Scientific Computing, Utrecht, February 17, 2014

Fourier Transforms
Wavelets
Theory and Applications

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Program

- Periodic Functions
- Function spaces
- Fourier Series
- Convergence
- Error Estimates
- Differential equations
- Discrete ℓ^2 spaces

$T > 0$

T-periodic functions

$f : \mathbb{R} \to \mathbb{C}$ is **T-periodic** if $f(t+T) = f(t) \quad \forall t \in \mathbb{R}$

Example. For each $k \in \mathbb{Z}$,
$t \rightsquigarrow \cos(2\pi t\frac{k}{T})$ and $t \rightsquigarrow \sin(2\pi t\frac{k}{T})$ are T-periodic.

Fourier: these are essentially all T-periodic functions:
each T-periodic function is in some sense a
linear combinations of these sines and cosines

T is the *length of the period*.

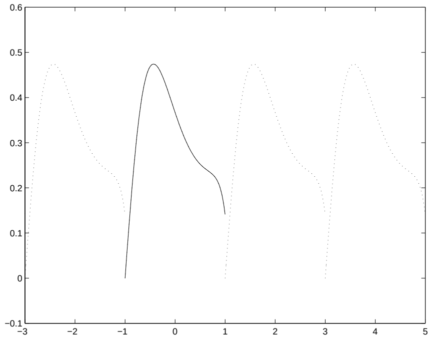
Note that $\exp(2\pi i t\frac{k}{T}) = \cos(2\pi t\frac{k}{T}) + i\sin(2\pi t\frac{k}{T})$

Functions on $[a, a+T]$ can be identified
with T-periodic functions:

If g is defined on $[a, a+T]$, then

$$f(t) \equiv g(t+kT) \quad (t \in \mathbb{R}) \quad k \in \mathbb{Z} \text{ s.t. } t+kT \in [a, a+T)$$

defines a T-periodic function and $f = g$ on $[a, a+T]$.



— graph g
$a = -1$, $T = 2$
. . . graph f,
f is 2-periodic

If f is T-periodic, then

$$\int_0^T f(t)\,\mathrm{d}t = \int_\tau^{\tau+T} f(t)\,\mathrm{d}t \qquad (\tau \in \mathbb{R})$$

For T-periodic, integrable functions f on $\mathbb{R}$ define

$$\|f\|_1 \equiv \frac{1}{T}\int_0^T |f(t)|\,\mathrm{d}t$$

The space of all complex-valued T-periodic functions for which $\|f\|_1 < \infty$ is denoted by $L^1_T(\mathbb{R})$.

$$\|f\|_2 \equiv \sqrt{\frac{1}{T}\int_0^T |f(t)|^2\,\mathrm{d}t}$$

The space of all complex-valued T-periodic functions for which $\|f\|_2 < \infty$ is denoted by $L^2_T(\mathbb{R})$.

Note. **We identify functions that coincide a.e.**.

$$\|f\|_1 \leq \|f\|_2 \leq \|f\|_\infty \equiv \text{ess-sup}\{|f(x)| \mid x \in \mathbb{R}\}$$

$$C_T(\mathbb{R}) \equiv \{f \in C(\mathbb{R}) \mid f \text{ is } T\text{-periodic}\} \subset L^2_T(\mathbb{R}) \subset L^1_T(\mathbb{R})$$

$L^2_T(\mathbb{R})$ is an inner product space w.r.t.

$$(f,g) \equiv \frac{1}{T}\int_0^T f(t)\,\overline{g(t)}\,\mathrm{d}t \qquad (f,g \in L^2_T(\mathbb{R}))$$

For each $k \in \mathbb{Z}$, put $\phi_k(t) \equiv \exp(2\pi i t \frac{k}{T}) \quad (t \in \mathbb{R})$.

Theorem. The ϕ_k form an orthonormal system in $L^2_T(\mathbb{R})$:

$$(\phi_k, \phi_j) = 0 \quad \text{if } k \neq j \quad \text{and} \quad \|\phi_k\|_2 = 1 \quad (j,k \in \mathbb{Z})$$

Fourier series

For $f \in L^1_T(\mathbb{R})$, put

$$\gamma_k(f) \equiv \frac{1}{T}\int_0^T f(t) e^{-2\pi i t \frac{k}{T}}\,\mathrm{d}t \qquad (k \in \mathbb{Z})$$

$\gamma_k(f)$ is the **kth Fourier coefficient**. For $n \in \mathbb{N}$,

$$S_n(f)(t) \equiv \sum_{k=-n}^{n} \gamma_k\, e^{2\pi i t \frac{k}{T}} \qquad (t \in \mathbb{R})$$

$S_n(f)$ is the **nth partial Fourier series**.
The formal infinite sum is the **Fourier series of f**:

$$f \sim \sum \gamma_k\, e^{2\pi i t \frac{k}{T}}.$$

Note. This is not statement on convergence!

Use $\exp(2\pi i t \frac{k}{T}) = \cos(2\pi t \frac{k}{T}) + i \sin(2\pi t \frac{k}{T})$ for a formulation in sines and cosines.

Fourier series

For $f \in L^2_T(\mathbb{R})$,

$$\gamma_k(f) \equiv \frac{1}{T}\int_0^T f(t) e^{-2\pi i t \frac{k}{T}}\,\mathrm{d}t = (f, \phi_k)$$

$\gamma_k(f)$ is the **kth Fourier coefficient**. For $n \in \mathbb{N}$,

$$S_n(f)(t) \equiv \sum_{k=-n}^{n} \gamma_k\, e^{2\pi i t \frac{k}{T}} = \sum_{|k| \leq n} (f, \phi_k)\, \phi_k$$

Note that $\quad S_n(f) \in C^{(\infty)}(\mathbb{R})$.

L^2-Convergence

Theorem. $\|S_n(f) - f\|_2 \to 0 \quad (n \to \infty)$ if $f \in L^2_T(\mathbb{R})$.

Therefore, for $f \in L^2_T(\mathbb{R})$, in the L^2_T-sense, we have that

$$f = \sum_{k=-\infty}^{\infty} \gamma_k(f)\, e^{2\pi i t \frac{k}{T}}$$

What about $\|S_n(f) - f\|_\infty$ or $|S_n(f)(t) - f(t)|$ for $n \to \infty$?

$\exists f \in C_T(\mathbb{R})$ **and a** $t \in \mathbb{R}$ **for which** $(S_n(f)(t))$ **diverges**:

additional smoothness is required for stronger convergence.

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of bounded variation.

A function f on $\mathbb{R}$ is of **bounded variation** (BV) if it is a finite linear combination of non-decreasing functions.

Example. $f(t) \equiv |t|$ on $[-1, +1]$.

Example. If $f(t) = f(0) + \int_0^t g(s)\,\mathrm{d}s$ with $g \in L^1_T(\mathbb{R})$, f is **absolutely continuous** (AC), then f is of BV: .

Proof. $f(t) = f(0) + \int_0^t g(s)\,\mathrm{d}s = f(0) + f_+(t) - f_-(t)$ with

$$f_+(t) \equiv \int_0^t \max(g(s), 0)\,\mathrm{d}s \quad \text{and} \quad f_-(t) \equiv \int_0^t \max(-g(s), 0)\,\mathrm{d}s$$

Uniform Convergence

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of bounded variation.

Theorem. $\|\sigma_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ iff $f \in C_T(\mathbb{R})$

Here, $$\sigma_n(f) \equiv \frac{1}{n} \sum_{j=0}^{n-1} S_n(f)$$ **Césaro sum**

Example. $f(t) = \cos(2\pi t \frac{1}{T})$. Then,

$$S_n(f) = \frac{1}{2}(e^{-2\pi i t/T} + e^{2\pi i t/T}) = f \quad (n \geq 1),$$

whereas $\sigma_n(f) = \frac{n-1}{n} f \qquad (n \in \mathbb{N})$.

Point-wise Convergence

Theorem. $S_n(f)(t) \to f(t) \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$
and f is of BV on $[t - \delta, t + \delta]$ for some $\delta > 0$.

Theorem.
$S_n(f)(t) \to \frac{1}{2}[f(t+) + f(t-)] \quad (n \to \infty)$ if $f \in L^1_T(\mathbb{R})$
and f is of BV on $[t - \delta, t + \delta]$ for some $\delta > 0$.

Here, $$f(t+) \equiv \lim_{\varepsilon > 0,\, \varepsilon \to 0} f(t + \varepsilon)$$ $$f(t-) \equiv \lim_{\varepsilon > 0,\, \varepsilon \to 0} f(t - \varepsilon)$$

and we assume that the (essential) limits exist.

Note that, for $f \in L^1_T(\mathbb{R})$, $f(t)$ is not well-defined.

Theorem. $\|S_n(f) - f\|_2 \to 0 \quad (n \to \infty)$ if $f \in L^2_T(\mathbb{R})$.

Theorem. $\|S_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ if $f \in C_T(\mathbb{R})$ **and** f is of bounded variation.

Theorem. $\|\sigma_n(f) - f\|_\infty \to 0 \quad (n \to \infty)$ iff $f \in C_T(\mathbb{R})$

Theorem.
$S_n(f)(t) \to \frac{1}{2}[f(t+) + f(t-)] \quad (n \to \infty)$ if $f \in L^1_T(\mathbb{R})$ **and** f is of BV on $[t-\delta, t+\delta]$ for some $\delta > 0$.

There is an $f \in C_T(\mathbb{R})$ and a $t \in \mathbb{R}$ for which $(S_n(f)(t))$ diverges.

Error estimates

Theorem. If $f \in C_T(\mathbb{R})$ and $f' \in L^1_T(\mathbb{R})$ then

$$\gamma_k(f) = \frac{T}{2\pi i k}\gamma_k(f') \quad (k \in \mathbb{Z}, k \neq 0)$$

Theorem. $f \in L^1_T(\mathbb{R})$.

$|\gamma_k(f)| \leq \|f\|_1 \leq \|f\|_\infty$

$\gamma_k(f) \to 0$ if $|k| \to \infty$. (**Riemann–Lebesgue**)

$|\gamma_k(f)| \leq \frac{1}{|k|^\ell}(\frac{T}{2\pi})^\ell \|f^{(\ell)}\|_1$ if $f \in C^{(\ell)}_T(\mathbb{R})$

Theorem. $f \in C^{(\ell)}_T(\mathbb{R})$

$\|S_n(f) - f\|_\infty \leq \frac{1}{n^{\ell-1}}(\frac{T}{2\pi})^\ell \|f^{(\ell)}\|_1$

If $f \in C^{(1)}_T(\mathbb{R})$ then $2\pi i k\,\gamma_k(f) = T\,\gamma_k(f')$

Differential equations

Turn differential equations into algebraic equations.

With $f \in C_T(\mathbb{R})$, $a, b, c \in \mathbb{C}$, find a T-periodic u s.t.

$$a\,u'' + b\,u' + c\,u = f$$

Solution.

$$\begin{aligned}\gamma_k(f) &= a\,\gamma_k(u'') + b\,\gamma_k(u') + c\,\gamma_k(u) \\ &= [a(\tfrac{2\pi i k}{T})^2 + b\,\tfrac{2\pi i k}{T} + c]\,\gamma_k(u)\end{aligned}$$

What about boundary conditions?

Applications. Electric circuits.

Parseval. $$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |\gamma_k(f)|^2 \qquad (f \in L^2_T(\mathbb{R}))$$

Consider $\ell^2(\mathbb{Z}) \equiv \{(\gamma_k)_{k\in\mathbb{Z}} \mid \gamma_k \in \mathbb{C}, \|(\gamma_k)\|_2 \equiv \sum |\gamma_k|^2 < \infty\}$

with inner product $<(\gamma_k), (\mu_k)> \equiv \sum \gamma_k \overline{\mu_k}$.

Riesz–Fischer. The Fourier transform $f \rightsquigarrow (\gamma_k(f))_{k\in\mathbb{Z}}$ identifies the inner product spaces $L^2_T(\mathbb{R})$ and $\ell^2(\mathbb{Z})$.

In particular, $(f, g) = <(\gamma_k(f)), (\gamma_k(g))> \quad (f, g \in L^2_T(\mathbb{R}))$.

Proof. $(\gamma_k) \in \ell^2(\mathbb{Z})$ then $(\sum_{|k|<n} \gamma_k \phi_k)_n$ Cauchy sequence in $L^2_T(\mathbb{R})$.
$\|f + \zeta g\|_2^2 = \|f\|_2^2 + 2\mathrm{Re}(\zeta(f,g)) + \|g\|_2^2$ for all $f, g \in L^2_T(\mathbb{R})$, $\zeta \in \mathbb{C}$.
$\|(\gamma_k) + \zeta(\mu_k)\|_2^2 = \|(\gamma_k)\|_2^2 + 2\mathrm{Re}(\zeta<(\gamma_k), (\mu_k)>) + \|(\mu_k)\|_2^2$ for all ....
Now, apply Parseval and take $\zeta = 1$ and $\zeta = i$.

$L^1_T(\mathbb{R}) \overset{\gamma(\cdot)}{\to} \ell^\infty(\mathbb{Z})$, $\|\gamma(f)\|_\infty \le \|f\|_1$, not surjective

$L^2(\mathbb{R}) \overset{\gamma(\cdot)}{\to} \ell^2(\mathbb{Z})$, $\|\gamma(f)\|_2 = \|f\|_2$, inversion exists.

Here, $\ell^\infty(\mathbb{Z}) \equiv \{(\gamma_k) \mid \|(\gamma_k)\|_\infty < \infty\}$ and $\gamma(f) \equiv (\gamma_k(f))$.