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Fourier Transforms
Wavelets
Theory and Applications

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Remark. If $I \subset \mathbb{R}^d$ and $f : I \to \mathbb{C}^\ell$ then

$$f = (f_1, \ldots, f_\ell)^\mathsf{T}$$

and **we can study the functions $f_i : I \to \mathbb{C}$ separately**.

However, there is no convenient way to restrict the analysis further, to functions defined on (a subset of) $\mathbb{R}$:
e.g., $x \rightsquigarrow f_1(x, x_2, \ldots, x_d)$ depends on $(x_2, \ldots, x_d)$!

Remark. A function $f : \mathbb{C} \to \mathbb{C}$ can be viewed as a function $f : \mathbb{R}^2 \to \mathbb{C}$.

Remark. If f is defined on a subset I of $\mathbb{R}^d$, then f can be extended to a function defined on $\mathbb{R}^d$, for instance, by defining $f(x) = 0$ for $x \notin I$ (or by periodicity).

Purpose

We want to analyse functions, reveal hidden structures.

Applications.

- De-noising, de-blurring
- Compression

Ex. For some $k \in \mathbb{Z}$ and $T > 0$, $f(t) = \sin(2\pi k t/T)$ for $t \in [0, 10]$.
Store $f(j\Delta t)$ for $j = 0, 1, \ldots, 10^5$ with $\Delta t = 10^{-4}$ (as on a CD).
Alternative, store k and T.
Compression also important to facilitate analysis.

- ...

Strategy

Find a suitable basis to represent the class of functions that are of interest.

(ϕ_k) (infinite set of) 'basisfunctions'.
Then $f = \sum_k \gamma_k \phi_k$ in some sense.

Find (ϕ_k) such that

1) $f \approx \sum_{k \in E} \gamma_k \phi_k$, with E finite (small) subset of indices k.
2) E is 'small' and can 'easily' be detected.
3) $\sum_{k \in E} \gamma_k \phi_k(t)$ can efficiently be computed.

1) Approximation, 2) Extraction, 3) Computation

Example. $f \in C([-1,1])$, $\phi_k(t) = t^k \quad (k \in \mathbb{N}_0, |t| \leq 1)$

Approximation. **Weierstrass**. $\forall \varepsilon > 0$

$\exists$ a polynomial p st $\forall t \in [-1,1]$, $|f(t) - p(t)| \leq \varepsilon$.

Extraction. **Taylor**. If f is sufficiently smooth:

$$p(t) = \sum_{j<k} \frac{t^j}{j!} f^{(j)}(0), \quad f(t) - p(t) = \frac{t^k}{k!} f^{(k)}(\xi).$$

Evaluation. **Horner**. If $p(t) = \gamma_0 + \gamma_1 t + \ldots + \gamma_k t^k$ then

$$p(t) = \gamma_0 + (\ldots(\gamma_{k-2} + (\gamma_{k-1} + \gamma_k t)t)t\ldots)t:$$

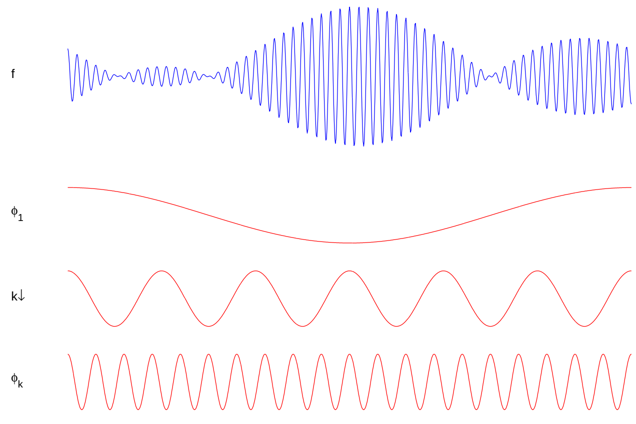
$s_0 = \gamma_k$, $s_j = \gamma_{k-j} + s_{j-1} t$ for $j = 1, \ldots, k$. Then $p(t) = s_k$.

Polynomials well suited for computing (but not t^k),
less suitable for analysis.

Example. $f \in C([0,1])$, $\phi_k(t) \equiv \cos(2\pi k t) = \phi(kt)$.

Reveals periodic structures in f:

test against ϕ_k $(k \in \mathbb{N}_0)$, i.e., compute $\int f(t)\phi_k(t)\,dt$



Applications Fourier analysis.

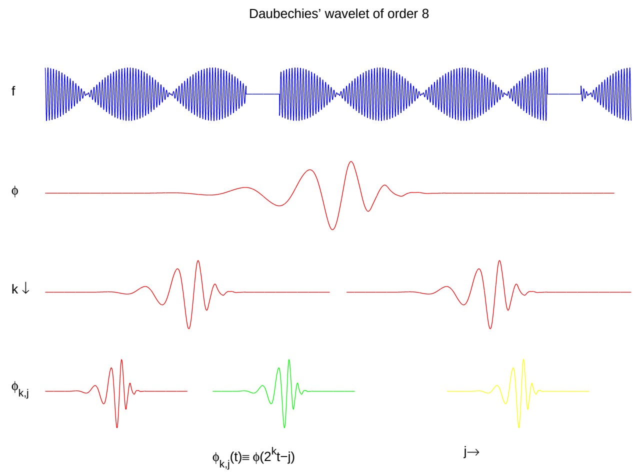
- Audio technique (equalizers, amplyfiers, tuner, CDs)
- MP3 and other audio compression techniques
- biology, ear, eye, ...
- radar, echo location, CT, MRI, ...
- Cristallography, Geophysics, ...
- denoising, deblurring of images, JPEG compression, MJPEG
- Theory (partial) differential equations

⋮

Example. $f \in C([0,1])$, $\phi_{k,j}(t) = \psi(2^k t - j)$.

Reveals periodic structures in f **and localized changes**:

compute $\int f(t)\phi_{k,j}(t)\,dt$ for $k, j \in E \subset \mathbb{Z}$



Application wavelet analysis.

As Fourier, tends to be more practical

- Storing and detection of fingerprints (to help police investigations)
- Computational techniques for partial differential equations

⋮

Example. $\phi_k(t) = t^k$ polynomials.

Example. $\phi_k(t) \equiv \cos(2\pi k t)$
Harmonic oscillations, Fourier modes

Example. **Wavelets**

Example. Bessel functions, . . .

Example. Splines (smooth, piece-wise polynomials)

Example. Finite element basis functions

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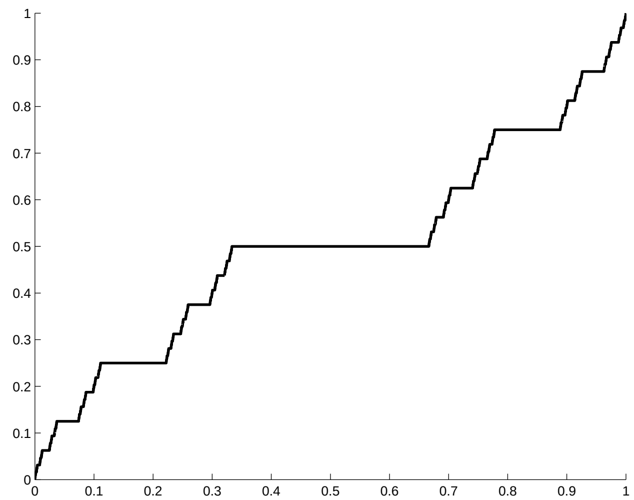
Preliminaries



Program

- Norms and inner products
- Convergence
- Almost everywhere
- Function spaces
- Point-wise convergence
- Function values
- Derivatives

Norms

Let $\mathcal{V}$ be a (real or) complex vector space.

A map $\|\cdot\|:\mathcal{V}\to[0,\infty)$ is a **norm** if

1) $\|f\|=0$ iff $f=0 \qquad (f\in\mathcal{V})$
2) $\|\lambda f\|=|\lambda|\,\|f\| \qquad (f\in\mathcal{V},\lambda\in\mathbb{C})$
3) $\|f+g\|\leq\|f\|+\|g\| \qquad (f,g\in\mathcal{V},\lambda\in\mathbb{C})$

Exercise.

Inner products

Let $\mathcal{V}$ be a (real or) complex vector space.

A map $(\cdot,\cdot):\mathcal{V}\times\mathcal{V}\to\mathbb{C}$ is an **inner product** if

1) $(f,f)\geq 0$, $\quad (f,f)=0$ iff $f=0 \qquad (f\in\mathcal{V})$
2) $(f,g)=\overline{(g,f)} \qquad (f,g\in\mathcal{V})$
3) $f\rightsquigarrow(f,g)$ is linear $\qquad (g\in\mathcal{V})$

Theorem. If $(\cdot,\cdot)$ is an inner product on $\mathcal{V}$, then $f\rightsquigarrow\sqrt{(f,f)}$ defines a norm on $\mathcal{V}$.

Example. $\|f\|_2=\sqrt{(f,f)}$ on $\mathcal{V}=C([a,b])$.

$\mathcal{V}$ is a space with norm $\|\cdot\|$.

A sequence (f_n) in $\mathcal{V}$ **converges** to an $f\in\mathcal{V}$ if

$$\lim_{n\to\infty}\|f_n-f\|=0$$

Exercise. $\mathcal{V}=C([0,1])$, $f_n(t)=t^n \qquad (n\in\mathbb{N},t\in[0,1])$.
Does (f_n) converge with respect to $\|\cdot\|_1$?
Does (f_n) converge with respect to $\|\cdot\|_\infty$?

Exercise. $\mathcal{V}=C([0,2])$, $f_n(t)=\min(t^n,1)$.
Does (f_n) converge with respect to $\|\cdot\|_1$?

(f_n) is a **Cauchy sequence** with respect to a norm $\|\cdot\|$

if $\quad \|f_n - f_m\| \to 0 \quad$ if $n > m$, $m \to \infty$

A space $\mathcal{V}$ with norm $\|\cdot\|$ is **complete** if each Cauchy sequence (f_n) in $\mathcal{V}$ converges to an $f \in \mathcal{V}$.

Exercise. $\mathcal{V} = C([0,2])$.

Is $\mathcal{V}$ complete wrt $\|\cdot\|_1$?

Is $\mathcal{V}$ complete wrt $\|\cdot\|_2$?

Is $\mathcal{V}$ complete wrt $\|\cdot\|_\infty$?

Can we complete $C([0,2])$ wrt the $\|\cdot\|_2$?
What kind of objects are contained in the completion?

Consider two functions f and g on $[a,b]$.

f and g coincide **almost everywhere** ($f = g$ a.e.)

if the set $\mathcal{N} \equiv \{t \in [a,b] \mid f(t) \neq g(t)\}$ on which they differ is negligible, i.e., has measure zero, i.e., $\int_a^b \chi_{\mathcal{N}}(t)\,\mathrm{d}t = 0$.

Example. Let $f(t) = 1$ for $t > 0$ and $f(t) = 0$ elsewhere, and let $\tilde{f}(t) = 1$ for $t \geq 0$ and $\tilde{f}(t) = 0$ elsewhere.
Then $f = \tilde{f}$ a.e..

Unless stated otherwise,

we will identify functions that coincide a.e.

For functions $f : [a,b] \to \mathbb{C}$

$$\|f\|_1 \equiv \int_a^b |f(t)|\,\mathrm{d}t, \qquad \|f\|_2 \equiv \sqrt{\int_a^b |f(t)|^2\,\mathrm{d}t}$$

$$\|f\|_\infty \equiv \text{ess-sup}\{|f(t)| \mid t \in [a,b]\}$$

Theorem. $\quad \|f\|_1 \leq \sqrt{b-a}\,\|f\|_2 \leq (b-a)\,\|f\|_\infty$

$L^1([a,b])$, $L^2([a,b])$, $L^\infty([a,b])$ is the space of all functions $f : [a,b] \to \mathbb{C}$ for which $\|f\|_1 < \infty$, $\|f\|_2 < \infty$, $\|f\|_\infty < \infty$, respectively, and we identify functions that coincide a.e..

$L^2([a,b])$ is an inner product space: $(f,g) \equiv \int_a^b f(t)\,\overline{g(t)}\,\mathrm{d}t$.

Theorem. $\quad C([a,b]) \subset L^\infty([a,b]) \subset L^2([a,b]) \subset L^1([a,b])$

Exercise. Show that all inclusions are strict.

(f_n) is a **Cauchy sequence** wrt a norm $\|\cdot\|$

if $\quad \|f_n - f_m\| \to 0 \quad$ if $n > m$, $m \to \infty$

Completeness Theorem.
The spaces $L^p([a,b])$, for $p = 1, 2, \infty$, are **complete** that is, if (f_n) is a **Cauchy sequence** in $L^p([a,b])$ then there is an $f \in L^p([a,b])$ such that $\lim_{n\to\infty} \|f_n - f\|_p = 0$.

Density Theorem. $C([a,b])$ is **dense** in $L^p([a,b])$ for $p = 1$ as well as for $p = 2$, i.e., for each $f \in L^p([a,b])$ and each $\varepsilon > 0$ there is a $g \in C([a,b])$ such that $\|f - g\|_p < \varepsilon$.

Exercise. $C([a,b])$ is **not** dense in $L^\infty([a,b])$
(with $f(t) = 1$ for $t > 0$ and $f(t) = -1$ for $t \leq 0$ ($|t| \leq 1$) show that $\|f - g\|_\infty \geq 1$ for all $g \in C([-1,+1])$.)

For sequences $(\gamma_k)_{k\in\mathbb{Z}}$ in $\mathbb{C}$. With $\gamma(k) = \gamma_k$, $\gamma : \mathbb{Z} \to \mathbb{C}$.

$$\left|\gamma\right|_1 \equiv \sum_{k=-\infty}^{\infty} |\gamma_k|, \quad \left|\gamma\right|_2 \equiv \sqrt{\sum_{k=-\infty}^{\infty} |\gamma_k|^2}, \quad \left|\gamma\right|_\infty \equiv \sup_{k\in\mathbb{Z}} |\gamma_k|$$

$\ell^1(\mathbb{Z})$, $\ell^2(\mathbb{Z})$, $\ell^\infty(\mathbb{Z})$ is the space of all sequences γ in $\mathbb{C}$ for which $\left|\gamma\right|_1 < \infty$, $\left|\gamma\right|_2 < \infty$, $\left|\gamma\right|_\infty < \infty$, resp.

$\ell^2(\mathbb{Z})$ is an inner product space: $<\gamma, \mu> \equiv \sum \gamma_k \overline{\mu_k}$.

Theorem. $\left|\gamma\right|_\infty \le \left|\gamma\right|_2 \le \left|\gamma\right|_1 \quad (\gamma : \mathbb{Z} \to \mathbb{C})$

$$\ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$$

For functions $f : \mathbb{R} \to \mathbb{C}$

$$\|f\|_1 \equiv \int_{-\infty}^{\infty} |f(t)|\, dt, \qquad \|f\|_2 \equiv \sqrt{\int_{-\infty}^{\infty} |f(t)|^2\, dt}$$

$$\|f\|_\infty \equiv \text{ess-sup}\{|f(t)| \mid t \in \mathbb{R}\}$$

$L^1(\mathbb{R})$, $L^2(\mathbb{R})$, $L^\infty(\mathbb{R})$ is the space of all functions $f : \mathbb{R} \to \mathbb{C}$ for which $\|f\|_1 < \infty$, $\|f\|_2 < \infty$, $\|f\|_\infty < \infty$, respectively, and we identify functions that coincide a.e..

$L^2(\mathbb{R})$ is an inner product space: $(f, g) \equiv \int_{-\infty}^{\infty} f(t)\, \overline{g(t)}\, dt$.

Exercise. Discuss the inclusions

$$C(\mathbb{R}) \subset L^\infty(\mathbb{R}) \subset L^2(\mathbb{R}) \subset L^1(\mathbb{R})$$

On $[a, b]$: $\quad C([a, b]) \subset L^\infty([a, b]) \subset L^2([a, b]) \subset L^1([a, b])$

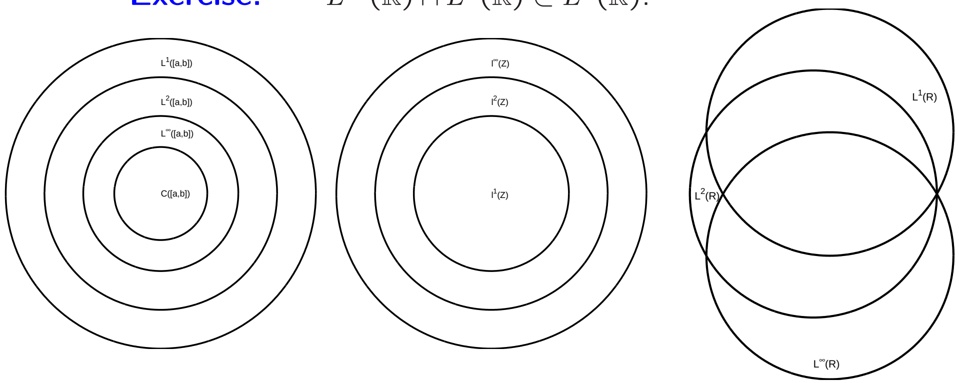
On $\mathbb{Z}$: $\quad \ell^1(\mathbb{Z}) \subset \ell^2(\mathbb{Z}) \subset \ell^\infty(\mathbb{Z})$

On $\mathbb{R}$: $\quad C(\mathbb{R}) \,??\, L^\infty(\mathbb{R}) \,??\, L^2(\mathbb{R}) \,??\, L^1(\mathbb{R})$

Explanation: $\|f\|_1 = \sum_{k\in\mathbb{Z}} \|f|_{[k,k+1]}\|_1$ for $f : \mathbb{R} \to \mathbb{C}$:

mixure of 'on $[a, b]$' and 'on $\mathbb{Z}$.

Exercise. $\quad L^\infty(\mathbb{R}) \cap L^1(\mathbb{R}) \subset L^2(\mathbb{R})$.



For $I = [a, b]$ or $I = \mathbb{R}$,

consider a sequence (f_n) in $L^1(\mathbb{R})$ and an $f \in L^1(\mathbb{R})$ st

$$\lim_{n\to\infty} f_n(t) = f(t) \qquad (t \in I).$$

The sequence **converges point-wise**.

Fatou's lemma. If there is a g st

$$g \in L^1(I) \quad \text{and} \quad |f_n(t)| \le |g(t)| \;\; (t \in I, n \in \mathbb{N}),$$

then $\quad \lim_{n\to\infty} f_n(t) = f(t) \;\; (t \in I) \;\; \Rightarrow \;\; \lim_{n\to\infty} \|f_n - f\|_1 = 0$

Function values

Note. Formally, $f(t)$ does not have a meaning.

However, if $f = g$ a.e. and g is continuous at t,
then $g(t)$ is well-defined and
Convention. With $f(t)$ **we will denote this value** $g(t)$.

More generally, we put $f(t+)$,
if $f = g$ a.e. for a function g that is left continuous at t
($\lim_{\varepsilon>0,\varepsilon\to 0} g(t+\varepsilon) = g(t)$). Then $f(t+)$ has the value $g(t)$.

Similarly,
$f(t-) = g(t)$ if $f = g$, a.e., and $\lim_{\varepsilon>0,\varepsilon\to 0} g(t-\varepsilon) = g(t)$

Weak Derivatives

Consider a function f on $[a, b]$. We will put f' if there is a function g on $[a, b]$ and a $c \in [a, b]$ such that

$$f(t) = f(c) + \int_c^t g(s)\,\mathrm{d}s \qquad (t \in [a, b]).$$

Then, f' will denote the function g.

g is unique if we identify functions that coincide a.e..

Theorem. If $f' \in L^1([a, b])$ then $f \in C([a, b])$.

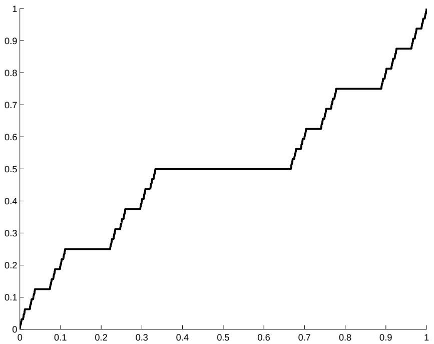
f is said to be **absolutely continuous** if $f' \in L^1([a, b])$.

Weak Derivatives

There is a **continuous non-decreasing** function f on $[0, 1]$ with $f(0) = 0$, $f(1) = 1$ such that

$$f'(t) = 0 \text{ for almost all } t \in [0, 1]:$$

Allthough most values $f'(t)$ exists, **f' does not exists!**



Integration by parts

If $f', g' \in L^1([a, b])$ then

$$\int_a^b f'(t)g(t)\,\mathrm{d}t = f(b)g(b) - f(a)g(a) - \int_a^b f(t)g'(t)\,\mathrm{d}t$$

It is essential that both f and g are continuous on $[a, b]$, the functions f' and g' need not be continuous.