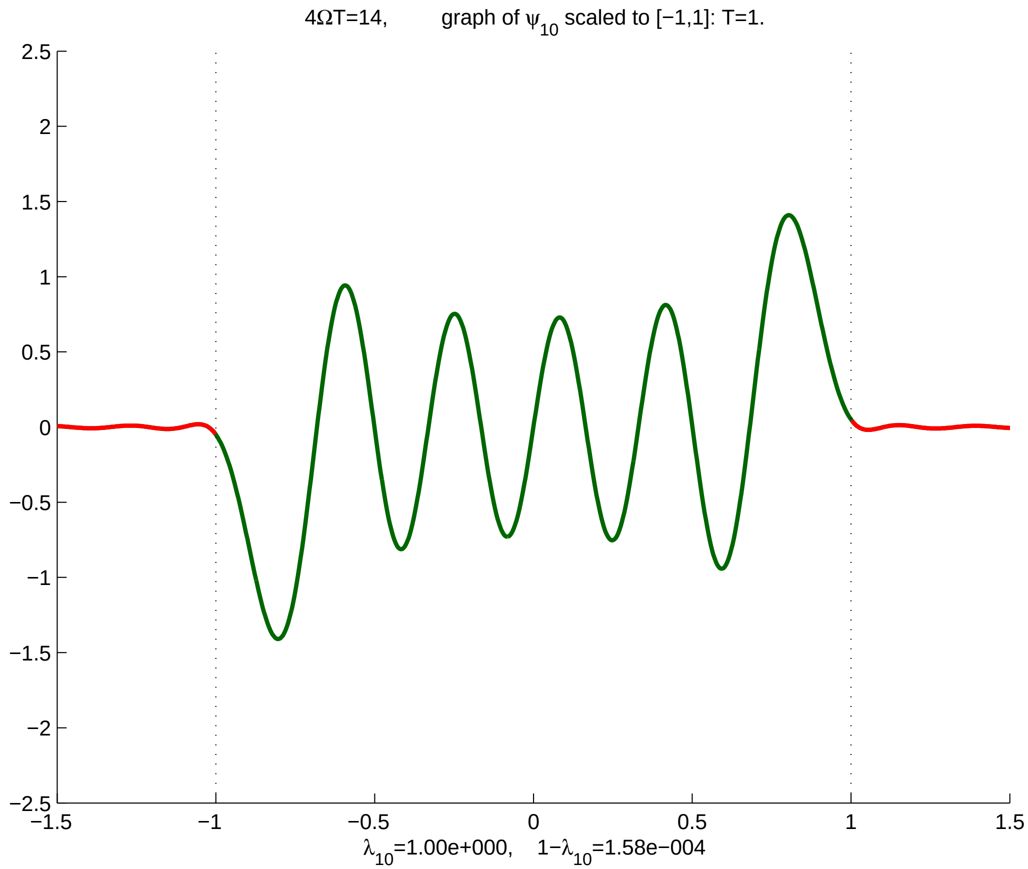
$4\Omega T = 14, \qquad \phi_8$

Eigenfunctions



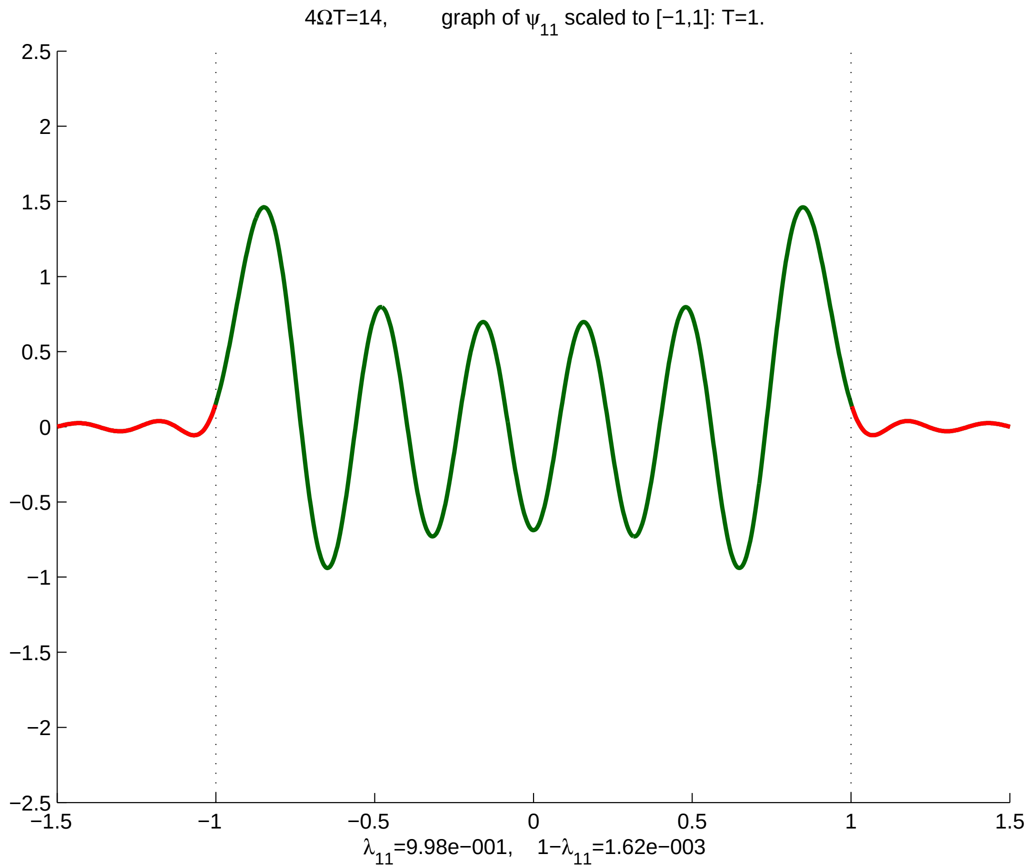
$4\Omega T = 14, \quad \phi_9$

Eigenfunctions



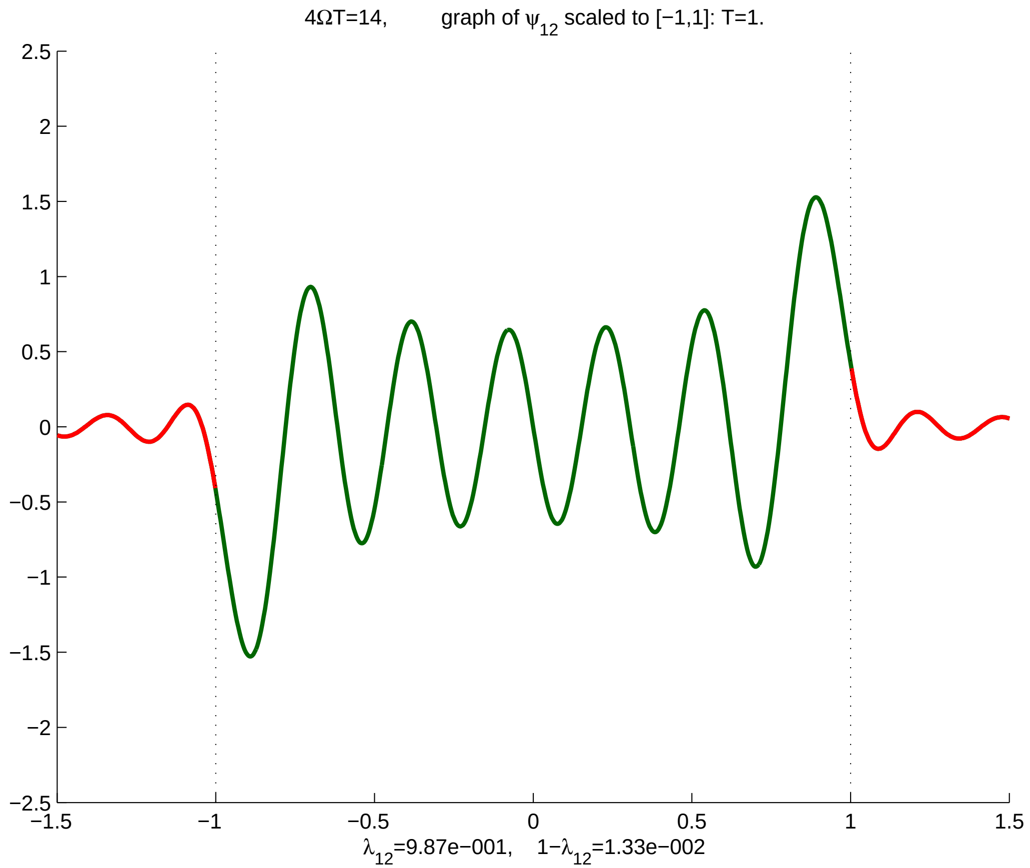
$4\Omega T = 14, \qquad \phi_{10}$

Eigenfunctions



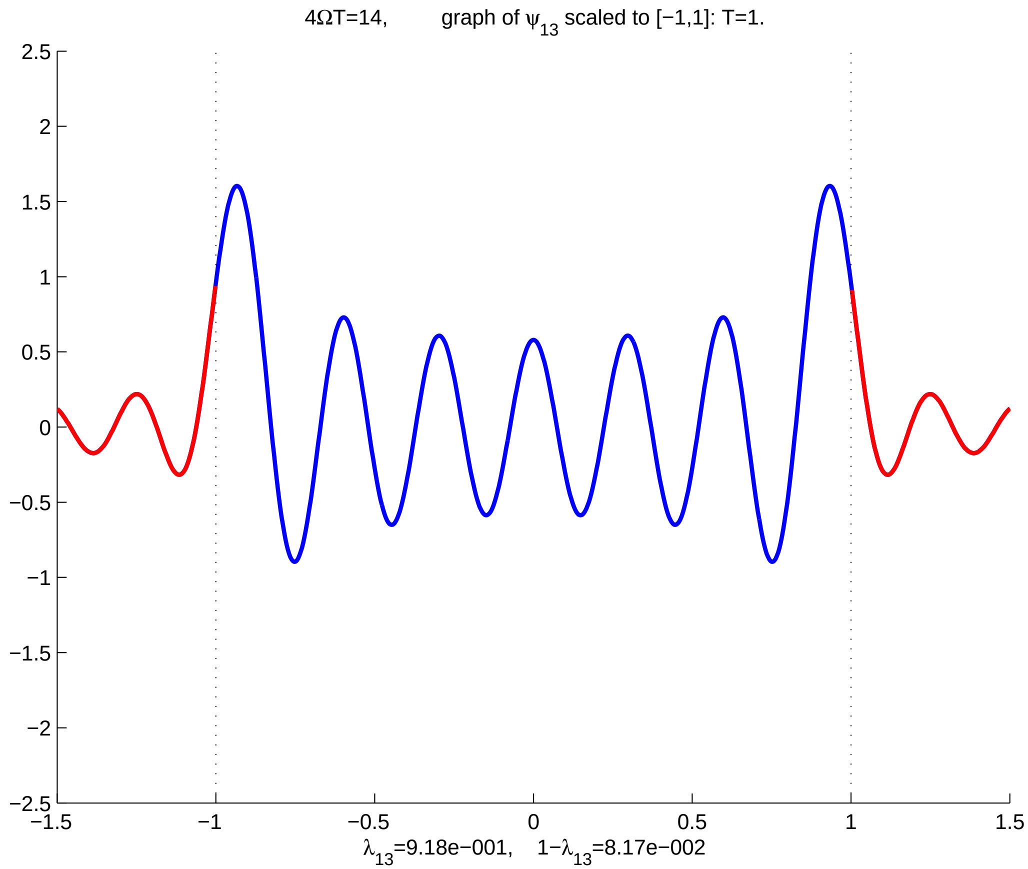
$4\Omega T = 14, \qquad \phi_{11}$

Eigenfunctions



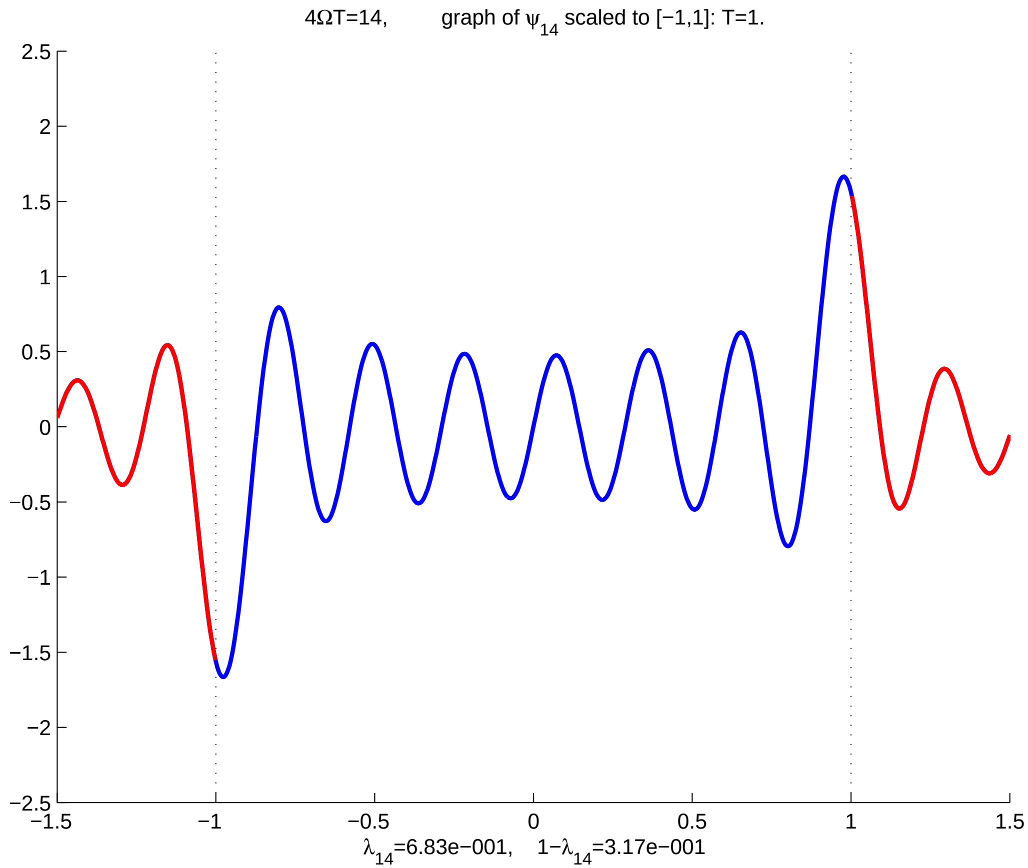
$4\Omega T = 14, \quad \phi_{12}$

Eigenfunctions



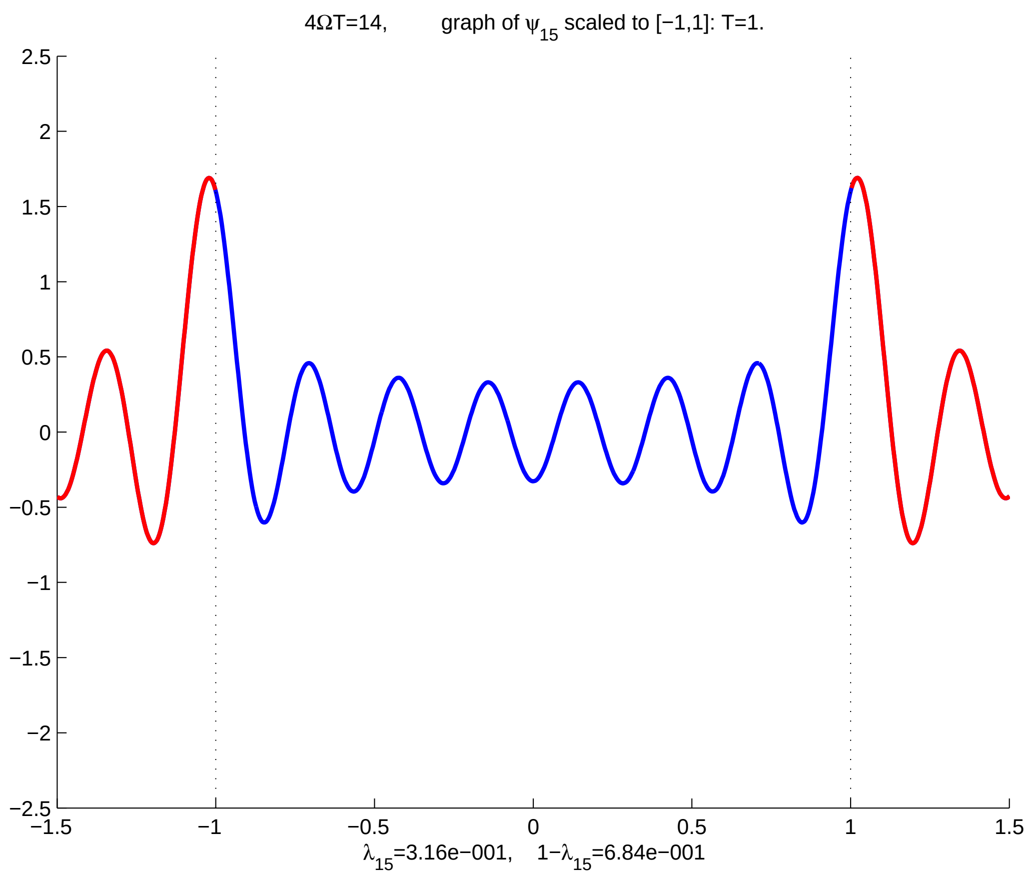
$4\Omega T = 14, \qquad \phi_{13}$

Eigenfunctions



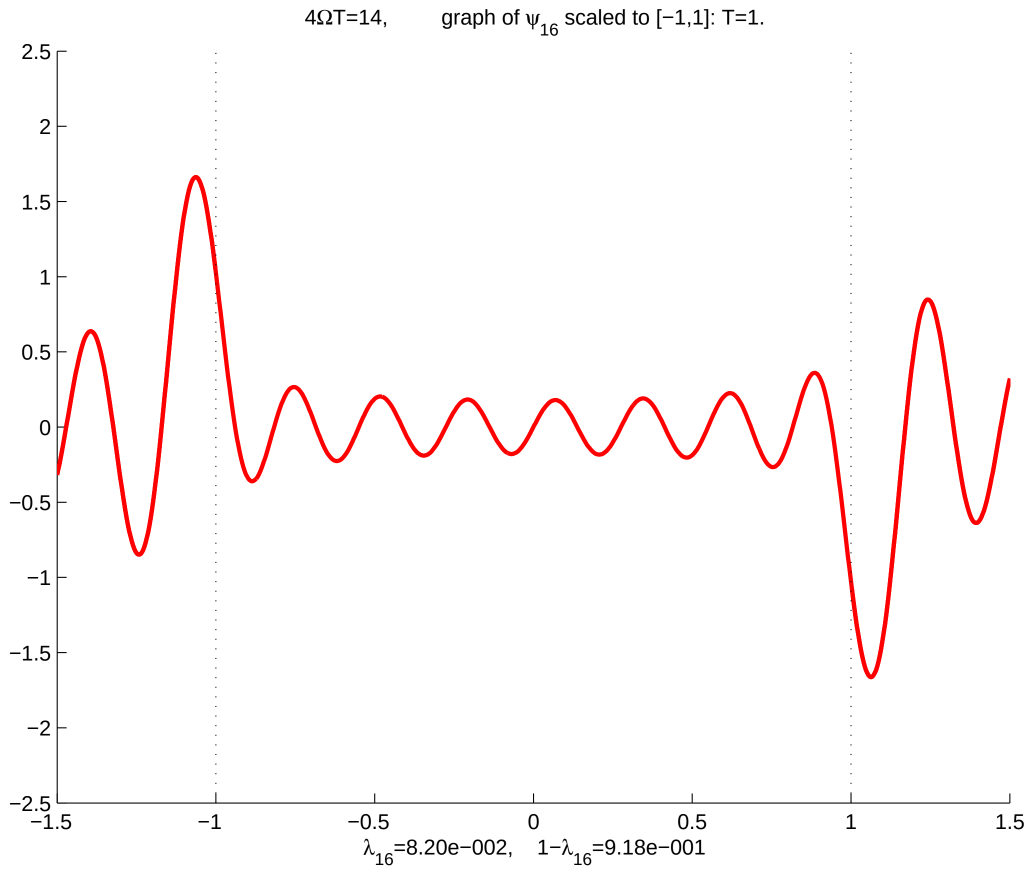
$4\Omega T = 14, \qquad \phi_{14}$

Eigenfunctions



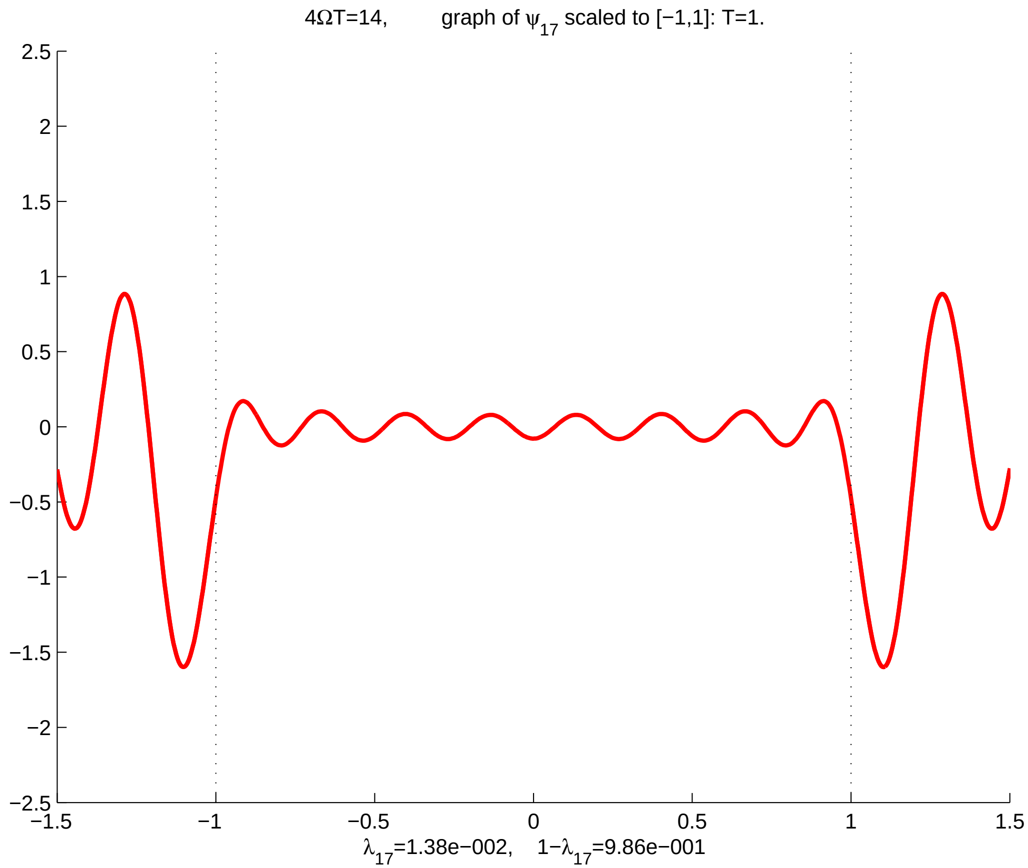
$4\Omega T = 14, \qquad \phi_{15}$

Eigenfunctions



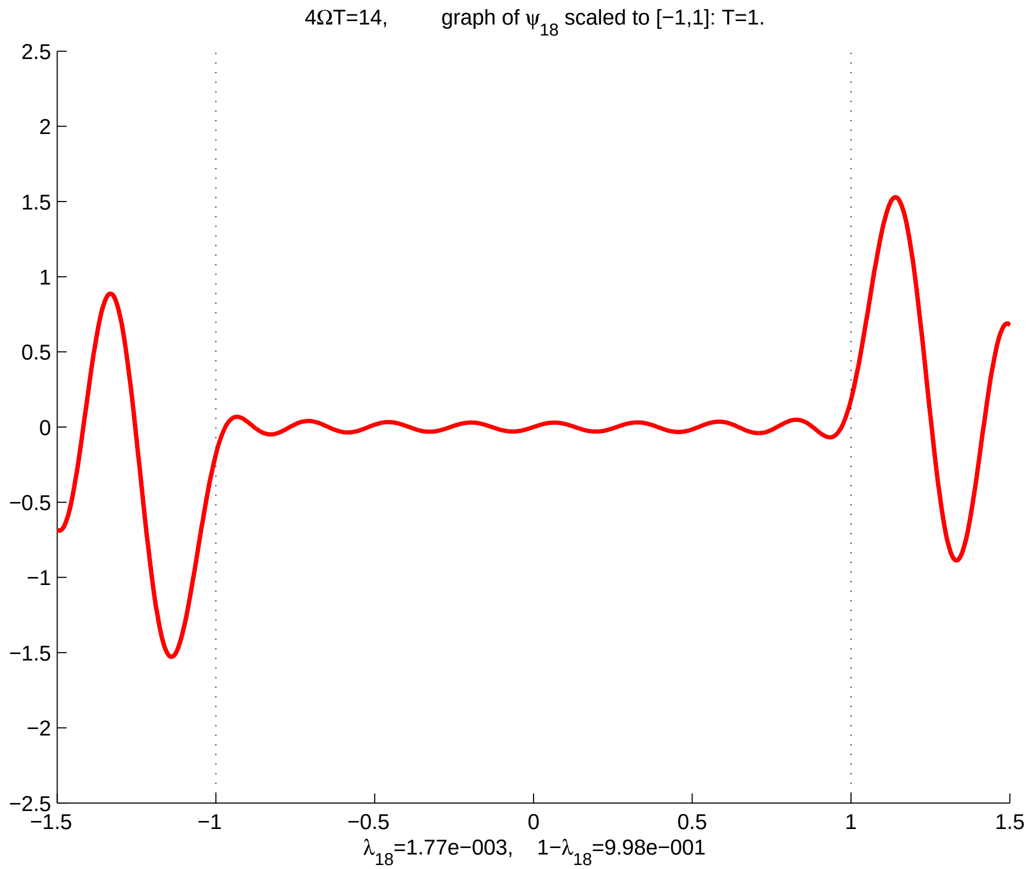
$4\Omega T = 14, \quad \phi_{16}$

Eigenfunctions



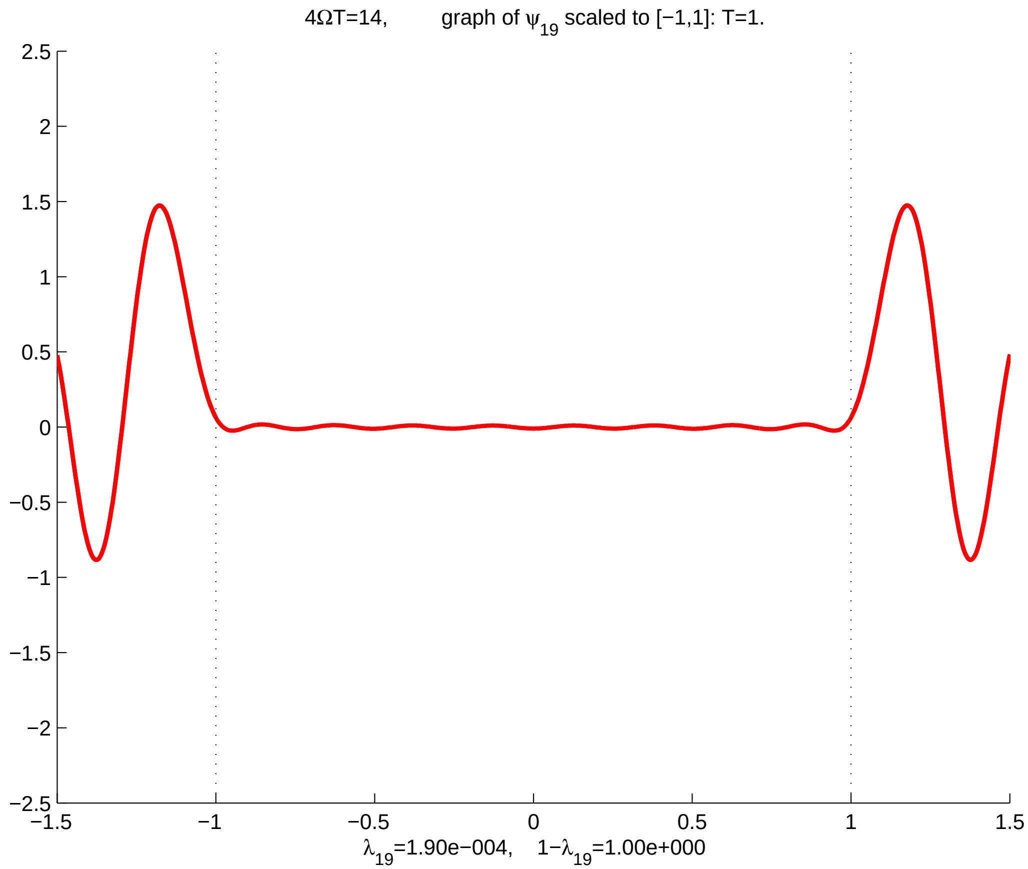
$4\Omega T = 14, \qquad \phi_{17}$

Eigenfunctions



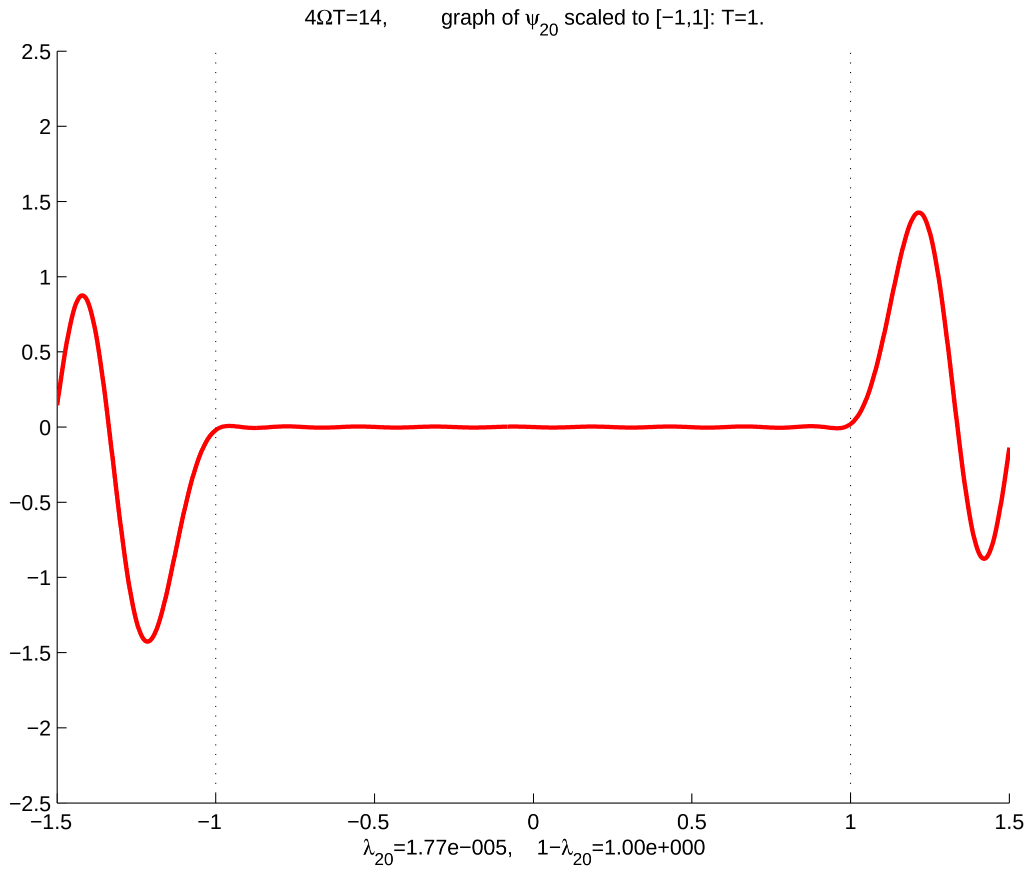
$4\Omega T = 14, \qquad \phi_{18}$

Eigenfunctions



$4\Omega T = 14, \quad \phi_{19}$

Eigenfunctions



$4\Omega T = 14, \quad \phi_{20}$

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$Df \equiv D_T(f) \equiv f\Pi_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$

$$BD\psi(t) = 2\Omega \int_{-T}^{T} \mathrm{sinc}(2\Omega(t-s))\,\psi(s)\,\mathrm{d}s = \lambda\psi(t)$$

Put $c \equiv 2\Omega T$ and $\phi(x) \equiv \psi(Tx)$. Then (with $s = Tx$)

$$c \int_{-1}^{1} \mathrm{sinc}(c(y-x))\,\phi(x)\,\mathrm{d}x = \lambda\phi(y)$$

Property. $\lambda_n \approx 1$ if $n < 4\Omega T - \ln(\Omega T)$,

$\lambda_n \approx 0$ if $n > 4\Omega T + \ln(\Omega T)$.

Discussion. *'$4\Omega T$ different signals from $\mathcal{B}$ can be packed on $[-T, T]$':* The dimension of the 'space' of signals in $\mathcal{B}$ that are concentrated in time in $[-T, +T]$ is $\approx 4\Omega T$.

Space as $\mathrm{span}\{\psi_k \mid 1 - \lambda_k < \varepsilon\}$

$$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\Pi_\Omega = \widehat{f}\}.$$

$$Df \equiv D_T(f) \equiv f\Pi_T \;,\; Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$$

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Property. $\lambda_n \approx 1$ if $n < 4\Omega T - \ln(\Omega T)$,

$\lambda_n \approx 0$ if $n > 4\Omega T + \ln(\Omega T)$.

Discussion. Results are mainly of theoretical interest. It is hard (unstable) to compute the ψ_k for large values of $4\Omega T$.

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\Pi_\Omega = \widehat{f}\}.$

$Df \equiv D_T(f) \equiv f\Pi_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$

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Put $c \equiv 2\Omega T$ and $\phi(x) \equiv \psi(Tx)$. Then (with $s = Tx$)

$$c \int_{-1}^{1} \mathrm{sinc}(c(y-x))\, \phi(x)\, \mathrm{d}x = \lambda\phi(y)$$

Let $BD\psi_k = \lambda_k\psi_k$, s.t. $\lambda_{i+1} < \lambda_i$ and $\|\psi_k\|_2 = 1$.

Theorem. • (ψ_k) forms an orthonormal basis of $\mathcal{B}$,

• $(\frac{1}{\sqrt{\lambda_k}}\psi\Pi_T)$ forms an orthonormal basis of $\{f\Pi_T \mid f \in \mathcal{B}\}$.

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\Pi_\Omega = \widehat{f}\}.$

$Df \equiv D_T(f) \equiv f\Pi_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$

$$BD\psi(t) = 2\Omega \int_{-T}^{T} \mathrm{sinc}(2\Omega(t-s))\,\psi(s)\,\mathrm{d}s = \lambda\psi(t)$$

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$$c \int_{-1}^{1} \mathrm{sinc}(c(y-x))\,\phi(x)\,\mathrm{d}x = \lambda\phi(y)$$

Let $BD\psi_k = \lambda_k\psi_k$, s.t. $\lambda_{i+1} < \lambda_i$ and $\|\psi_k\|_2 = 1$.

Theorem.

$$f = \sum_j \frac{\beta_j}{\lambda_j}\psi_k \quad \text{with} \quad \beta_j \equiv \int_{-T}^{T} f(t)\psi_k(t)\,\mathrm{d}t \qquad (f \in \mathcal{B})$$

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\Pi_\Omega = \widehat{f}\}.$

$Df \equiv D_T(f) \equiv f\Pi_T\ ,\ Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$

$$BD\psi(t) = 2\Omega \int_{-T}^{T} \mathrm{sinc}(2\Omega(t-s))\,\psi(s)\,\mathrm{d}s = \lambda\psi(t)$$

Put $c \equiv 2\Omega T$ and $\phi(x) \equiv \psi(Tx)$. Then (with $s = Tx$)

$$c\int_{-1}^{1} \mathrm{sinc}(c(y-x))\,\phi(x)\,\mathrm{d}x = \lambda\phi(y)$$

Let $BD\psi_k = \lambda_k\psi_k$, s.t. $\lambda_{i+1} < \lambda_i$ and $\|\psi_k\|_2 = 1$.

Theorem.

$$f = \sum_j \frac{\beta_j}{\lambda_j}\psi_k \quad \text{with} \quad \beta_j \equiv \int_{-T}^{T} f(t)\psi_k(t)\,\mathrm{d}t \qquad (f \in \mathcal{B})$$

Proof. Use Hilbert theory: BDB is a compact Hermitian operator on the Hilbert space $\mathcal{B}$ (close subspace $L^2(\mathbb{R})$).

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\,\Pi_\Omega = \widehat{f}\}.$

$Df \equiv D_T(f) \equiv f\,\Pi_T$, $Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$

$$BD\psi(t) = 2\Omega \int_{-T}^{T} \mathrm{sinc}(2\Omega(t-s))\,\psi(s)\,\mathrm{d}s = \lambda\psi(t)$$

Put $c \equiv 2\Omega T$ and $\phi(x) \equiv \psi(Tx)$. Then (with $s = Tx$)

$$c\int_{-1}^{1} \mathrm{sinc}(c(y-x))\,\phi(x)\,\mathrm{d}x = \lambda\phi(y)$$

Let $BD\psi_k = \lambda_k\psi_k$, s.t. $\lambda_{i+1} < \lambda_i$ and $\|\psi_k\|_2 = 1$.

Theorem.

$$f = \sum_j \frac{\beta_j}{\lambda_j}\psi_k \quad \text{with} \quad \beta_j \equiv \int_{-T}^{T} f(t)\psi_k(t)\,\mathrm{d}t \qquad (f \in \mathcal{B})$$

Discussion. f can be reconstructed from $f\Pi_T$ if $f \in \mathcal{B}$.

Ill conditioned (for $\lambda_k \approx 0$).

Remedy. Restrict to $\lambda_k \approx 1$.

$\mathcal{B} \equiv \mathcal{B}_\Omega \equiv \{f \in L^2(\mathbb{R}) \mid \widehat{f}\Pi_\Omega = \widehat{f}\}.$

$Df \equiv D_T(f) \equiv f\Pi_T \; , \; Bf \equiv B_\Omega(f) \equiv f * \widehat{\Pi}_\Omega \quad (f \in L^2(\mathbb{R}))$

$$BD\psi(t) = 2\Omega \int_{-T}^{T} \mathrm{sinc}(2\Omega(t-s))\, \psi(s)\, \mathrm{d}s = \lambda\psi(t)$$

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Let $BD\psi_k = \lambda_k\psi_k$, s.t. $\lambda_{i+1} < \lambda_i$ and $\|\psi_k\|_2 = 1$.

Theorem.

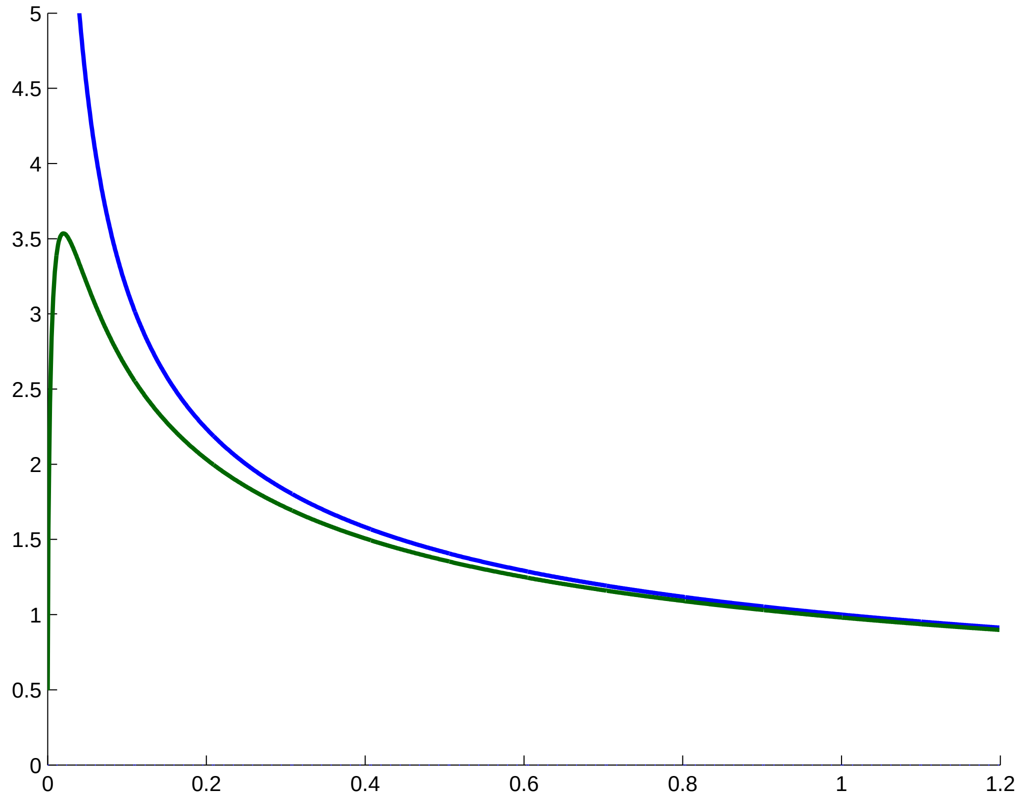
$$f = \sum_j \frac{\beta_j}{\lambda_j}\psi_k \quad \text{with} \quad \beta_j \equiv \int_{-T}^{T} f(t)\psi_k(t)\, \mathrm{d}t \qquad (f \in \mathcal{B})$$

Discussion. f can be reconstructed from $f\Pi_T$ if $f \in \mathcal{B}$.

Ill conditioned (for $\lambda_k \approx 0$).

Remedy. Solve $f^{\mathrm{r}} = \mathrm{argmin}_{g\in\mathcal{B}}(\|g - f\Pi_T\|_2^2 + \tau\|g\|_2^2)$

Eigenvalues



Program

- Filters
- Finite Impulse Response Filters
- Windows
- Signals of finite duration and bounded bandwidth?
- Infinite Impulse Response Filters
- Analog filters (hardware)
- Digital filters (software)

Infinite Impulse Response filters?

Part of the problems with the **FIR** filters come from the fact that the filters have a bounded (finite) time domain.

The technique of windowing in time domain is still usefull **if** (long) delays are allowed.
For instance, in Imaging (where, in the above discussion we should read 'space' for 'time'), where we have the complete (blurred, noicy) image (signal f) available. The technique might not be useful in case the signal that has to be processed 'comes in' in time: then the signal is only partially available or we have to 'wait' too long.

Infinite Impulse Response filters?

Part of the problems with the **FIR** filters come from the fact that the filters have a bounded (finite) time domain.

Can we create filters with unbounded domain (**IIR**) that nevertheless forms the output from 'local' information?

Note that this may not be impossible since a signal of bounded bandwidth is completely determined by its values at any (non empty) time interval.

This suggests to exploit the smoothness of the input signal (of bounded bandwidth).

Program

- Filters
- Finite Impulse Response Filters
- Windows
- Signals of finite duration and bounded bandwidth?
- Infinite Impulse Response Filters
- Analog filters (hardware)
- Digital filters (software)

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \quad (\star)$$

The higher order derivatives in g represent **feedback** to the system. They give infinite impulse response.

Systems of this form can be realised in electronic circuits. Coupled second order differential equations can be formed into higher dimensional coupled first order systems. Also, by elimination, coupled second order differential equations can formed into one dimensional higher order systems.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \quad (\star)$$

FT of $(\star)$ leadsto

$$p(2\pi i\omega)\,\widehat{g}(\omega) = q(2\pi i\omega)\,\widehat{f}(\omega),$$

where, for $\zeta \in \mathbb{C}$,

$$p(\zeta) \equiv a_0 + a_1\zeta + \ldots + a_k\zeta^k \quad \text{and} \quad q(\zeta) \equiv b_0 + b_1\zeta + \ldots + b_m\zeta^m$$

Let a_j be such that $p(\zeta) \neq 0$ for all $\zeta \in \{2\pi i\omega \mid \omega \in \mathbb{R}\}$.

Then, $H(\omega) \equiv \dfrac{q(2\pi i\omega)}{p(2\pi i\omega)} \in C^\infty(\mathbb{R})$ and bounded if $m \leq k$.

$H \in L^2(\mathbb{R})$ if $m < k$. Then $H = \widehat{h}$ for some $h \in L^2(\mathbb{R})$.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

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Let a_j be such that $p(\zeta) \neq 0$ for all $\zeta \in \{2\pi i\omega \mid \omega \in \mathbb{R}\}$.

Then, $H(\omega) \equiv \dfrac{q(2\pi i\omega)}{p(2\pi i\omega)} \in C^\infty(\mathbb{R})$ and bounded if $m \leq k$.

$H \in L^2(\mathbb{R})$ if $m < k$. Then $H = \widehat{h}$ for some $h \in L^2(\mathbb{R})$.

Does h belong to $L^1(\mathbb{R})$ (to guarantee that g is L^2 if f is)?

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

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Let a_j be such that $p(\zeta) \neq 0$ for all $\zeta \in \{2\pi i\omega \mid \omega \in \mathbb{R}\}$.

Then, $H(\omega) \equiv \dfrac{q(2\pi i\omega)}{p(2\pi i\omega)} \in C^\infty(\mathbb{R})$ and bounded if $m \leq k$.

$H \in L^2(\mathbb{R})$ if $m < k$. Then $H = \widehat{h}$ for some $h \in L^2(\mathbb{R})$.

Note. $H(\omega) = \mathbf{c}^*(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b}$ is of the above form. [Ex.3.11]

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Theorem. Let $m < k$. Then $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$:

$$f \in L^2(\mathbb{R}) \quad \Rightarrow \quad \widehat{g} = \widehat{f} H \quad \& \quad g = f * h \in L^2(\mathbb{R}).$$

Proof. Factorise p to see that for some $\gamma_1, \ldots, \gamma_k \in \mathbb{C}$

$$\frac{q(\zeta)}{p(\zeta)} = \sum_{j=1}^{k} \frac{\gamma_j}{(\zeta - \lambda_j)^{\mu(j)}}.$$

Here, $\lambda_1, \ldots, \lambda_k$ are the zeros of p counted according to multiplicity, $\quad \mu(j) \equiv \#\{i \mid i \leq j, \lambda_i = \lambda_j\}$.

The zeros of p are the **poles** of the filter,

the zeros of q are the **zeros** of the filter.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

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$$f \in L^2(\mathbb{R}) \qquad \Rightarrow \qquad \widehat{g} = \widehat{f} H \qquad \& \qquad g = f * h \in L^2(\mathbb{R}).$$

Proof. It suffices to show that, for $j \in \mathbb{N}$, the function

$$H(\omega) \equiv \frac{1}{(2\pi i \omega - \lambda)^j} \qquad (\omega \in \mathbb{R})$$

is the **FT** of an h in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ if $\lambda \in \mathbb{C}$, $\lambda \notin i\mathbb{R}$.

Clearly, $H \in L^2(\mathbb{R})$. Hence, $h \in L^2(\mathbb{R})$.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

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Clearly, $H \in L^2(\mathbb{R})$. Hence, $h \in L^2(\mathbb{R})$.

If $\mathrm{Re}(\lambda) < 0$, then h is a scalar multiple of [Ex.3.3]

$$\begin{cases} t^{j-1} e^{\lambda t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \quad (h \text{ is causal!}) \end{cases}$$

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Theorem. Let $m < k$. Then $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$:

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Proof. It suffices to show that, for $j \in \mathbb{N}$, the function

$$H(\omega) \equiv \frac{1}{(2\pi i\omega - \lambda)^j} \qquad (\omega \in \mathbb{R})$$

is the **FT** of an h in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ if $\lambda \in \mathbb{C}$, $\lambda \notin i\mathbb{R}$.

Clearly, $H \in L^2(\mathbb{R})$. Hence, $h \in L^2(\mathbb{R})$.

If $\mathrm{Re}(\lambda) > 0$, then h is a scalar multiple of [Ex.3.3]

$$\begin{cases} t^{j-1}\, e^{\lambda t} & \text{for } t \le 0 \\ 0 & \text{for } t > 0 \end{cases}$$

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

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Theorem. Let $m < k$. Then $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$:

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is the **FT** of an h in $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ if $\lambda \in \mathbb{C}$, $\lambda \notin i\mathbb{R}$.

Clearly, $H \in L^2(\mathbb{R})$. Hence, $h \in L^2(\mathbb{R})$.

In all cases $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. $\square$ [Ex.3.3]

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Theorem. Let $m < k$. Then $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$:

$$f \in L^2(\mathbb{R}) \quad \Rightarrow \quad \widehat{g} = \widehat{f} H \quad \& \quad g = f * h \in L^2(\mathbb{R}).$$

Theorem. Let $m < k$.

The filter is causal $\Leftrightarrow$ the poles are in $\mathbb{C}^-$.

Poles are the zeros of p. $\mathbb{C}^- \equiv \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$ is the left half of the complex plane.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

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Proof. See the proof of the preceding theorem. $\square$

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

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The filter needs a start.

Suppose $f(t) = 0$ for all $t < 0$. Then

$$g(0) = g'(0) = \ldots = g^{(k-1)} = 0$$

seems a reasonable choice.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Theorem. Let $m < k$. Then $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$:

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The filter needs a start.

Suppose $f(t) = 0$ for all $t < 0$. Then

$$g(0) = g'(0) = \ldots = g^{(k-1)} = 0$$

holds for $g = f * h$ $\Leftrightarrow$ the filter is causal.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Theorem. Let $m < k$. Then $h \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$:

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Theorem. Let $m < k$.

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Poles are the zeros of p. $\mathbb{C}^- \equiv \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$ is the left half of the complex plane.

Property. h is real if the coefficients a_j and b_j are real.

Proof. $f \approx$ real pulse $\Rightarrow g$ real $\Rightarrow h \approx g$ real.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

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Poles are the zeros of p. $\mathbb{C}^- \equiv \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$ is the left half of the complex plane.

Property. h is real if the coefficients a_j and b_j are real.

h real and causal $\Rightarrow H$ is even $(H(-\omega) = \overline{H(\omega)})$, not-real

time/group delays are an issue!

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Example 1. $g + \frac{1}{2\pi\Omega} g' = f$.

Then, $\quad p(\zeta) = 1 + \frac{1}{2\pi\Omega}\zeta, \quad q(\zeta) = 1, \quad H(\omega) = \frac{1}{1+i\frac{\omega}{\Omega}}$

with gain $\quad |H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^2}}$

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Example 3. $g + (\frac{1}{2\pi\Omega})^k g^{(k)} = f$.

Then, $\quad p(\zeta) = 1 + (\frac{1}{2\pi\Omega}\zeta)^k, \quad q(\zeta) = 1, \quad H(\omega) = \frac{1}{1+(i\frac{\omega}{\Omega})^k}$

with gain $\quad |H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^{2k}}}$.

Note that for large(r) k:

if $|\omega| < \Omega$, then $|\frac{\omega}{\Omega}|^{2k} \approx 0$ and $|H(\omega)| \approx 1$

if $|\omega| > \Omega$, then $|\frac{\Omega}{\omega}|^{2k} \approx 0$ and $|H(\omega)| \approx 0$

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

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with gain $\quad |H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^2}}$

Example 2. $g - \frac{1}{2\pi\Omega} g' = f$.

Then, $\quad p(\zeta) = 1 - \frac{1}{2\pi\Omega}\zeta, \quad q(\zeta) = 1, \quad H(\omega) = \frac{1}{1-i\frac{\omega}{\Omega}}$

with gain $\quad |H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^2}}$

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

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Examples. (1) $g + \frac{1}{2\pi\Omega} g' = f$. (2) $g - \frac{1}{2\pi\Omega} g' = f$.

Same gain.

Pole (1) in $\mathbb{C}^-$, pole (2) in $\mathbb{C}^+$: (1) causal, (2) not causal.

Note. All filters are essentially of the above form: see the proof of the "L^1-theorem".

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Examples. (1) $g + \frac{1}{2\pi\Omega} g' = f$. (2) $g - \frac{1}{2\pi\Omega} g' = f$.

Let g be the $L^2(\mathbb{R})$ solution.
Suppose g is perturbed at time t_0, that is,

- $\widetilde{g}$ satisfies the ODE,
- $\widetilde{g}(t) = g(t)$ for $t < t_0$,
- $\widetilde{g}(t_0) = g(t_0) + \varepsilon$.

Here we assumed that we obtained the output $g(t)$ at time t by solving the ODE (following the increasing time t): this was the purpose of this type of filters.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

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- $\widetilde{g}(t) = g(t)$ for $t < t_0$,
- $\widetilde{g}(t_0) = g(t_0) + \varepsilon$.

Then $(\widetilde{g} - g)(t) = \varepsilon\, e^{\lambda_1 (t - t_0)}$ for $t \geq t_0$.
Here λ_1 is the zero of p.

(1) $\Rightarrow \lambda_1 = -\frac{1}{2\pi\Omega} < 0$ and $|(\widetilde{g} - g)(t)| \to 0$ for $t \to \infty$.

(2) $\Rightarrow \lambda_1 = +\frac{1}{2\pi\Omega} < 0$ and $|(\widetilde{g} - g)(t)| \to \infty$ for $t \to \infty$.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Conclusion. Let $m < k$.
The filter is **stable** (perturbations do not have a lasting effect) if and only if the poles are in $\mathbb{C}^-$.

To avoid discussions on what effects are acceptable (how long, how large?), a formal definition of stability is introduced.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \quad (\star)$$

Definition. Let $m < k$.
The filter is **stable** if and only if all poles are in $\mathbb{C}^-$
(that is, $\lambda \in \mathbb{C}$ & $\quad p(\lambda) = 0 \quad \Rightarrow \quad \text{Re}(\lambda) < 0$.)

Theorem. Let $m < k$.
The filter is stable $\Leftrightarrow$ the filter is causal.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

With $p(\zeta) \equiv a_0 + \ldots + a_k \zeta^k$ and $q(\zeta) \equiv b_0 + \ldots + b_m \zeta^m$,

put $\quad H(\omega) \equiv |H(\omega)|\, e^{-i\phi(\omega)} \equiv \frac{q(2\pi i\omega)}{p(2\pi i\omega)}$.

Summary. Polynomials p and q should be such that

1) For technical realisation: p and q are real (real coeff.)
2) degr(p) $>$ degr(q)
3) For caus. and stab.: $\lambda \in \mathbb{C}$ & $p(\lambda) = 0 \;\Rightarrow$ Re$(\lambda) < 0$
4) For requested filtering: $\quad |H| \approx \Pi_\Omega$
5) For acceptable group/time delay; $\phi(\omega) \approx \ldots$

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

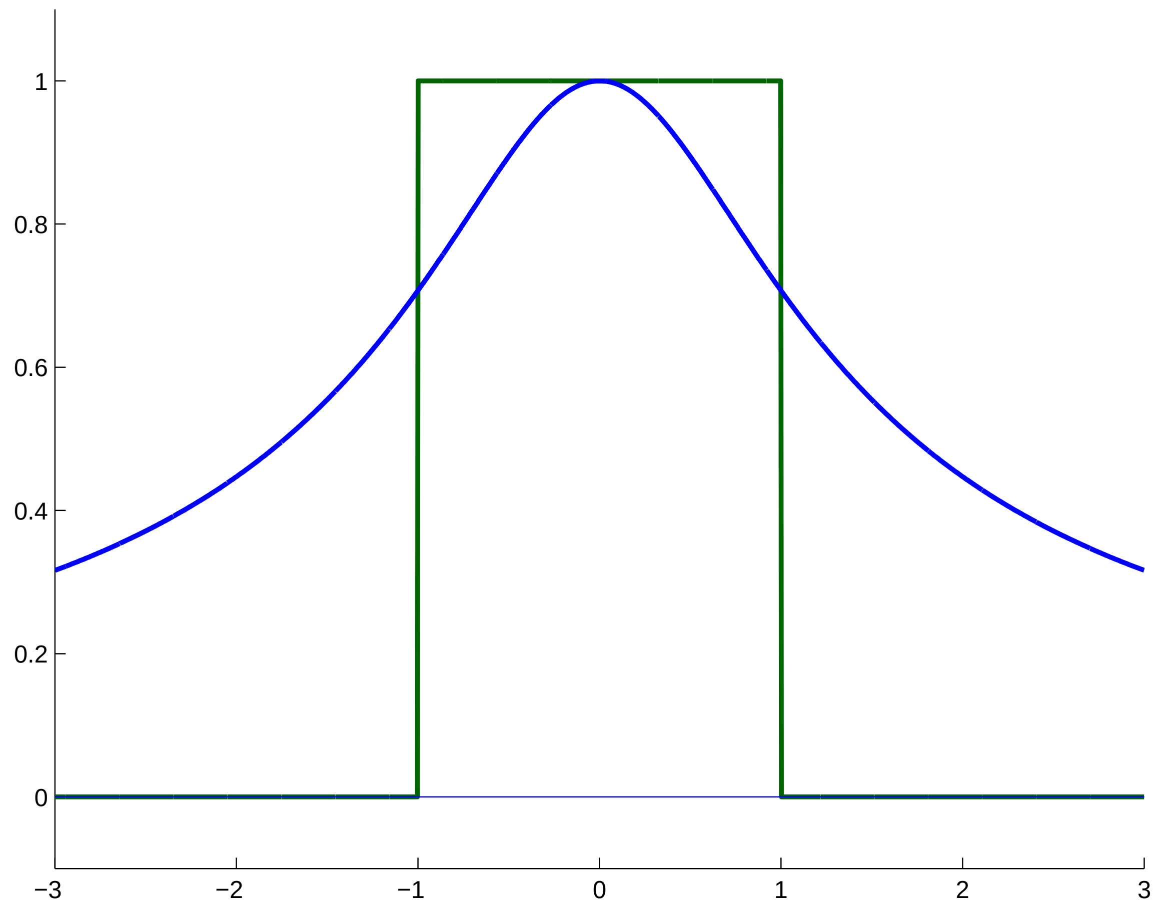
For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \qquad (\star)$$

Example. $g + (\frac{1}{2\pi\Omega})^k g^{(k)} = f$.

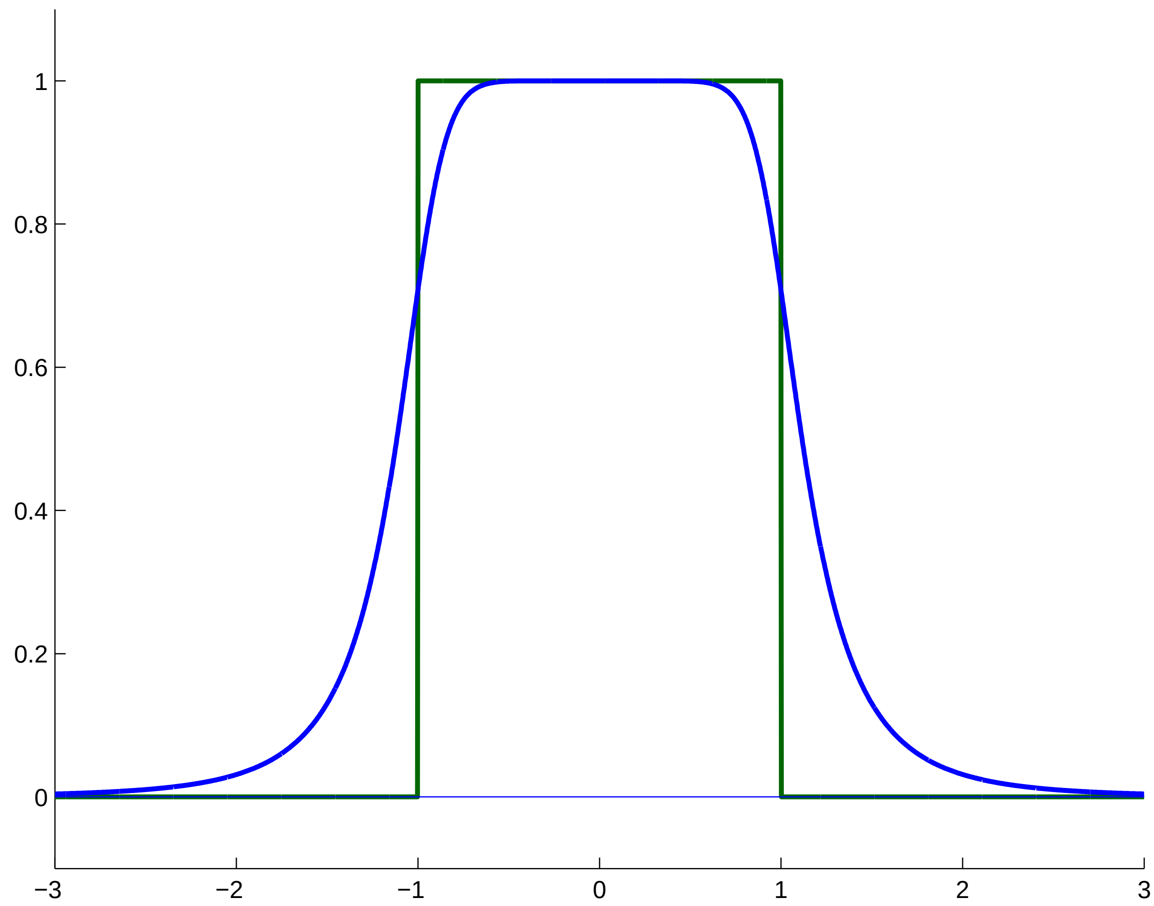
Then, $p(\zeta) = 1 + (\frac{1}{2\pi\Omega}\zeta)^k$ with gain $|H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^{2k}}}$.

Butterworth filter



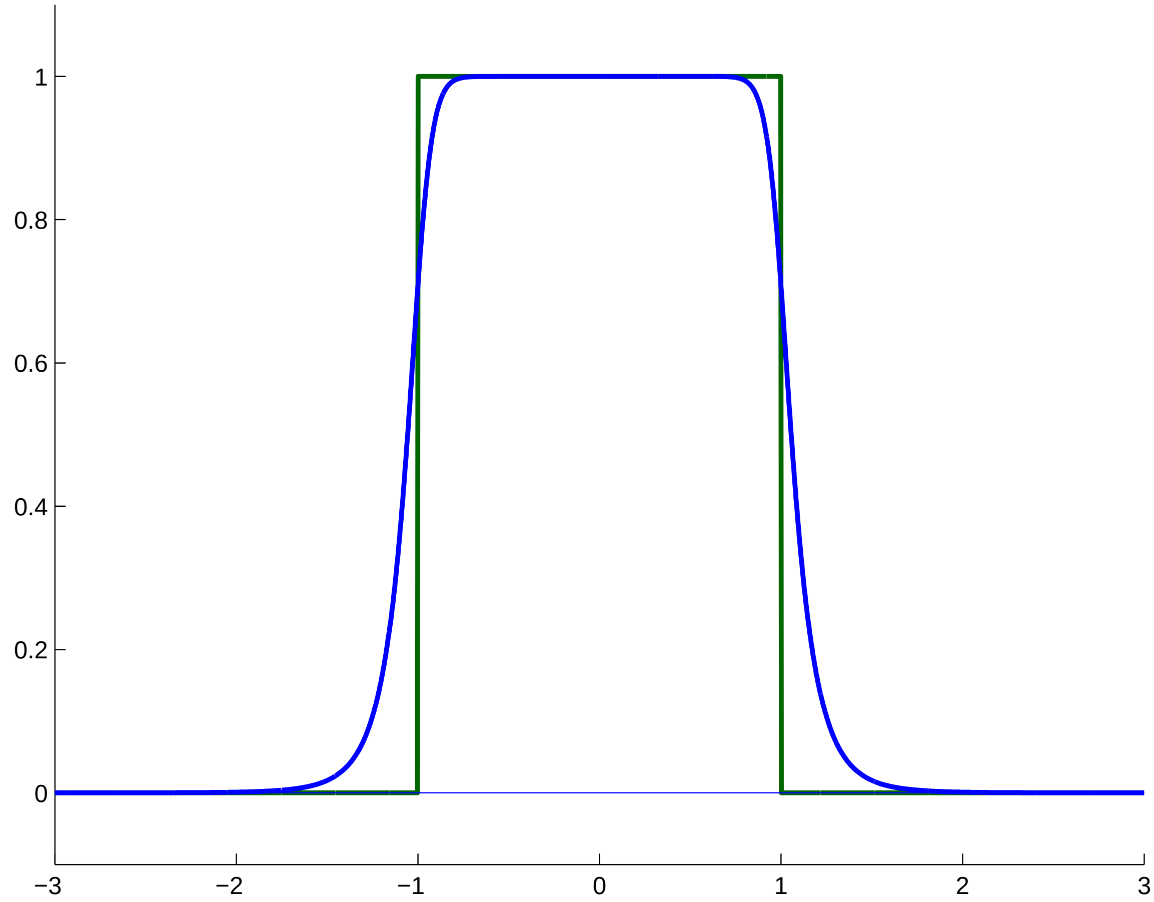
$|H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^{2k}}}$ (blue), $k = 1$. Here we took $\Omega = 1$.

Butterworth filter



$|H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^{2k}}}$ (blue), $k = 5$. Here we took $\Omega = 1$.

Butterworth filter



$|H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^{2k}}}$ (blue), $k = 10$. Here we took $\Omega = 1$.

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

For a given input signal f, the output g is such that

$$a_0 g + a_1 g' + \ldots + a_k g^{(k)} = b_0 f + b_1 f' + \ldots + b_m f^{(m)} \quad (\star)$$

Example. $g + (\frac{1}{2\pi\Omega})^k g^{(k)} = f$.

Then, $p(\zeta) = 1 + (\frac{1}{2\pi\Omega}\zeta)^k$ with gain $|H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^{2k}}}$.

The gain is fine, but

the filter *is unstable and not causal for* $k > 2$

(if $k = 2$ then p has even zeros on $i\,\mathbb{R}$).

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

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Butterworth filters:

- stable, real coefficients
- $q = 1$, $\qquad \dfrac{1}{|p(2\pi i\omega)|} = \dfrac{1}{\sqrt{1 + \left|\frac{\omega}{\Omega}\right|^{2k}}}$

Given $a_0, a_1, \ldots, a_k$ and $b_0, \ldots, b_m$ in $\mathbb{R}$.

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Butterworth filters:

- stable, real coefficients
- $q = 1, \qquad \dfrac{1}{|p(2\pi i\omega)|} = \dfrac{1}{\sqrt{1 + \left|\frac{\omega}{\Omega}\right|^{2k}}}$

Example. $g + \frac{\sqrt{2}}{2\pi\Omega} g' + (\frac{1}{2\pi\Omega})^2 g^{(2)} = f.$

$\Rightarrow \quad p(2\pi\Omega\zeta) = 1 + \sqrt{2}\,\zeta + \zeta^2,$ gain $|H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^4}}.$

Given $a_0, a_1, \dots, a_k$ and $b_0, \dots, b_m$ in $\mathbb{R}$.

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Example. $g + \frac{\sqrt{2}}{2\pi\Omega} g' + (\frac{1}{2\pi\Omega})^2 g^{(2)} = f$.

$\Rightarrow \quad p(2\pi\Omega\zeta) = 1 + \sqrt{2}\,\zeta + \zeta^2$, gain $|H(\omega)| = \frac{1}{\sqrt{1+|\frac{\omega}{\Omega}|^4}}$.

$p(2\pi\Omega\lambda_i) = 0 \;\Rightarrow \lambda_2 = \overline{\lambda}_1$ & $2\mathrm{Re}(\lambda_i) = \lambda_1 + \lambda_2 = -\sqrt{2}$.

Chebyshev filters.

- stable, real coefficients
- $q = 1$, $$\frac{1}{|p(2\pi i\omega)|} = \frac{1}{\sqrt{1 + \varepsilon^2 T_k^2\left(\frac{\omega}{\Omega}\right)}}$$

Here, T_k is the kth degree **Chebyshev polynomial**. [Ex.2.8]

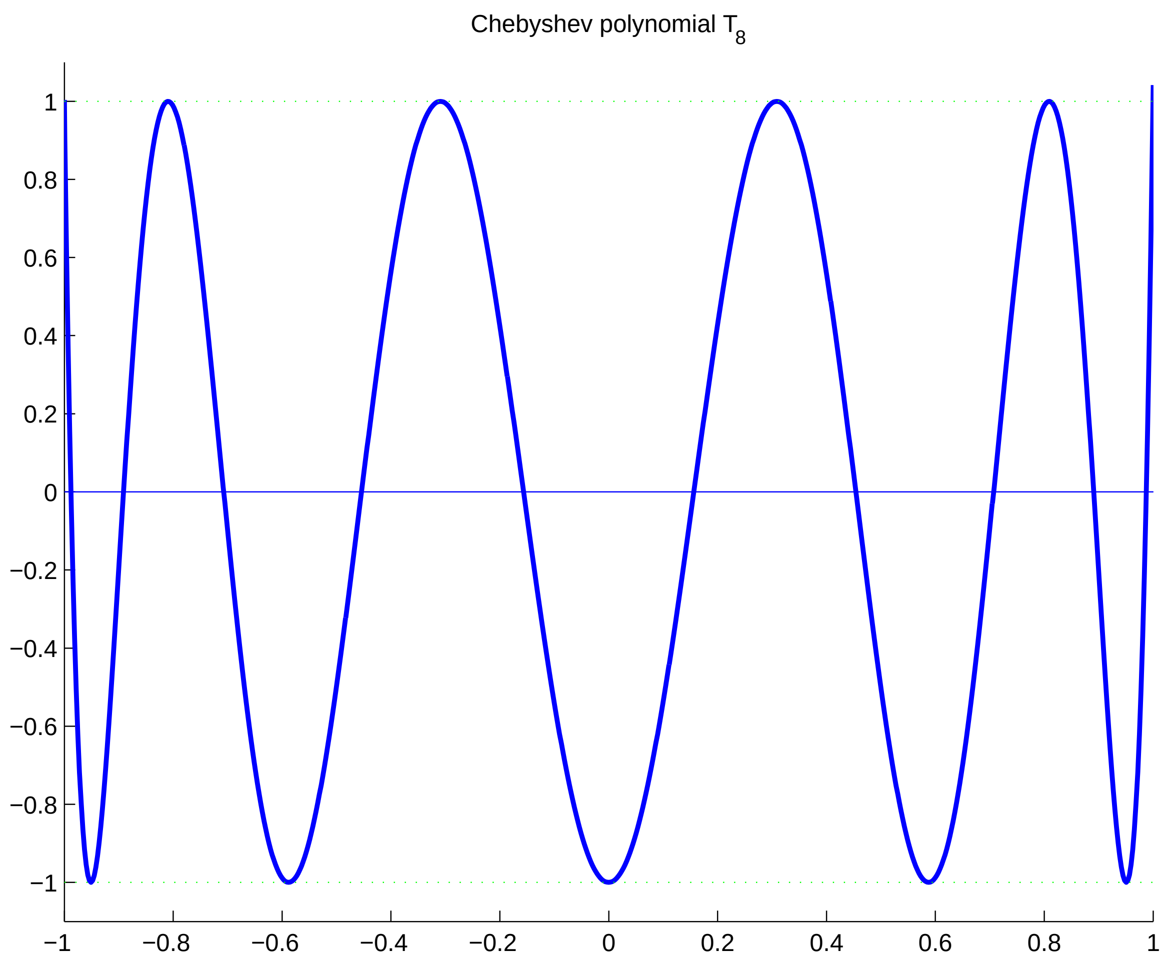
Property.

- T_k is a real polynomial of degree k
- $|T_k(x)| \leq 1$ for all $x \in [-1, +1]$,
- $|T_k(x)| \geq |P(x)|$ for all $x, |x| > 1$ and

all polynomials P of degree $\leq k$

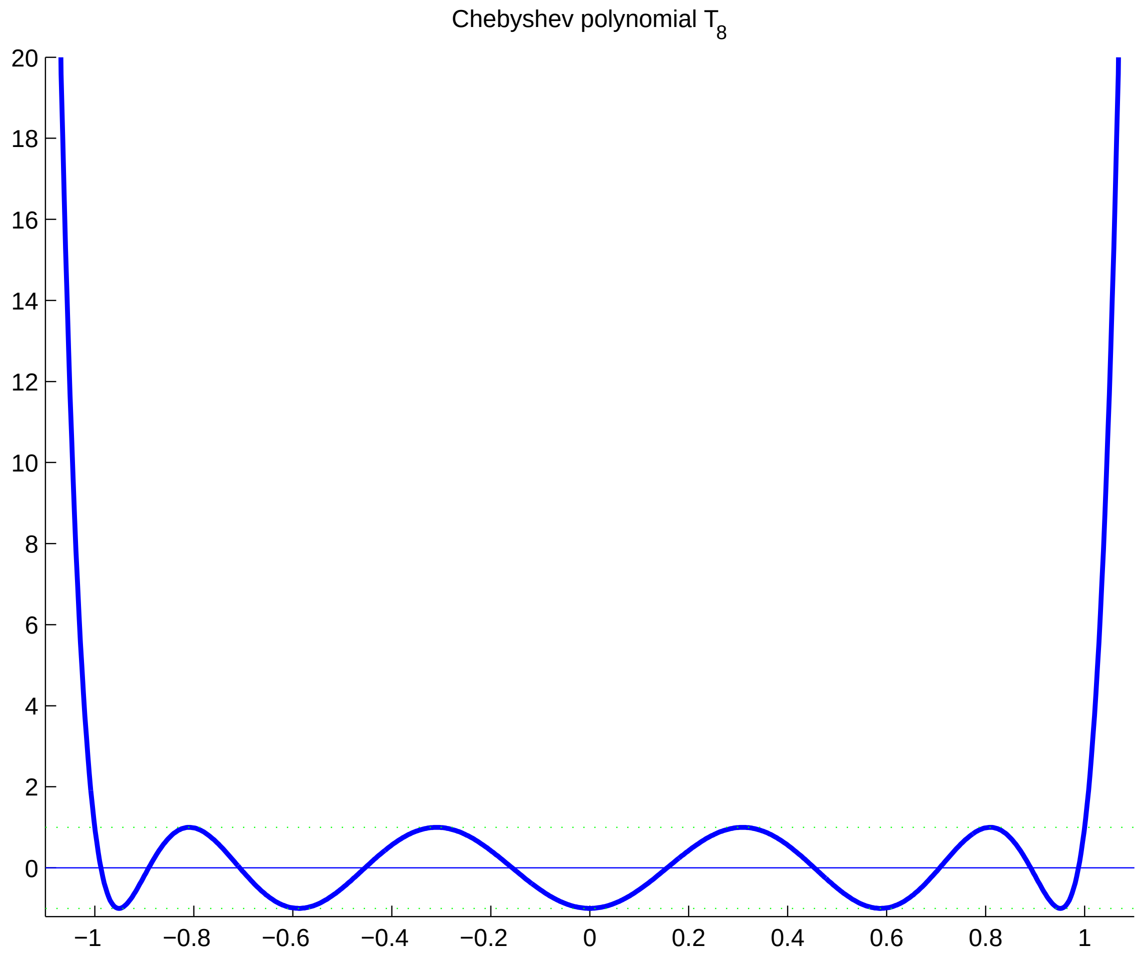
for which $\sup\{|P(x)| \mid x \in [-1, 1]\} \leq 1$

Chebyshev filters



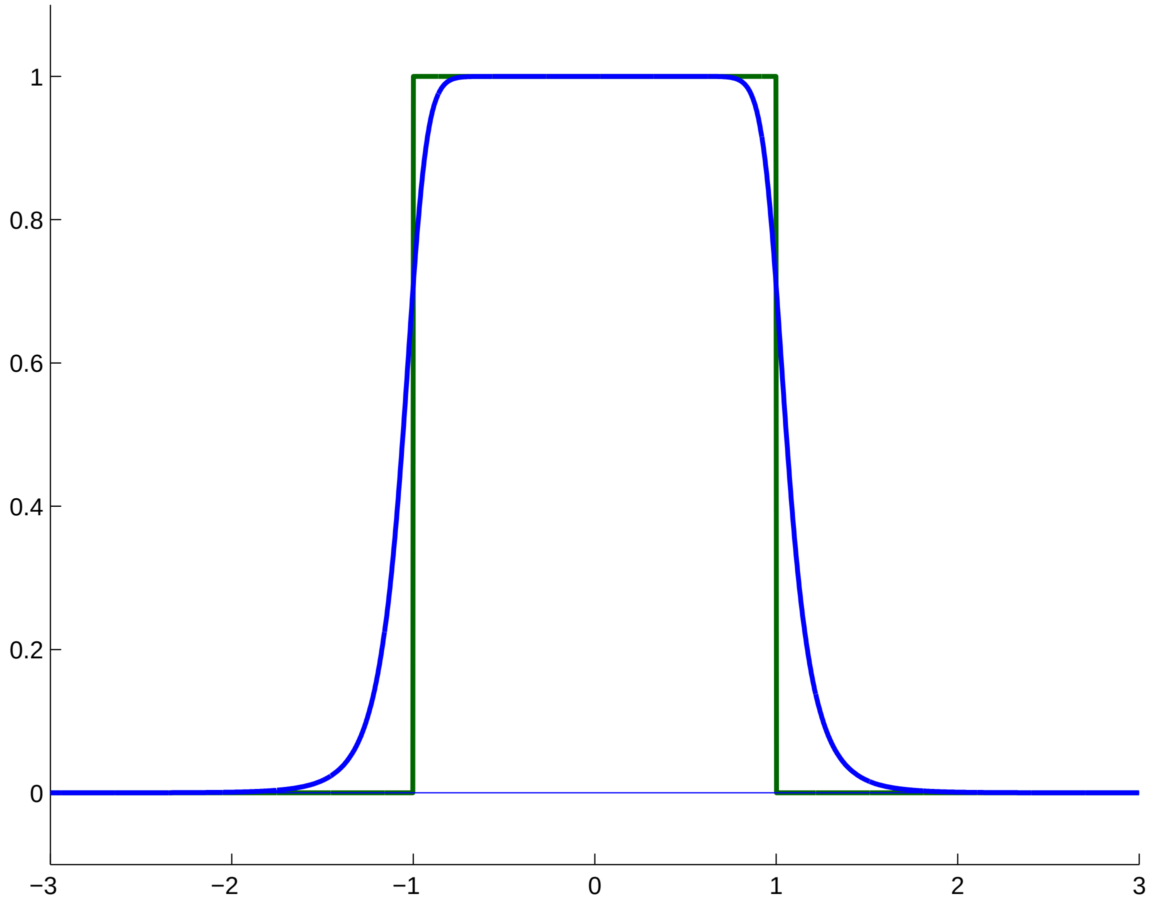
Cheb. pol. T_k of degree $k = 8$

Chebyshev filters



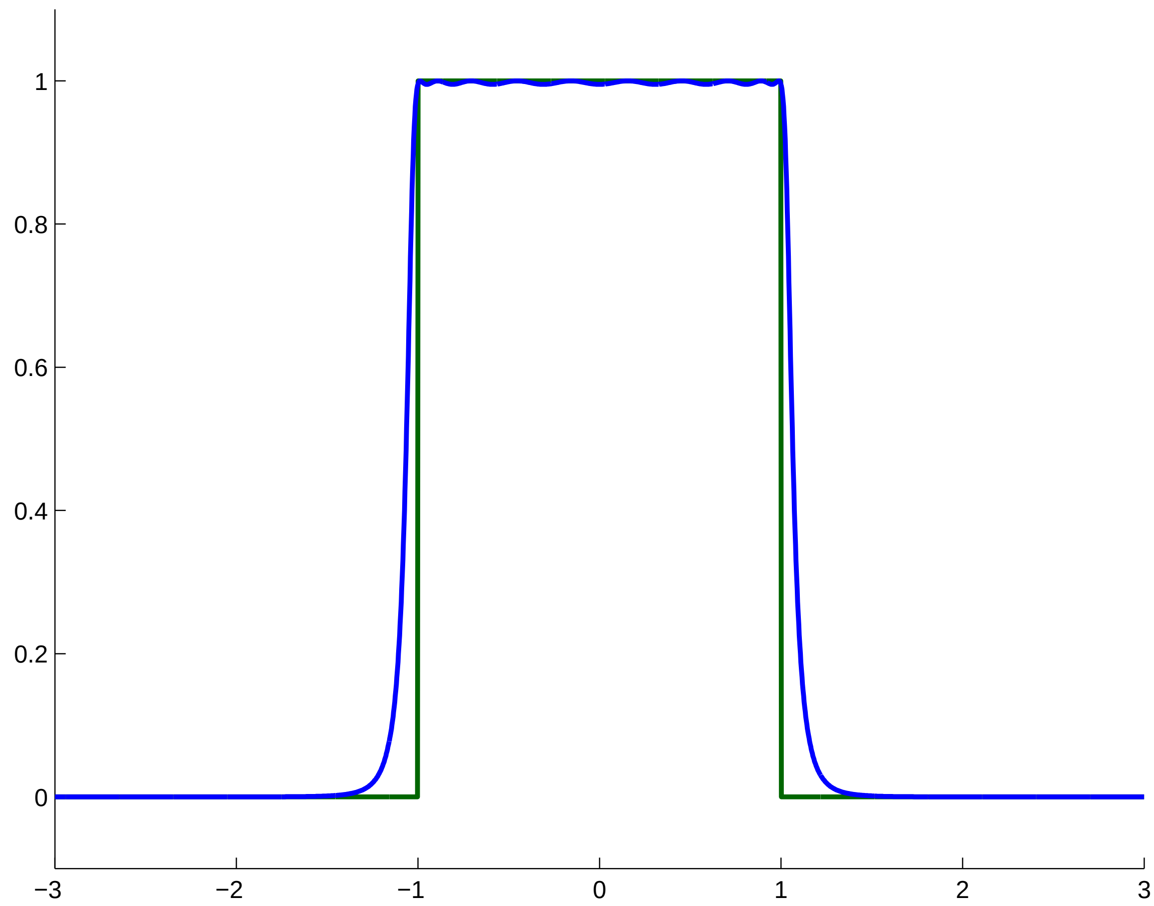
Cheb. pol. T_k of degree $k = 8$

Butterworth filter



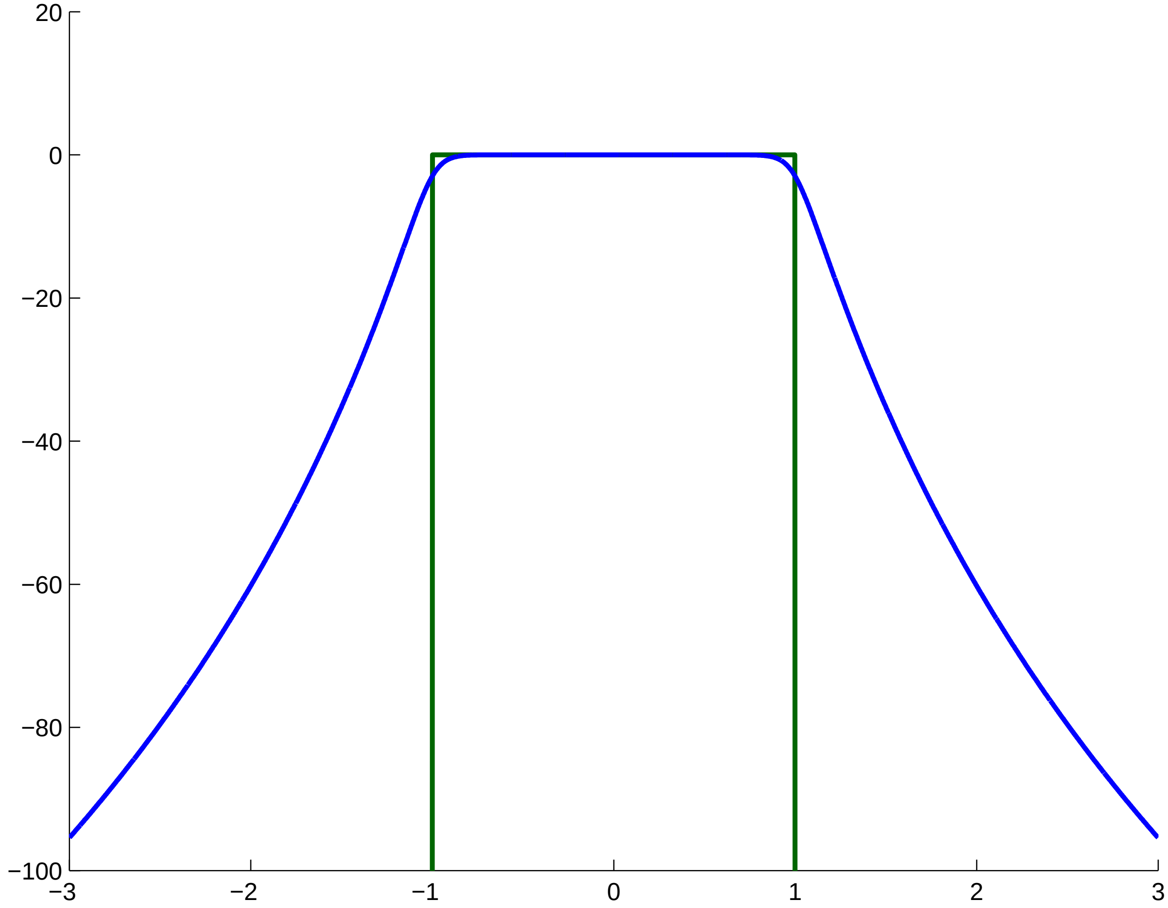
Gain Butterworth filter of degree $k = 10$

Chebyshev filters



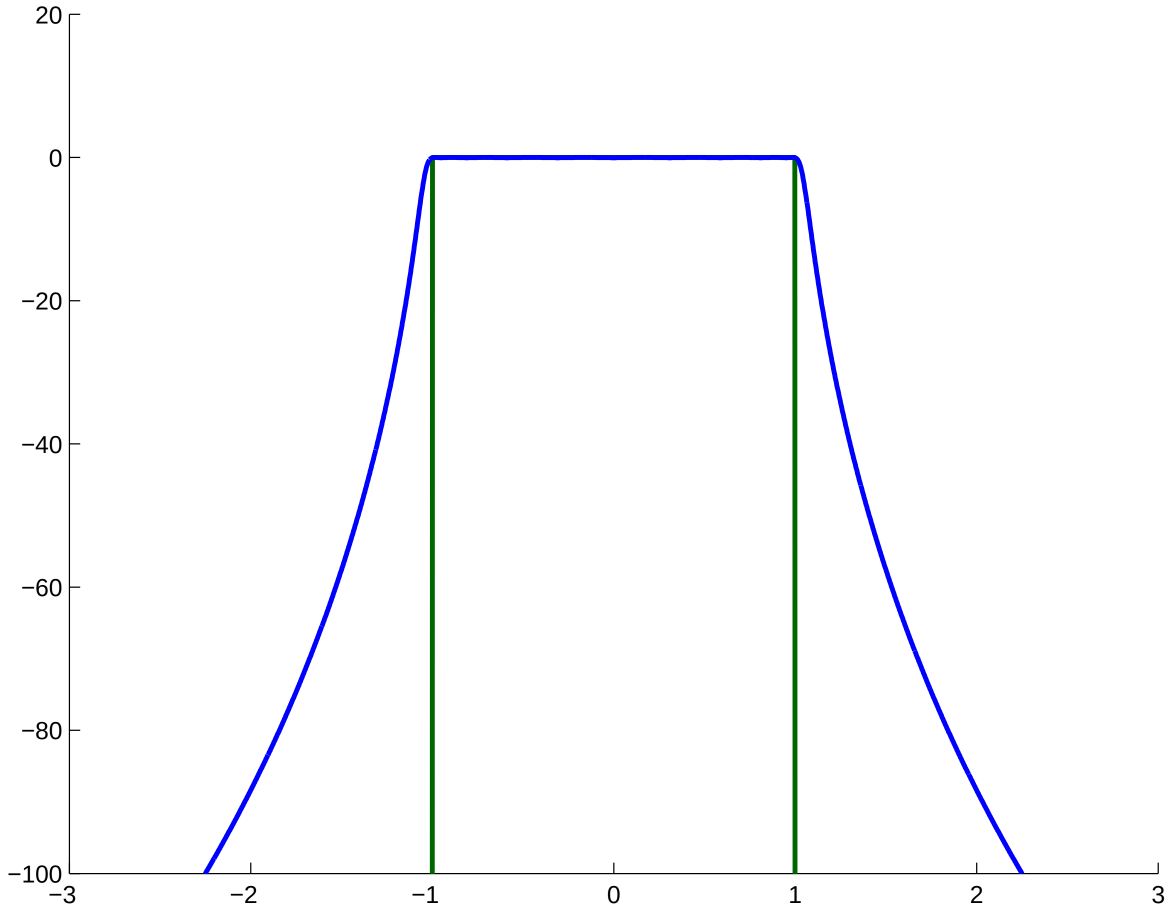
Gain Chebyshev filter of degree $k = 8$

Butterworth filter



Gain Butterworth filter of degree $k = 10$, dB scale

Chebyshev filters



Gain Chebyshev filter of degree $k = 8$, dB scale

Windowing versus analogue filtering

Windowing.

+ Real frequency response function

− Long filter (long impulse response function) (requiring information from "past" as well as "future")

Analogue filtering.

+ "Short" filters

+ No "future" information needed

− Non-real frequency response function

− Stability issues

Program

- Filters
- Finite Impulse Response Filters
- Windows
- Signals of finite duration and bounded bandwidth?
- Infinite Impulse Response Filters
- Analog filters (hardware)
- Digital filters (software)

We consider discrete signals $\boldsymbol{f} \equiv (\ldots, f_0, f_1, f_2, \ldots) \in \ell^2(\mathbb{Z})$

The values f_k can be obtained by sampling a function F on $\mathbb{R}$:

$f_n \equiv F(n\Delta t)$ with sampling frequency $\frac{1}{\Delta t} = 2\Omega$
where Ω the bandwidth of the signal F.

We consider discrete signals $\boldsymbol{f} \equiv (\ldots, f_0, f_1, f_2, \ldots) \in \ell^2(\mathbb{Z})$

*We wish to construct Infinite Impulse Response (**IIR**) filters that rely on "local information".*

We consider discrete signals $\boldsymbol{f} \equiv (\ldots, f_0, f_1, f_2, \ldots) \in \ell^2(\mathbb{Z})$

Recall

$$\widehat{f}(\omega) \equiv \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \omega n} \qquad \Leftrightarrow \qquad f_n = \int_0^1 \widehat{\boldsymbol{f}}(\omega) \, e^{2\pi i \omega n} \, \mathrm{d}\omega$$

If $\boldsymbol{f}$ is from sampling F, then the formula

$$f_n = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \widehat{\boldsymbol{f}}(\omega) \, e^{2\pi i \omega n \Delta t} \, d\omega$$

is more appropriate.

To simplify notation, we (took and) take $\Omega = \frac{1}{2}$.
This corresponds to scaling of
the ω-axis by 2Ω and the t-axis by $\Delta t = \frac{1}{2\Omega}$:

$$G(t) = F(t\Delta t) \qquad \Leftrightarrow \qquad \widehat{G}(\omega) = \widehat{\boldsymbol{f}}(2\Omega\omega).$$

G has bandwidth $\frac{1}{2}$, G is to be sampled at $t = n$.

We consider discrete signals $\boldsymbol{f} \equiv (\ldots, f_0, f_1, f_2, \ldots) \in \ell^2(\mathbb{Z})$

Recall

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Given $\alpha_0, \ldots, \alpha_k$ and $\beta_0, \ldots, \beta_m$ in $\mathbb{R}$, $\alpha_0 \neq 0$
the output g satisfies

$$\alpha_0 g_n = (\beta_0 f_n + \ldots + \beta_m f_{n-m}) - (\alpha_1 g_{n-1} + \ldots + \alpha_k g_{n-k})$$

We consider discrete signals $\boldsymbol{f} \equiv (\ldots, f_0, f_1, f_2, \ldots) \in \ell^2(\mathbb{Z})$

Recall

$$\widehat{f}(\omega) \equiv \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \omega n} \qquad \Leftrightarrow \qquad f_n = \int_0^1 \widehat{\boldsymbol{f}}(\omega)\, e^{2\pi i \omega n} \, \mathrm{d}\omega$$

With $\boldsymbol{\alpha} \equiv (\alpha_0, \ldots, \alpha_k)$, $\boldsymbol{\beta} \equiv (\beta_0, \ldots, \beta_m)$, $\alpha_0 \neq 0$,

the output g satisfies $\qquad \boldsymbol{\alpha} * \boldsymbol{g} = \boldsymbol{\beta} * \boldsymbol{f} \qquad (\star)$

The digital filter has

m **feed-forward stages** and
k **feed-backward stages**.
k is the **order** of the filter.

If $k = 0$ then the filter is Finite Impulse Response (**FIR**).

We consider discrete signals $\boldsymbol{f} \equiv (\ldots, f_0, f_1, f_2, \ldots) \in \ell^2(\mathbb{Z})$

Recall

$$\widehat{f}(\omega) \equiv \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i \omega n} \qquad \Leftrightarrow \qquad f_n = \int_0^1 \widehat{\boldsymbol{f}}(\omega) \, e^{2\pi i \omega n} \, \mathrm{d}\omega$$

With $\boldsymbol{\alpha} \equiv (\alpha_0, \ldots, \alpha_k)$, $\boldsymbol{\beta} \equiv (\beta_0, \ldots, \beta_m)$, $\alpha_0 \neq 0$,

the output g satisfies $\qquad \boldsymbol{\alpha} * \boldsymbol{g} = \boldsymbol{\beta} * \boldsymbol{f} \qquad (\star)$

DFT of $(\star)$ leadsto

$$p(\bar{z})\widehat{\boldsymbol{g}}(\omega) = q(\bar{z})\widehat{\boldsymbol{f}}(\omega) \quad \text{with} \quad z = e^{2\pi i \omega}$$

$p(\zeta) \equiv \alpha_0 + \ldots + \alpha_k \zeta^k$ and $q(\zeta) \equiv \beta_0 + \ldots + \beta_m \zeta^m$ $(\zeta \in \mathbb{C})$.

Let α be such that $p(\zeta) \neq 0$ for all $\zeta \in \mathbb{C}, |\zeta| = 1$.

Then $\quad H(\omega) = \dfrac{q(\bar{z})}{p(\bar{z})} \quad$ for $\quad z \equiv e^{2\pi i \omega}$.

H is 1-periodic, continuous and bounded.

$H \in L_1^2(\mathbb{R})$, whence $H = \widehat{\boldsymbol{h}}$ for some $\boldsymbol{h} \in \ell^2(\mathbb{Z})$

Theorem. $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Hence, $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h} \in \ell^2(\mathbb{Z})$ if $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.

Proof. For some $\gamma_j, \widetilde{\gamma}_j \in \mathbb{C}$, $\mu(j) \in \mathbb{N}$, we have

$$\frac{q(\zeta)}{p(\zeta)} = \sum_{j=0}^{m} \widetilde{\gamma}_j \, \zeta^j + \sum_{j=0}^{k} \frac{\gamma_j}{(\zeta - \lambda_j)^{\mu(j)}} \quad (\zeta \in \mathbb{C}).$$

Here, λ_j are the zeros of p. They are counted according to multiplicity. The $1/\lambda_j$ are the **poles** of the filter.

Suffices to consider $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$, $\mu \in \mathbb{N}$ and show

$$H(\omega) \equiv \frac{1}{(\bar{z} - \lambda)^{\mu}} = \sum_{n \in \mathbb{Z}} h_n \, \bar{z}^{\,n} \qquad (z = e^{2\pi i \omega})$$

for some $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Then $H(\omega) = \widehat{\boldsymbol{h}}(\omega)$.

Note that now there are no restrictions on the degree of q in relation to the degree of p.

Theorem. $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Hence, $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h} \in \ell^2(\mathbb{Z})$ if $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.

Proof. Let $\lambda \in \mathbb{C}$, $|\lambda| \neq 1$. We will show that

$$\frac{1}{\bar{z} - \lambda} = \sum_{n \in \mathbb{Z}} h_n \, \bar{z}^{\,n} \qquad (z = e^{2\pi i \omega})$$

for some $\boldsymbol{h} = (h_n) \in \ell^1(\mathbb{Z})$. (Exercise: similar proof if $\mu > 1$.)

$$\frac{1}{\bar{z} - \lambda} = -\frac{1}{\lambda} \frac{1}{1 - \bar{z}/\lambda} = \frac{1}{\bar{z}} \frac{1}{1 - \lambda/\bar{z}}$$

If $|\lambda| > 1$, then $|\bar{z}/\lambda| = 1/|\lambda| < 1$, and

$$-\frac{1}{\lambda} \frac{1}{1 - \bar{z}/\lambda} = -\sum_{n=0}^{\infty} \frac{1}{\lambda^{n+1}} \bar{z}^{\,n} \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{1}{|\lambda|^{n+1}} < \infty.$$

Hence, $h_n = \frac{1}{\lambda^{n+1}}$ $(n \geq 0)$, $h_n = 0$ $(n < 0)$, $\boldsymbol{h} \in \ell^1(\mathbb{Z})$.

Note that $\boldsymbol{h}$ is causal.

Theorem. $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Hence, $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h} \in \ell^2(\mathbb{Z})$ if $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.

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$$\frac{1}{\bar{z} - \lambda} = -\frac{1}{\lambda} \frac{1}{1 - \bar{z}/\lambda} = \frac{1}{\bar{z}} \frac{1}{1 - \lambda/\bar{z}}$$

If $|\lambda| < 1$, then $|\lambda/\bar{z}| = |\lambda| < 1$, and

$$\frac{1}{\bar{z}} \frac{1}{1 - \lambda/\bar{z}} = \sum_{n=-\infty}^{-1} \frac{1}{\lambda^{n+1}} \bar{z}^{\,n} \quad \text{and} \quad \sum_{n=-\infty}^{-1} \frac{1}{|\lambda|^{n+1}} < \infty.$$

Hence, $h_n = \frac{1}{\lambda^{n+1}}$ $(n < 0)$, $h_n = 0$ $(n \geq 0)$, $\boldsymbol{h} \in \ell^1(\mathbb{Z})$.

Note that $\boldsymbol{h}$ is not causal.

Theorem. $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Hence, $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h} \in \ell^2(\mathbb{Z})$ if $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.

Theorem. The filter is causal $\Leftrightarrow$ the poles are in $\mathcal{S}$.

Here $\mathcal{S} \equiv \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$.

Proof. See the proof of the preceding theorem.

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To start the filter, suppose $f_j = 0$ for $j < 0$. Then

$$g_{-k+1} = g_{-k+2} = \ldots = g_{-1} = 0$$

holds for $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h}$ $\Leftrightarrow$ the filter is causal.

Theorem. $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Hence, $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h} \in \ell^2(\mathbb{Z})$ if $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.

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Let $\boldsymbol{g} \in \ell^2(\mathbb{Z})$ be the output for input $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.
Suppose g_{n_0} is perturbed, that is,

- $\widetilde{\boldsymbol{g}}$ satisfies the recurrence relations for $n \neq n_0$,
- $\widetilde{g}_n = g_n$ for $n < n_0$,
- $\widetilde{g}_{n_0} = g_{n_0} + \varepsilon$.

Here we assume that we obtained $\boldsymbol{g}$ by recursively solving the recurrence relations $(\star)$.

Theorem. $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Hence, $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h} \in \ell^2(\mathbb{Z})$ if $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.

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- $\widetilde{g}_n = g_n$ for $n < n_0$,
- $\widetilde{g}_{n_0} = g_{n_0} + \varepsilon$.

Then,

- $\widetilde{\boldsymbol{g}} - \boldsymbol{g}$ satisfies the recurrence for $\boldsymbol{f} \equiv \mathbf{0}$, $n \neq n_0$,
- $\widetilde{g}_n - g_n = 0$ for $n < n_0$,
- $\widetilde{g}_{n_0} - g_{n_0} = \varepsilon$.

Theorem. $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Hence, $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h} \in \ell^2(\mathbb{Z})$ if $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.

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- $\widetilde{g}_{n_0} = g_{n_0} + \varepsilon$.

Example. $p(\zeta) = \alpha_0 + \alpha_1 \zeta$ and $q(\zeta) = 1$.

Then $p(-\alpha_0/\alpha_1) = 0$. With $\lambda \equiv -\alpha_1/\alpha_0$, we have

$$\alpha_0 \lambda^{n+1} + \alpha_1 \lambda^n = \lambda^n(\alpha_0 \lambda + \alpha_1) = 0 :$$

$\widetilde{g}_n - g_n = \varepsilon \lambda^{n-n_0}$ for $n \geq n_0$.

Theorem. $\boldsymbol{h} \in \ell^1(\mathbb{Z})$. Hence, $\boldsymbol{g} = \boldsymbol{f} * \boldsymbol{h} \in \ell^2(\mathbb{Z})$ if $\boldsymbol{f} \in \ell^2(\mathbb{Z})$.

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- $\widetilde{g}_n = g_n$ for $n < n_0$,
- $\widetilde{g}_{n_0} = g_{n_0} + \varepsilon$.

Then, for some $\delta_1, \ldots, \delta_k$, we have that

$$\widetilde{g}_n - g_n = \sum_{j=1}^{k} \delta_j \, n^{\mu(j)} \lambda_j^{-n} \qquad (n \geq n_0 - k + 1).$$

The error $\widetilde{g}_n - g_n$ vanishes for $n \to \infty$ $\Leftrightarrow$ all λ_j in $\mathcal{S}$.

Here, $1/\lambda_j$ zero of p.

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Theorem. The filter is causal $\Leftrightarrow$ the poles are in $\mathcal{S}$.

Here $\quad \mathcal{S} \equiv \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$.

Definition. The filter is **stable** if all poles are in $\mathcal{S}$,
that is, $p(1/\lambda) = 0 \;\Rightarrow |\lambda| < 1$.

Theorem. The filter is stable $\Leftrightarrow$ it is causal.

$$\alpha_0 + \alpha_1 g_{n-1} + \ldots + \alpha_k g_{n-k} = \beta_0 f_n + \ldots + \beta_m f_{n-m}$$

$$p(\zeta) \equiv \alpha_0 + \ldots + \alpha_k \zeta^k, \qquad q(\zeta) \equiv \beta_0 + \ldots + \beta_m \zeta^m,$$

Put $\quad H(\omega) \equiv |H(\omega)|\, e^{-i\phi(\omega)} \equiv \frac{q(\bar{z})}{p(\bar{z})}$ with $z = e^{2\pi i \omega}$.

Summary. Polynomials p and q should be such that

1) For technical realisation: p and q are real (real coeff.)
2) For caus. and stab.: $\lambda \in \mathbb{C}$ & $p(\lambda) = 0 \;\Rightarrow |\lambda| > 1$
3) For requested filtering: $\quad |H| \approx \Pi_\Omega$
4) For acceptable group/time delay; $\phi(\omega) \approx \ldots$

Note.
There is no restriction on the degree of the polynomial q.

$$\alpha_0 + \alpha_1 g_{n-1} + \ldots + \alpha_k g_{n-k} = \beta_0 f_n + \ldots + \beta_m f_{n-m}$$

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Discussion. The stability/causality restriction $|\lambda| > 1$ on the zeros of the polynomial p seems a bit odd: because, the familiar stability condition for difference equation is $|\lambda| < 1$. This is explained from the fact that $e^{-2\pi i\omega n}$ is used for the Fourier transform: changing $-n$ into $+n$ (or, equivalently, reversing time $t = n$), leads to the usual stability condition.

The familiar looking condition can also be recovered by changing the "order" of the pol. terms: with $N \equiv \max(m, k)$, put $Q(\zeta) \equiv \zeta^N q(1/\zeta)$ and $P(\zeta) \equiv \zeta^N p(1/\zeta) \quad (\zeta \in \mathbb{C})$.

Then,
- P and Q are polynomials of degree N,
- $H(\omega) = \frac{Q(z)}{P(z)}$ with $z = e^{2\pi i\omega}$
- $P(\lambda) = 0 \;\Leftrightarrow p(1/\lambda) = 0$.

From analogue to digital

Analogue filters can easily be transformed into digital ones using:

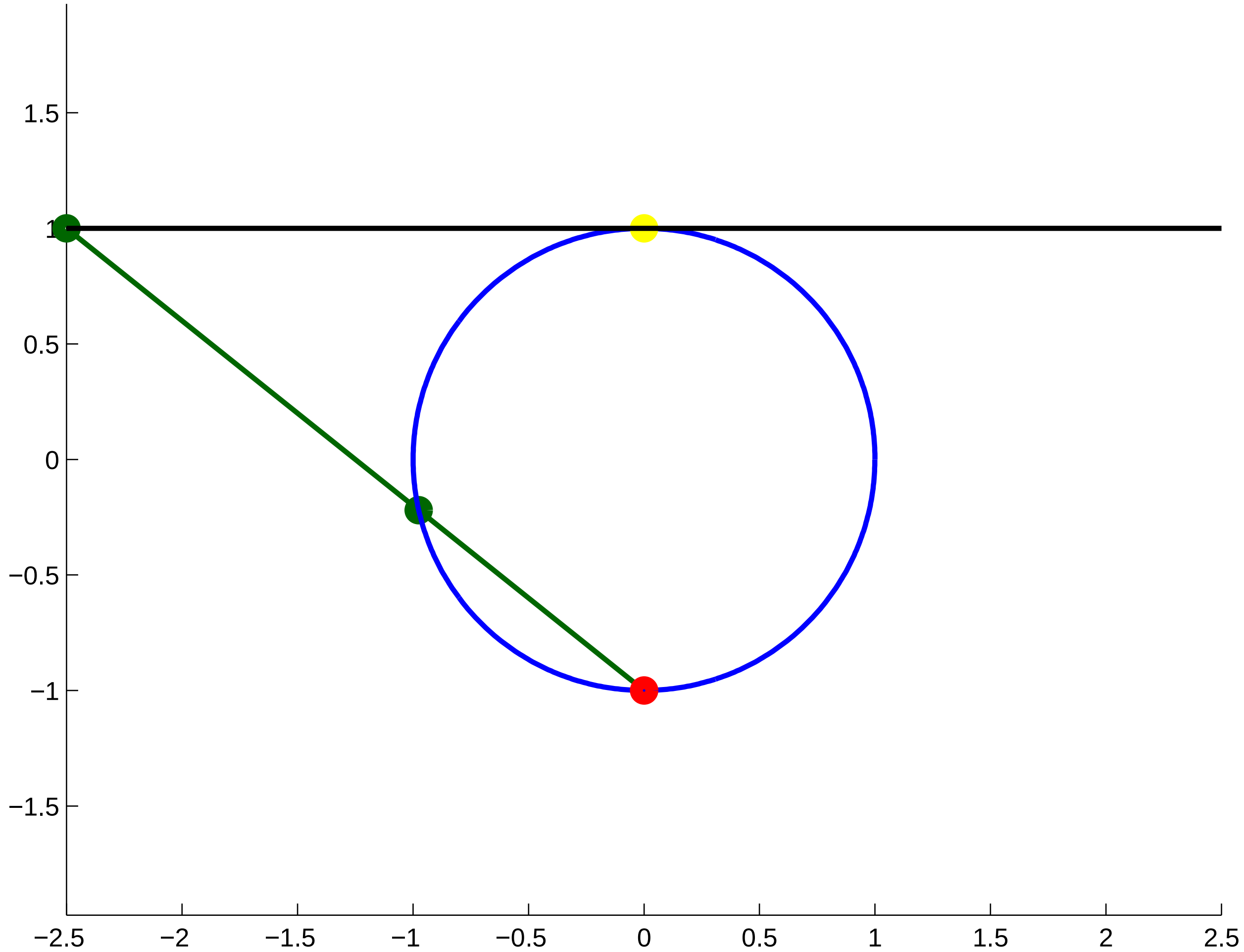
$$\zeta \equiv \gamma Z(z) \quad \text{with} \quad Z(z) \equiv \frac{z-1}{z+1} \quad (z \in \mathbb{C})$$

Z is **Cayley's transform**. It is a conformal (i.e., analatyic with non-zero derivative) bijection, mapping

- $\mathbb{C}\setminus\{-1\}$ onto $\mathbb{C}\setminus\{1\}$,
- $\{z \in \mathbb{C} \mid |z| < 1\}$ onto $\{\zeta \in \mathbb{C} \mid \text{Re}(\zeta) < 0\}$ and
- $\{z \in \mathbb{C} \mid |z| = 1, z \neq -1\}$ onto $i\mathbb{R}$
- ◦ -1 to ∞.

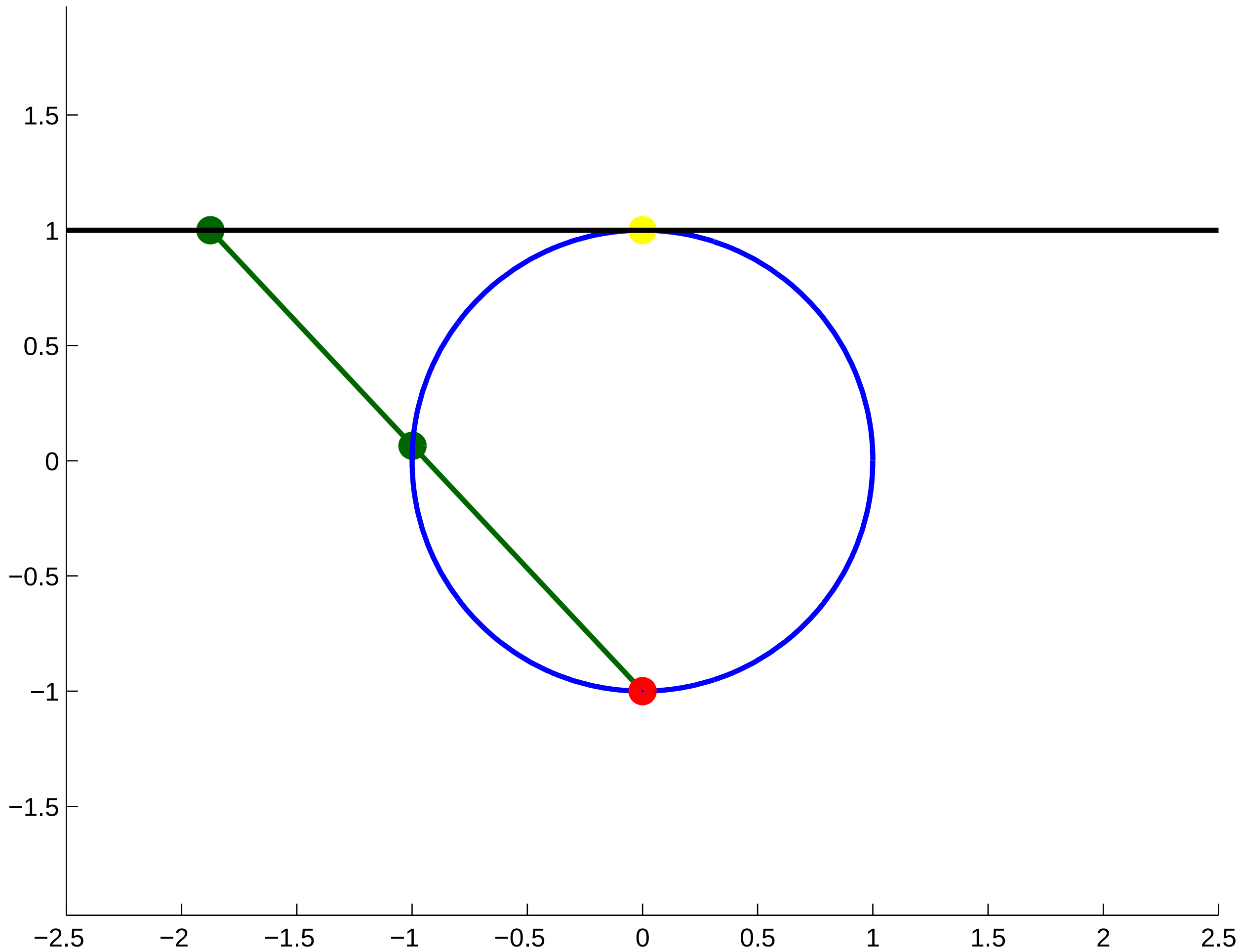
We select $\gamma < 0$ and use its size to scale the $i\mathbb{R}$ axis.

Cayley's transform



If we identify (the green dots) the complex plane with the unit sphere, $2\zeta \rightsquigarrow (\frac{\zeta}{1+|\zeta|^2}, \frac{1-|\zeta|^2}{1+|\zeta|^2})^{\mathsf{T}}$ (0 = north-pole, ∞ - south-pole), then the Cayley transform rotates the sphere by rotating the southern hemisphere to the northern one.

Cayley's transform



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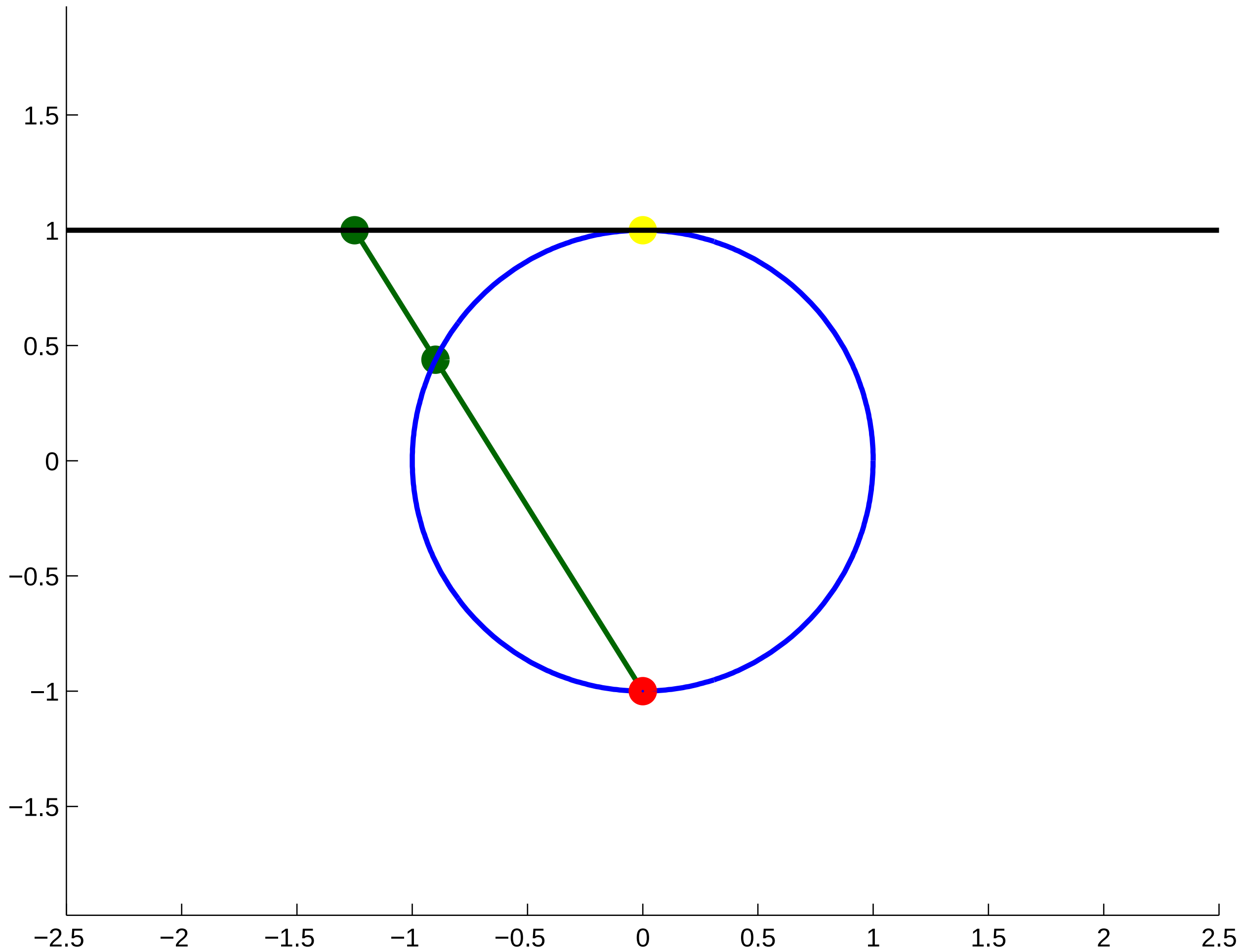

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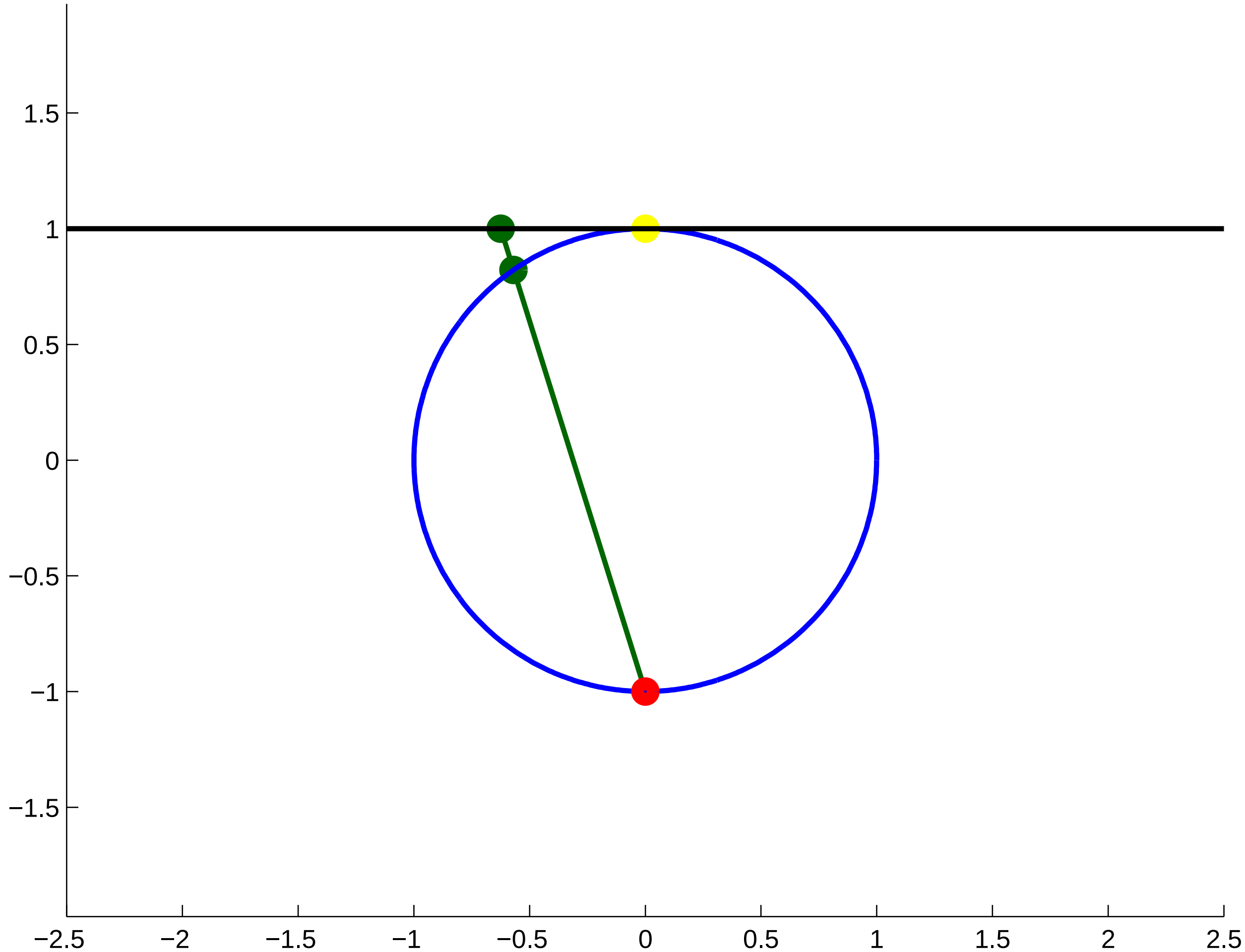

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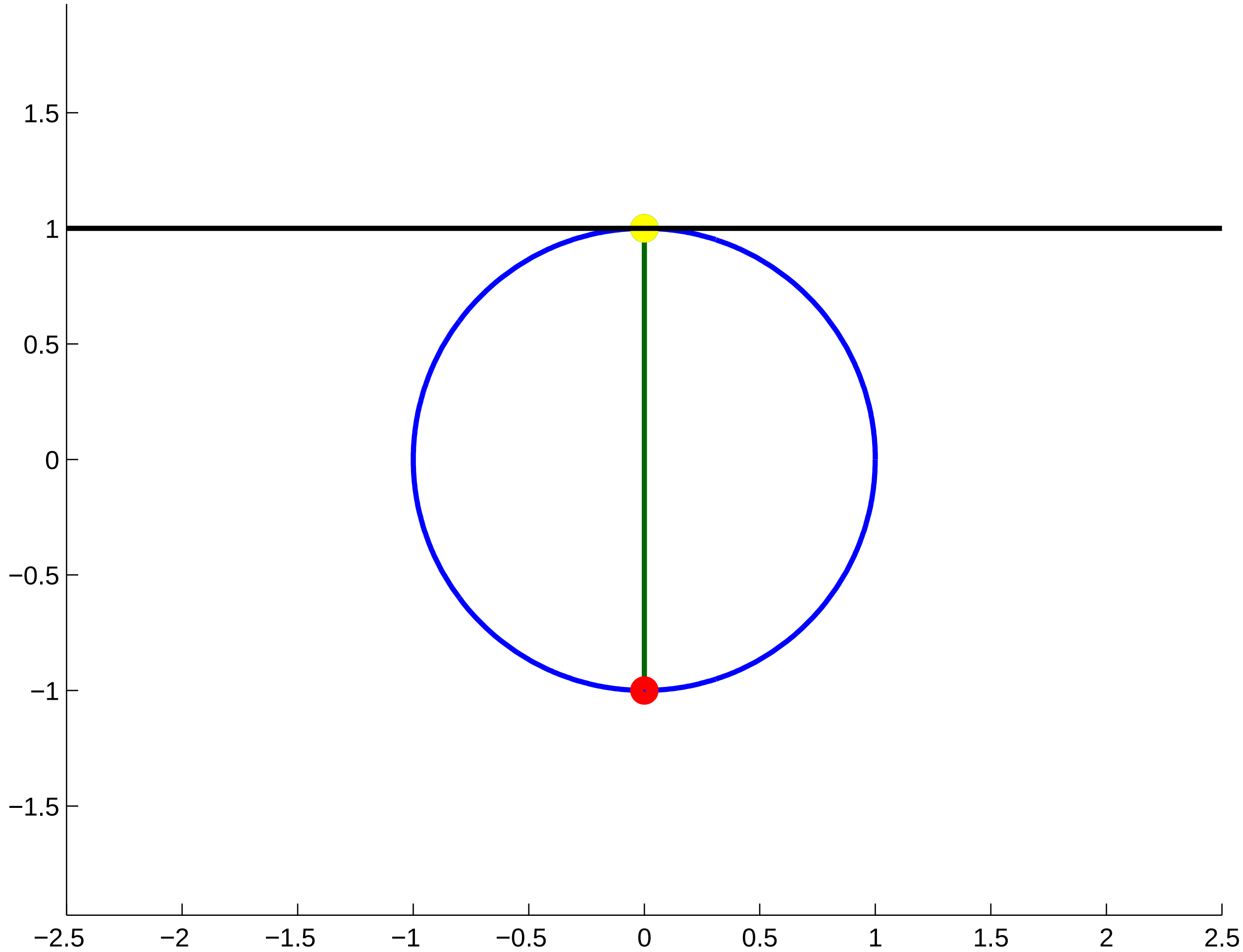

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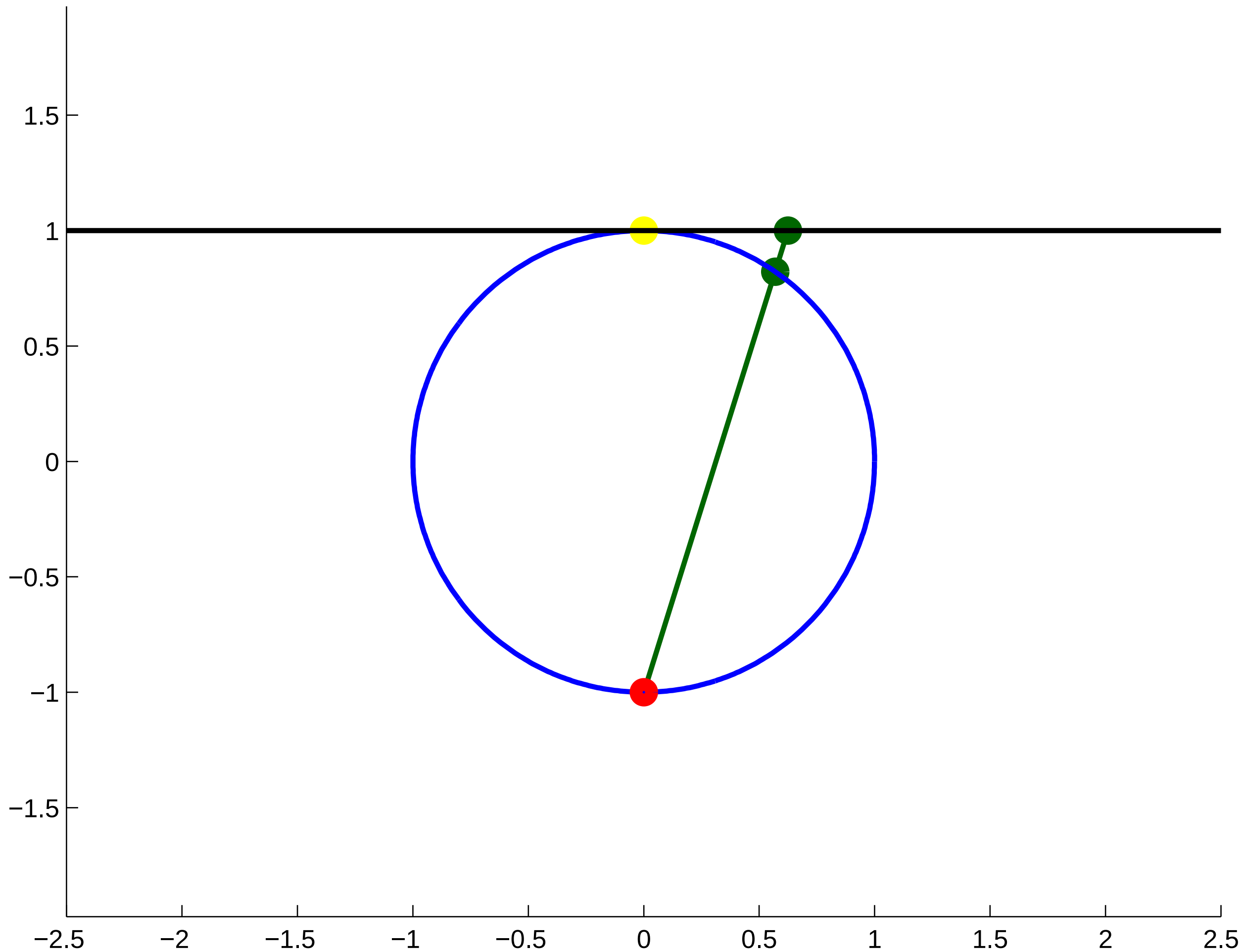

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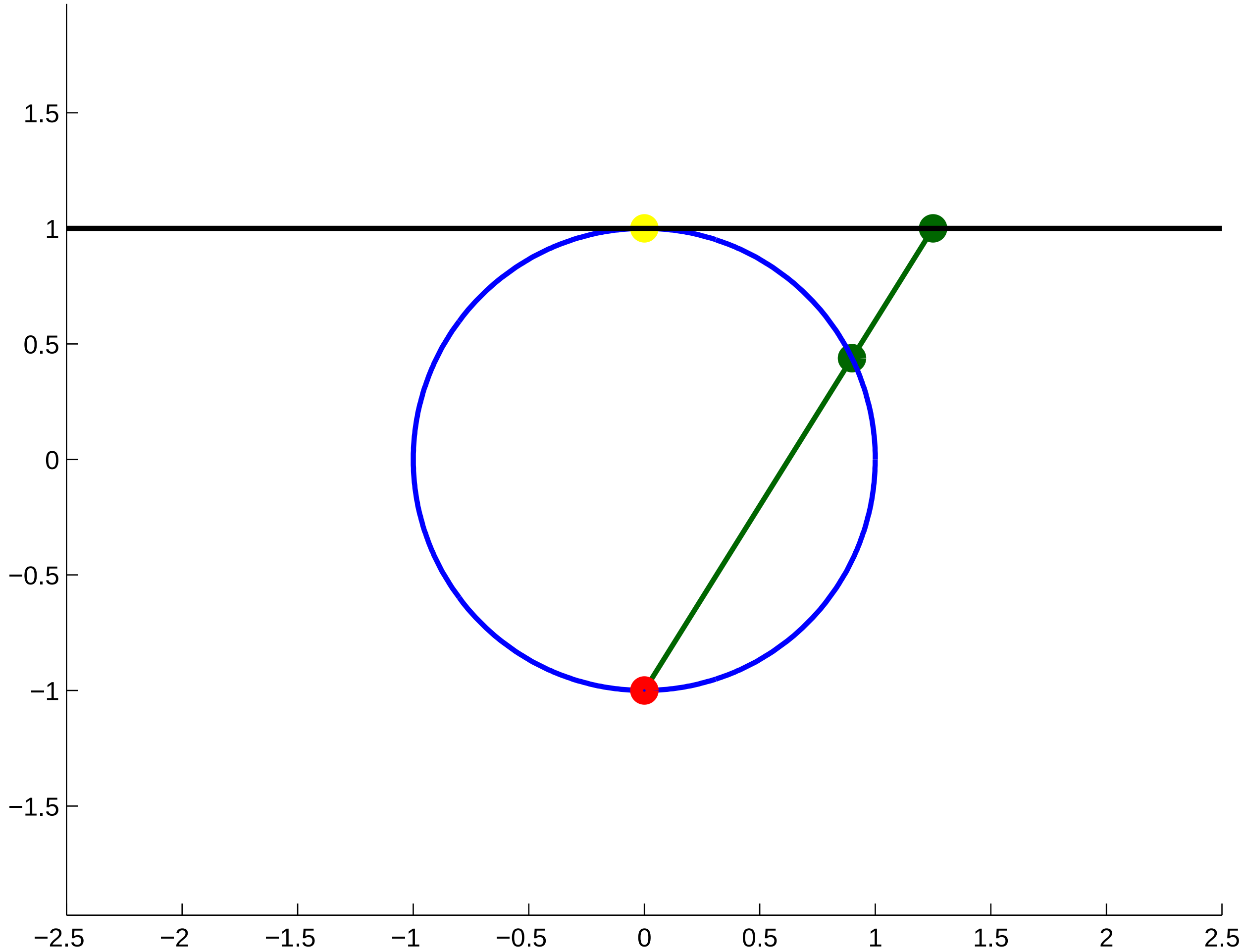

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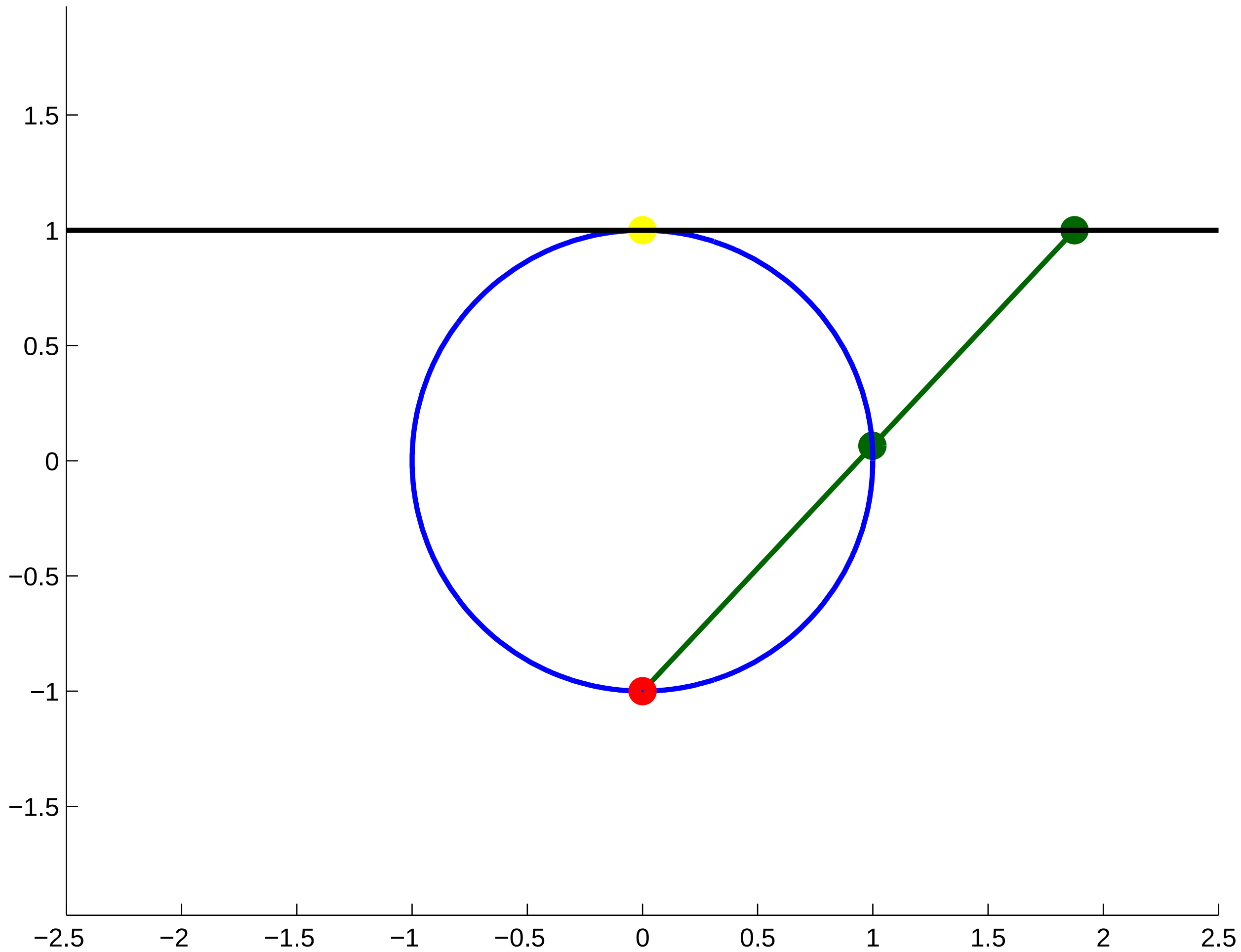

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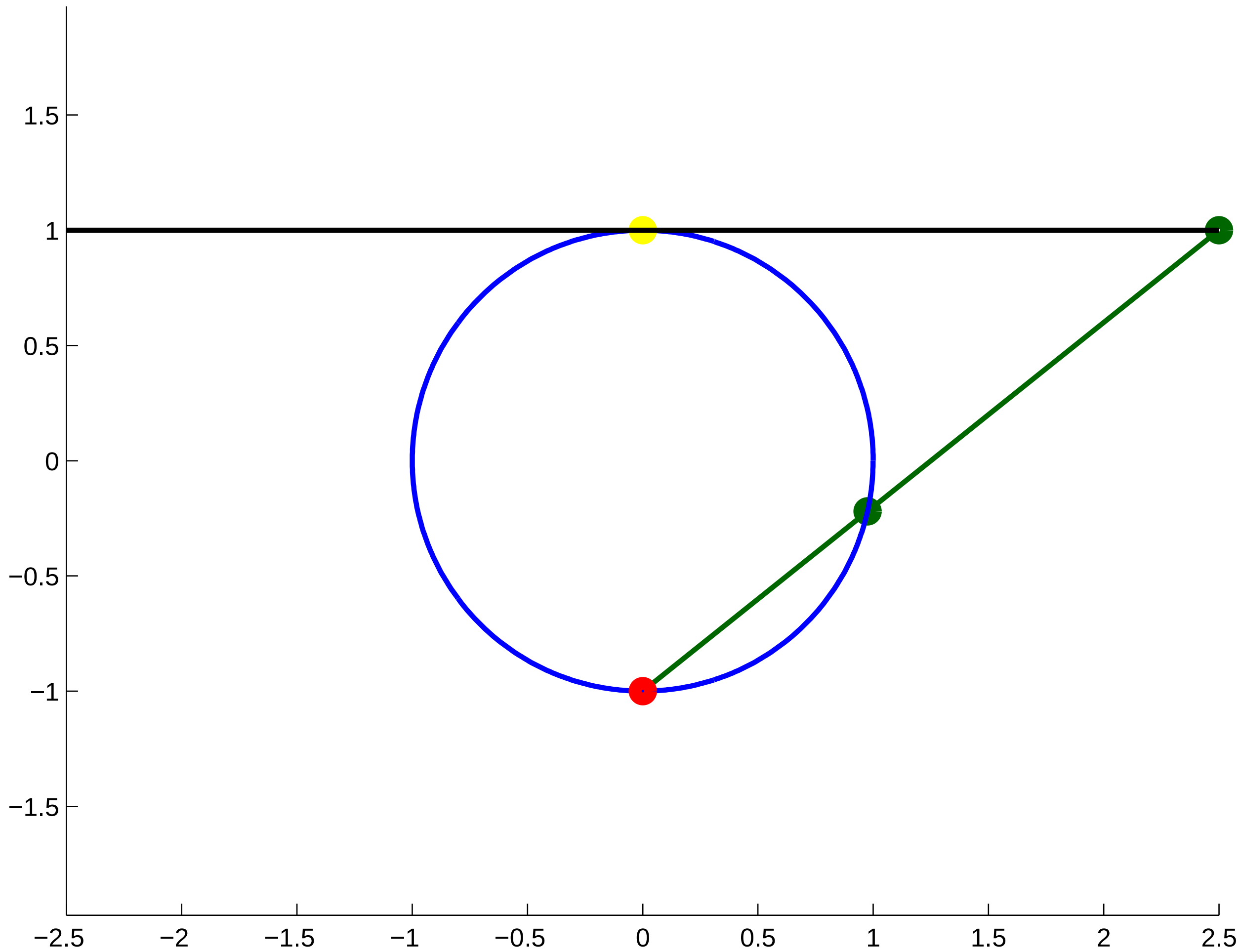

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From analogue to digital

Analogue filters can easily be transformed into digital ones using:

$$\zeta \equiv \gamma Z(z) \quad \text{with} \quad Z(z) \equiv \frac{z-1}{z+1} \quad (z \in \mathbb{C})$$

Properties. Select $\gamma < 0$. Put $\omega \equiv -\frac{\gamma}{2\pi}\tan(\pi v)$ $(v \in \mathbb{R})$.

- $z = e^{-2\pi i v} \quad \Leftrightarrow \quad \zeta = -i\gamma\tan(\pi v) = 2\pi i\omega.$
- $\mathrm{Re}(\zeta) < 0 \quad \Leftrightarrow \quad |z| > 1.$

If, for $V > 0$, we are interested in approximating $\Pi_V(v)$, then we can try to approximate $\Pi_\Omega(\omega)$ for $\Omega \equiv -\frac{\gamma}{2\pi}\tan \pi V$.

Nate that $\gamma = -\frac{2\pi}{\tan(\pi V)}$ might be an attractive scaling then.

From analogue to digital

Analogue filters can easily be transformed into digital ones using:

$$\zeta \equiv \gamma Z(z) \quad \text{with} \quad Z(z) \equiv \frac{z-1}{z+1} \quad (z \in \mathbb{C})$$

Consider a stable analogue filter $A(\zeta) \equiv \dfrac{q(\zeta)}{p(\zeta)}$:

with $H(\omega) \equiv A(2\pi i\omega)$, we have

- $|H(\omega)| \approx \Pi_\Omega(\omega)$, • $A(\zeta) = \infty \;\Leftrightarrow\; \mathrm{Re}(\zeta) < 0$.

With $D(z) \equiv A(\zeta)$ and $\widetilde{H}(v) \equiv D(e^{-2\pi i v})$,

D is a stable digital filter:

- $D(z) = \dfrac{\widetilde{q}(z)}{\widetilde{p}(z)}$ for some polynomials $\widetilde{p}$ and $\widetilde{q}$,
- $D(e^{-2\pi i v}) = A(2\pi i\omega)$, $\quad |\widetilde{H}(v)| \approx \Pi_\Omega(\omega) = \Pi_V(v)$,
- $D(z) = \infty \;\Leftrightarrow\; A(\zeta) = \infty \;\Leftrightarrow\; |z| > 1$.