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Fourier Transforms
Wavelets
Theory and Applications

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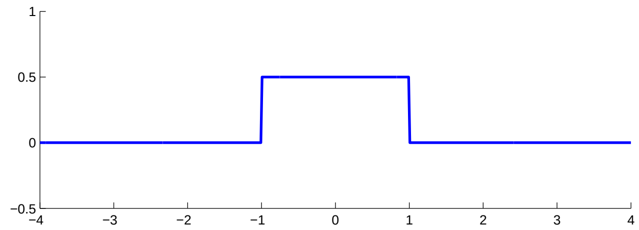
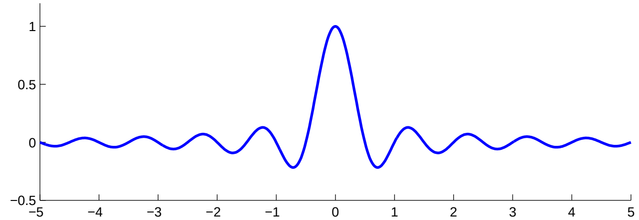
Fourier Integrals





Program

- Heuristic
- Fourier transform for L^1 functions
- Derivatives
- Fourier transform for L^2 functions
- Extensions
- Duality observations

$f : \mathbb{R} \to \mathbb{C}$ st $\|f\|_1 \equiv \int |f(t)|\,\mathrm{d}t = \int_{-\infty}^{+\infty} |f(t)|\,\mathrm{d}t < \infty.$

With $$\gamma_k^T \equiv \frac{1}{T}\int_{-T/2}^{T/2} f(t) e^{-2\pi i t \frac{k}{T}}\,\mathrm{d}t \qquad (k \in \mathbb{Z}),$$

and $f \in C^{(1)}(\mathbb{R})$, $$f(t) = \sum_{k\in\mathbb{Z}} \gamma_k^T e^{2\pi i t \frac{k}{T}} \qquad (|t| < T/2)$$

(restrict f to $[-T/2, T/2]$, extend T-periodic, use Th. 2.4.b)

With $$\widehat{f}(\omega) \equiv \int f(t) e^{-2\pi i t \omega}\,\mathrm{d}t$$

we have that $T\gamma_k^T \approx \widehat{f}(\frac{k}{T})$. Hence, (Riemann sum)

$$f(t) \approx \sum_{k\in\mathbb{Z}} \frac{1}{T}\widehat{f}(\frac{k}{T}) e^{2\pi i t \frac{k}{T}} \approx \int \widehat{f}(\omega) e^{2\pi i t \omega}\,\mathrm{d}\omega$$

Conjecture. $f(t) = \widehat{\widehat{f}}(-t)$.

$$f:\mathbb{R}\to\mathbb{C} \text{ st } \|f\|_1 \equiv \int |f(t)|\,\mathrm{d}t = \int_{-\infty}^{+\infty} |f(t)|\,\mathrm{d}t < \infty$$

$$\boxed{\widehat{f}(\omega) \equiv \int f(t)e^{-2\pi i t\omega}\,\mathrm{d}t \quad (\omega\in\mathbb{R})}$$

Theorem. $\|f\|_1 < \infty$

- $\widehat{f}$ is **bounded**: $\|\widehat{f}\|_\infty \le \|f\|_1$.
- $\widehat{f}$ is **uniformly continuous**:
  $\sup_\omega |\widehat{f}(\omega+\delta) - \widehat{f}(\omega)| \to 0$ if $\delta \to 0$.
- $\widehat{f}$ **vanishes at** ∞: $\widehat{f}(\omega)\to 0$ if $|\omega|\to\infty$.

$$L^1(\mathbb{R}) = \{f:\mathbb{R}\to\mathbb{C} \mid \|f\|_1<\infty\}, \text{ norm } \|\cdot\|_1$$

$$C_\infty(\mathbb{R}) = \{g\in C(\mathbb{R}) \mid g \text{ vanishes at } \infty\}, \text{ norm } \|\cdot\|_\infty.$$

$$f\in L^1(\mathbb{R}) \;\Rightarrow\; \widehat{f}\in C_\infty(\mathbb{R}) \quad \text{and} \quad \|\widehat{f}\|_\infty \le \|f\|_1$$

Corollary. $f\in L^1(\mathbb{R})$ with support bounded by T,
then $\widehat{f}$ is **analytic** on $\mathbb{R}$, i.e., $\widehat{f}\in C^{(\infty)}(\mathbb{R})$ and

$$\widehat{f}(\omega) = \sum_{k=0}^{\infty} \frac{\omega^k}{k!}\widehat{f}^{(k)}(0) \quad (\omega\in\mathbb{R}).$$

To be precise,
with (Taylor's theorem on Taylor series)

$$\widehat{f}(\omega) = \sum_{k=0}^{n-1} \frac{\omega^k}{k!}\widehat{f}^{(k)}(0) + \frac{\omega^n}{n!}\widehat{f}^{(n)}(\xi)$$

for some ξ in between 0 and ω,
we have that

$$\left|\frac{\omega^n}{n!}\widehat{f}^{(n)}(\xi)\right| \le \frac{(2\pi T\omega)^n}{n!}\|f\|_1 \to 0 \quad \text{if } n\to\infty.$$

Applications

- **Differential equations.**
- **Insight** Smoothness f relates to decrease $\widehat{f}$ at ∞
- **New concept of derivative.**

Differential equations.

See exercises.

Insight

First note that

$$f, tf, t^2 f, \dots, t^n f \in L^1(\mathbb{R}) \quad \Leftrightarrow \quad (1+|t|)^n f \in L^1(\mathbb{R}).$$

Therefore,

$(1+|t|)^n f \in L^1(\mathbb{R})$, then $\widehat{f} \in C^{(k)}(\mathbb{R})$ for $k = 0, \dots, n$.

$f, f', \dots, f^{(n)} \in L^1(\mathbb{R})$, then $(1+|\omega|)^n \widehat{f}$ bounded.

- *'Size' of f at ∞ determines smoothness of $\widehat{f}$.*
- *Smoothness of f determines 'size' of $\widehat{f}$ at ∞.*

$\widehat{\cdot}$ identifies $L^2(\mathbb{R})$ with $L^2(\mathbb{R})$ (see later): 'size' of f at ∞ corresponds to smoothness of $\widehat{f}$.

New concept of derivative.

For the moment (see later), assume that

$\widehat{\cdot}$ identifies $L^2(\mathbb{R})$ with $L^2(\mathbb{R})$.

If $(1+|\omega|)^n \widehat{f} \in L^2(\mathbb{R})$ then, $\forall\, k = 0, \dots, n$, $\omega^k \widehat{f} \in L^2(\mathbb{R})$ and $\exists\, g \in L^2(\mathbb{R})$ st $\widehat{g} = (2\pi i\omega)^k \widehat{f}$. Denote $f^{(k)} \equiv g$.

Consistent. If $f, \dots, f^{(k)} \in L^1(\mathbb{R})$, then $g = f^{(k)}$.

Let $\gamma > 0$. Suppose $(1+|\omega|)^\gamma \widehat{f} \in L^2(\mathbb{R})$.
Then, $\exists g \in L^2(\mathbb{R})$ st $\widehat{g} = (2\pi i\omega)^\gamma \widehat{f}$. Denote $f^{(\gamma)} \equiv g$.
$f^{(\gamma)}$ is a **pseudo (or fractional) derivative** of f.

$$H^{(\gamma)} \equiv \{f \mid (1+|\omega|)^\gamma \widehat{f} \in L^2(\mathbb{R})\}$$

is the **Sobolev space** of order γ.

$f: \mathbb{R} \to \mathbb{C}$ st $\|f\|_2 \equiv \sqrt{\int |f(t)|^2 \,\mathrm{d}t} < \infty$: $f \in L^2(\mathbb{R})$.

Note that $f_n \equiv f\Pi_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \quad (n \in \mathbb{N})$.

Lemma. $\|f_n\|_2 = \|\widehat{f_n}\|_2$ & $(\widehat{f_n})$ is Cauchy in $L^2(\mathbb{R})$.

$$\exists g \in L^2(\mathbb{R}) \text{ st } \|\widehat{f_n} - g\|_2 \to 0 \quad (n \to \infty).$$

Proposition. $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R}) \quad \Rightarrow \quad g = \widehat{f}$.

Definition. $\widehat{f} \equiv g$. **Plancherel.** $\|\widehat{f}\|_2 = \|f\|_2$.

For $f \in L^2(\mathbb{R})$, we also put $\widehat{f}(\omega) = \int f(t) e^{-2\pi i t\omega} \,\mathrm{d}\omega$.

Theorem. $f \in L^2(\mathbb{R})$ then $\|\widehat{f}\|_2 = \|f\|_2$, and

$$\widehat{f}(\omega) = \int f(t) e^{-2\pi i t\omega} \,\mathrm{d}\omega, \qquad f(t) = \int \widehat{f}(\omega)\, e^{+2\pi i t\omega} \,\mathrm{d}\omega.$$

Interpretation. $f(t) = \int \widehat{f}(\omega)\, e^{2\pi i t\omega} \,\mathrm{d}\omega$:

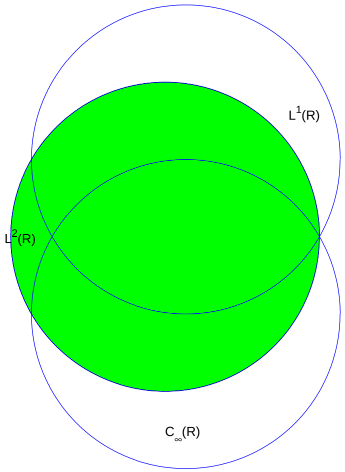
f is a superposition of harmonic oscilations:

with $\widehat{f}(\omega) = |\widehat{f}(\omega)|\, e^{2\pi i \phi(\omega)}$,

$|\widehat{f}(\omega)|$ is the **amplitude** of the oscilation
with **frequency** ω,
$\phi(\omega)$ is the **phase**.

$L^1(\mathbb{R}) \xrightarrow{\widehat{\ }} C_\infty(\mathbb{R})$, $\quad \|\widehat{f}\|_\infty \leq \|f\|_1$, $\quad$ not surjective

$L^2(\mathbb{R}) \xrightarrow{\widehat{\ }} L^2(\mathbb{R})$, $\quad \|\widehat{f}\|_2 = \|f\|_2$, $\quad$ inversion exists.



Interpretation. $f(t) = \int \widehat{f}(\omega)\, e^{2\pi i t\omega}\, \mathrm{d}\omega$:

f is a superposition of harmonic oscilations:

with $\qquad \widehat{f}(\omega) = |\widehat{f}(\omega)|\, e^{2\pi i t\phi(\omega)}$,

$|\widehat{f}(\omega)|$ is the **amplitude** of the oscilation
with **frequency** ω,
$\phi(\omega)$ is the **phase**.

Let $\nu \in \mathbb{R}$ be a frequency. Can the function ϕ_ν, with

$$\phi_\nu(t) \equiv e^{2\pi i t\nu} \qquad (t \in \mathbb{R})$$

be viewed as a superposition of harmonic oscilations?

Can $\phi_\nu(t) \equiv e^{2\pi i t\nu}$ be viewed as a superpos. of harm. osc.?

The Dirac δ function

$$e^{2\pi i t\nu} = \int \delta_\nu(\omega)\, e^{2\pi i t\omega}\, \mathrm{d}\omega \qquad (t \in \mathbb{R})$$

Here δ_ν is the **Dirac δ function** or **point measure** at ν defined by the following two properties:

$$\delta_\nu(\omega) = 0 \text{ for all } \omega \neq \nu \quad \text{and}$$
$$\int \delta_\nu(\omega)\, g(\omega)\, \mathrm{d}\omega = g(\nu) \quad (g \in C(\mathbb{R})).$$

δ_ν can be view as some **weak limit** of, e.g., $\frac{1}{2\varepsilon}\Pi_\varepsilon$ for $\varepsilon \to 0$.

In some sense $\widehat{\phi_\nu} = \delta_\nu$ and $\phi_\nu(t) = \widehat{\delta_\nu}(-t)$.

Application of the Dirac δ-function.

Suppose f is $C^{(1)}$ on both $(-\infty, \tau)$ and (τ, ∞) and $f(\tau+)$ and $f(\tau-)$ exists. Then, with $\alpha \equiv f(\tau+) - f(\tau-)$,

$$f(t) = f(0) + \int_0^t \big(f'(s) + \alpha\delta_\tau(s)\big)\, \mathrm{d}s \qquad (t \in \mathbb{R}).$$

The function $f' + \alpha\delta_\tau$ can be viewed as the derivative of f.

Exercise. Consider the approximate derivatives $\partial_{\Delta t} f$:

$$\partial_{\Delta t} f(t) \equiv \frac{f(t+\Delta t) - f(t-\Delta t)}{2\Delta t}.$$

Show that the behaviour for $(\partial_{\Delta t} f)$ for $\Delta t \to 0$ is consistent with the point of view that $f' + \alpha\delta_\tau$ is the derivative of f and the definition of δ_τ. Pay special attention to t's for which $\tau \in (t - \Delta t, t + \Delta t)$

Application of the Dirac δ-function.

Exercise. For $\lambda \in \mathbb{C}$, Re$(\lambda) \neq 0$,

consider the differential equation

$$f'(t) = \lambda f(t) \ (t \in \mathbb{R},\, t \neq 0), \quad f(0-) = 0,\ f(0+) = 1$$

- Solve this eq. for an $f \in L^2(\mathbb{R})$ (if exist).
- Is the eq. equivalent to

$$f \in L^2(\mathbb{R}) \quad \text{st} \quad f' = \lambda f + \delta_0$$

- Use Fourier transform to show that

$$\widehat{f}(\omega) = \frac{1}{2\pi i\omega - \lambda} \quad (\omega \in \mathbb{R})$$

- Discuss the situation for Re$(\lambda) < 0$ and Re$(\lambda) > 0$.

$f \in L^2(\mathbb{R})$.

Energy:

$$E \equiv \int |f(t)|^2 \,\mathrm{d}t = \int |\widehat{f}(\omega)|^2 \,\mathrm{d}\omega.$$

Energy center:

$$t_0 \equiv \tfrac{1}{E} \int t|f(t)|^2 \,\mathrm{d}t, \qquad \omega_0 \equiv \tfrac{1}{E} \int \omega|\widehat{f}(\omega)|^2 \,\mathrm{d}\omega.$$

Spread:

$$\sigma_t^2 \equiv \tfrac{1}{E} \int (t-t_0)^2|f(t)|^2 \,\mathrm{d}t, \quad \sigma_\omega^2 \equiv \tfrac{1}{E} \int (\omega-\omega_0)^2|\widehat{f}(\omega)|^2 \,\mathrm{d}\omega.$$

Heisenberg uncertainty principle.

$$\sigma_t\sigma_\omega \geq \tfrac{1}{4\pi}.$$

$$\sigma_t\sigma_\omega = \tfrac{1}{4\pi} \ \Leftrightarrow \ f(t) = c\, e^{\gamma(t-t_0)^2} \ (t \in \mathbb{R})$$

Duality

f	$\Rightarrow$	$\widehat{f}$
real		even
even		real
smooth		rapid decrease at ∞
rapid decrease at ∞		smooth
localized		spread out