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Fourier Transforms
Wavelets
Theory and Applications

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Program

- Convolution products
- Correlation
- Radar

$$f * h(t) = \int_{-\infty}^{+\infty} f(t-s)h(s)\,\mathrm{d}s$$

Well-defined if $f \in L^p(\mathbb{R}), h \in L^q(\mathbb{R})$ $(p, q \in [1, \infty],\ \frac{1}{p} + \frac{1}{q} = 1)$

Then $\|f * h\|_\infty \leq \|f\|_p\,\|h\|_q$ and $f * h$ is uniformly continuous.

Application. Narrow Gaussians,

$$h(x, y) = \frac{1}{\delta} e^{-\pi \frac{1}{\delta^2}(x^2+y^2)},$$

are used to smooth ('**denoise**') pictures:

$$f_{\text{denoised}} = (f + n) * h$$

(using the 2d-variant of the convolution product).

The idea here is the $n * h \approx 0$ if n is noise.

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Then $\|f * h\|_\infty \leq \|f\|_p\,\|h\|_q$ and $f * h$ is uniformly continuous.

Application. With $B_0(x) \equiv 1$ if $x \in [0, 1]$, $B_0(x) \equiv 0$ elsewhere, define

$$B_k \equiv B_0 * B_{k-1} \qquad (k \in \mathbb{N})$$

[Ex.6.5]

Theorem. For all $k \in \mathbb{N}$:

- $B_k \in C^{(k-1)}(\mathbb{R})$
- On $[j, j+1]$, B_k is a polynomial of degree k $(j \in \mathbb{Z})$.
- $B_k(x) > 0$ for all $x \in (0, k+1)$
- $B_k(x) = 0$ for all $x \notin (0, k+1)$.

The B_k are **basis splines** or **Box splines** of degree k.

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Then $\|f*h\|_\infty \leq \|f\|_p \, \|h\|_q$ and $f*h$ is uniformly continuous.

Notation. $f_s(t) \equiv f(t-s)$; s is a **delay**. $f^\top(t) = \overline{f(-t)}$.

Prop. $f * h(t) = (h, f_t^\top)$, $\|f\|_p = \|\overline{f}\|_p = \|f_t\|_p = \|f^\top\|_p$.

$$(f * h, g) = (h, f^\top * g).$$

$f \in L^p, h \in L^1$. Then $\|f * h\|_p \leq \|h\|_1 \, \|f\|_p$.

Theorem. $f, h \in L^1(\mathbb{R}) \cup L^2(\mathbb{R})$. Then

$$\widehat{(f * h)} = \widehat{f} \cdot \widehat{h}$$

Application. h and $f * h$ are known. Construct f.

Solution. In principle

$$\widehat{f} = \frac{\widehat{f * h}}{\widehat{h}}.$$

Discussion. The received signal $f * h$ (and h?) will be affected by noise: received $f * h + n$.

Remedy. **(Tikhonov) Regularise**: for some appropriate **regularisation parameter** τ (which one?)

$$f^{\mathrm{r}} \equiv \mathrm{armin}_g \left(\|g * h - [f * h + n]\,\|_2^2 + \tau \|g\|_2^2 \right)$$

(and combine with filtering).

Application.

$$(h, f_t) = \int \overline{f(s-t)}\, h(s)\,\mathrm{d}s = f^\top * h(t)$$

The map $f \odot h(t) \equiv (h, f_t)$
is called the **correlation product** of f and h:

it tests how much h is correlated to a shifted variant of f.

Note that the correlation product is the *adjoint* of the convolution product:

$$(f * g, h) = (g, f^\top * h).$$

Wiener-Khintchini Theorem.

$$(f \odot h)^\wedge = \widehat{h}\,\overline{\widehat{f}}, \qquad (f \odot f)^\wedge = |\widehat{f}|^2.$$