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Fourier Transforms
Wavelets
Theory and Applications

Gerard Sleijpen

Universiteit Utrecht
Department of Mathematics

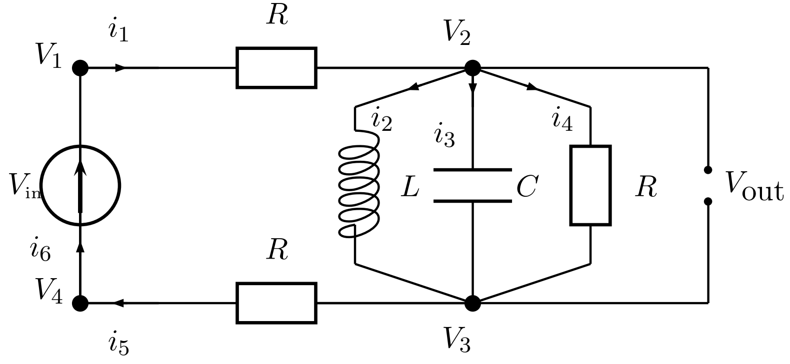
http://www.staff.science.uu.nl/∼sleij101/

Program

- Electronic Circuits
- MRI
- Diffraction
- CT

Electronic circuits

Example.



Larger examples in computer chips, with up to $5\,10^8$ electronic components (2011: Intel's dual-core i5).

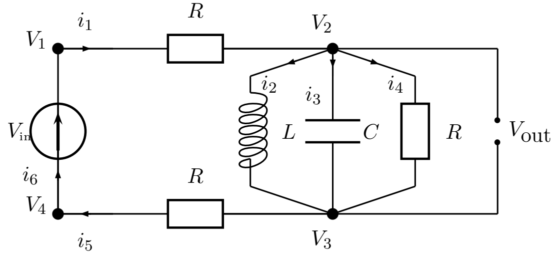
Definition. A **directed graph** is a collection of vertices (v_i) (points) and edges (e_j) (lines) with a direction.

An electronic network can be described by a directed graph, where each edge contains exactly one electronic component, as a resistor, capacitor, inductor, etc.. At vertex v_i we have Voltage V_i, in edge e_j is an electrical current i_j, with i_j positive if the current is in line with the direction of edge e_j and negative if it is in opposite direction.

One way of describing a directed graph is by an
Definition. Incidence matrix G:
- the ith row of **G** corresponds to the ith vertex v_i;
- the jth column of G corresponds to the jth edge e_j;
- if edge i_j connects v_k and v_ℓ with v_k first, then G has value $+1$ at entry (k, j) and -1 at (ℓ, j), while all other entries in the jth column have value 0.

We collect the voltages in a vector **V**, and the currents in a vector **i**, with vector indices corresponding to the index of the vertices and edges, respectively.

Example.



$$\mathbf{G} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ -1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}$$

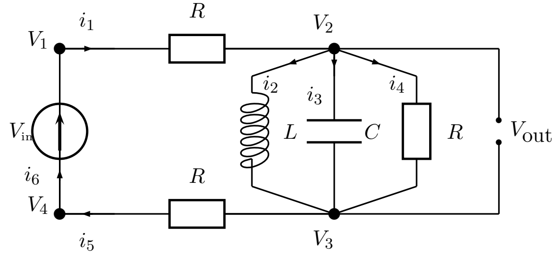
The size of the incidence matrix is $k \times n$, where k is the number of vertices and n the number of edges, that is, of electronic components. $k = 4$ and $n = 6$ in the present example. $n \approx 5\,10^8$ in Intels dual-core i5.

Kirchhoff's laws

$\mathbf{Gi} = \mathbf{0}$ expresses **Kirchhoff's law of currents** stating that the inflow of the currents at a vertex equals the outflow at that vertex.

Kirchhoff's law of voltages is automatically fulfilled. This law states that in any closed loop (sub-circuit) the sum of the voltage differences is 0.

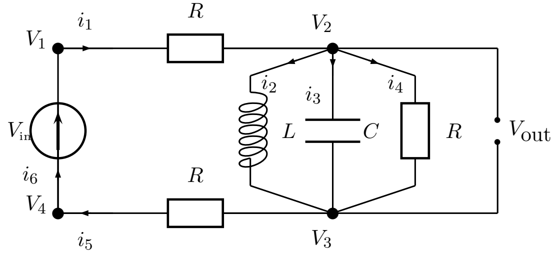
Example.



$(V_1 - V_2) + (V_2 - V_3) + (V_3 - V_4) + (V_4 - V_1) = 0,$
$(V_2 - V_3) + (V_3 - V_2) = 0, \ \ldots$

Note that $\mathbf{G}^\top\mathbf{V}$ is the vector of Voltage differences across the edges.

Example.

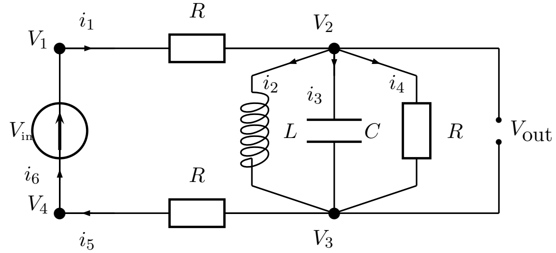


$$\mathbf{G}^\top\mathbf{V} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = \begin{bmatrix} V_1 - V_2 \\ V_2 - V_3 \\ V_2 - V_3 \\ V_2 - V_3 \\ V_3 - V_4 \\ V_4 - V_1 \end{bmatrix}.$$

Note. $\mathbf{G}$ is not of full rank: $\text{rank}(\mathbf{G}) = k - 1$. This follows from the fact that $\mathbf{G}^\top\mathbf{1} = \mathbf{0}$: The value of $\mathbf{G}^\top\mathbf{V}$ does not change by adding the same constant to all V_i.

Note that $\mathbf{G}^\top\mathbf{V}$ is the vector of Voltage differences across the edges.

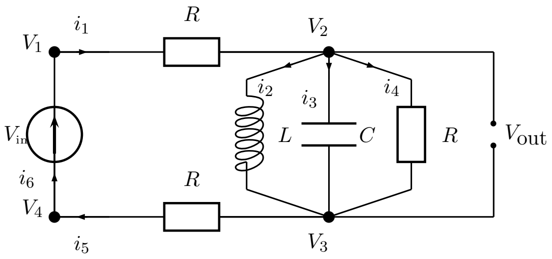
Example.



$$\mathbf{G}^\top\mathbf{V}' = \begin{bmatrix} V_1' - V_2' \\ V_2' - V_3' \\ V_2' - V_3' \\ V_2' - V_3' \\ V_3' - V_4' \\ V_4' - V_1' \end{bmatrix} = \begin{bmatrix} R_1\, i_1' \\ L_2\, i_2'' \\ \frac{1}{C_3}\, i_3 \\ R_4\, i_4' \\ R_5\, i_5' \\ R_6\, i_6' \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ V_{\text{in}}' \end{bmatrix}$$

Note that $\mathbf{G}^\top\mathbf{V}$ is the vector of Voltage differences across the edges.

Example.



$$\mathbf{G}^\top\mathbf{V}' = \mathbf{R}\mathbf{I}' + \widetilde{\mathbf{C}}\mathbf{I} + \mathbf{L}\mathbf{I}'' + \mathbf{e}\,u, \qquad \text{where}$$

$$\mathbf{R} \equiv \begin{bmatrix} R_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & R_6 \end{bmatrix}, \quad \widetilde{\mathbf{C}} \equiv \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{C_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{L} = \dots$$

The currents and the voltages in the electronic network satisfy the relations

$$\begin{cases} \mathbf{G}\,\mathbf{i} = \mathbf{0} \\ \mathbf{G}^\top\mathbf{V}' = \mathbf{R}\,\mathbf{i}' + \widetilde{\mathbf{C}}\,\mathbf{i} + \mathbf{L}\,\mathbf{i}'' + \mathbf{e}\,u \end{cases}$$

or, with $\mathbf{J} \equiv \mathbf{i}'$, we can turn the second order differential equation into two coupled first order differential equations:

$$\begin{cases} \mathbf{G}\,\mathbf{i}' = \mathbf{0} \\ \mathbf{G}^\top\mathbf{V}' - \mathbf{R}\,\mathbf{i}' - \mathbf{L}\,\mathbf{J}' = \widetilde{\mathbf{C}}\,\mathbf{i} + \mathbf{e}\,u \\ \mathbf{i}' = \mathbf{J} \end{cases}$$

Combine these three relations into one first order diff.eq.

$$\begin{bmatrix} \mathbf{0} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^\top & -\mathbf{R} & -\mathbf{L} \\ \mathbf{0} & \mathrm{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix}' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \\ \mathbf{0} \end{bmatrix} u$$

Here, a $\mathbf{0}$ in the block matrices represent a matrix of zeros of matching size, a $\mathbf{0}$ in the block vector is a vector of appropriate size, I is the $n \times n$ identity matrix.

The currents and the voltages in the electronic network satisfy the relation

$$\begin{bmatrix} \mathbf{0} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^\top & -\mathbf{R} & -\mathbf{L} \\ \mathbf{0} & \mathrm{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix}' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \\ \mathbf{0} \end{bmatrix} u$$

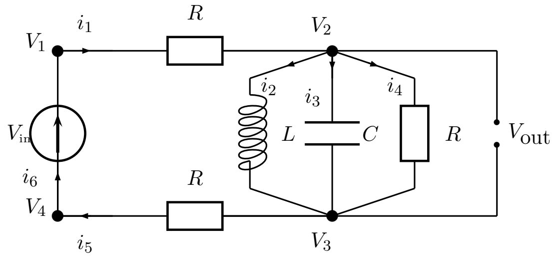
Uniqueness. If we have a solution, then adding a constant to the Voltages at all vertices (the same constant) is also a solution. We therefore, fix one of the Voltages to 0 (i.e., connect that vertex to the earth).
We incorporate this scaling into the model by replacing the $k \times k$ left upper block of $\mathbf{0}$ in the matrix at the left by $\mathbf{E}$, a $k \times k$ matrix of zeros except at the diagonal position (ℓ, ℓ) where $\mathbf{E}$ has entry 1. This means that we fix V_ℓ to 0.
Note that this does not affect the values of the i_j: because, since $\mathbf{G}$ does not have full rank, the other rows (other than the ℓth) determine the values of the i_j.

The currents and the voltages in the electronic network satisfy the relation $\quad \mathbf{B}\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{b}\,u, \quad$ where

$$\mathbf{B} \equiv \begin{bmatrix} \mathbf{E} & \mathbf{G} & \mathbf{0} \\ \mathbf{G}^\top & -\mathbf{R} & -\mathbf{L} \\ \mathbf{0} & \mathrm{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{A} \equiv \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{C}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{I} \end{bmatrix}, \quad \mathbf{x} \equiv \begin{bmatrix} \mathbf{V} \\ \mathbf{i} \\ \mathbf{J} \end{bmatrix}, \quad \mathbf{b} \equiv \begin{bmatrix} \mathbf{0} \\ \mathbf{e} \\ \mathbf{0} \end{bmatrix}.$$

The output

Example.



With $\mathbf{c}^\top \equiv (0, 1, -1, 0, \mathbf{0}^\top, \mathbf{0}^\top)^\top$ we have that

$$\mathbf{c}^\top\mathbf{x} = V_2 - V_3$$

Here $\mathbf{0}$ is the n-vector of zeros.

Dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\, u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

$k+2n$ is the **number of states** or **order** of the system,

$t \rightsquigarrow \mathbf{x}(t)$ is the **state** of the system,

$\mathbf{b}$ is the **input** or **control** vector, $\mathbf{c}$ is the **output** vector,

$t \rightsquigarrow u(t)$ is the **control function**,

$t \rightsquigarrow y(t)$ is the **output of the system**.

Dynamical system

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$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\, u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

Theorem. If $u \in L^2(\mathbb{R})$, then $y(t) = \int H(\omega)\, \widehat{u}(\omega) e^{2\pi i \omega t}\, \mathrm{d}\omega$

with $$H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i \omega \mathbf{B})^{-1}\mathbf{b}$$

H is the **response** or **transfer** function. It describes the response of the system to an harmonic oscillation (at the input). The amplitude (at the output) of such an oscillation with frequency ω is amplified with $|H(\omega)|$ and the **phase is shifted** by $\phi(\omega)$ with $\phi(\omega) \in [0, 2\pi)$ such that

$$H(\omega) = |H(\omega)|\, e^{i\phi(\omega)}.$$

Dynamical system

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$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\, u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

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H is the **response** or **transfer** function.

The graph of $\omega \rightsquigarrow |H(\omega)|$ ($\omega \in [0, \infty)$ along the horizontal axis, $|H(\omega)|$ along the vertical axis on **Decibel scale** (Db), i.e., $20\log_{10}$-scale) is called the **Bode plot** of the transfer function.

The curve in the complex plain described by $\omega \rightsquigarrow H(\omega)$ also gives useful information. Note that a point on this curve does not reveal the corresponding value of ω: it relates $|H(\omega)|$ to $\phi(\omega)$.

Stability of dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\, u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t) \end{cases}$$

Consider an **eigenpair** $(\lambda, \mathbf{v})$ of the matrix pair $(\mathbf{A}, \mathbf{B})$:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{B}\mathbf{v}.$$

Suppose that at time t_0 the solution $\mathbf{x}$ is perturbed by $\varepsilon\mathbf{v}$, i.e., $\widetilde{\mathbf{x}}$ satisfies

$$\begin{cases} \mathbf{B}\widetilde{\mathbf{x}}'(t) = \mathbf{A}\widetilde{\mathbf{x}}(t) + \mathbf{b}\, u(t), \\ \widetilde{\mathbf{x}}(t) = \mathbf{x} \text{ for } t < t_0, \\ \widetilde{\mathbf{x}}(t_0) = \mathbf{x}(t_0) + \varepsilon\mathbf{v}. \end{cases}$$

Then, the error $\mathbf{e} \equiv \widetilde{\mathbf{x}} - \mathbf{x}$ satisfies

$$\mathbf{B}\mathbf{e}' = \mathbf{A}\mathbf{e} \quad \text{and} \quad \mathbf{e}(t_0) = \varepsilon\mathbf{v}$$

Hence, $\mathbf{e}(t) = \varepsilon e^{\lambda t}\mathbf{v}$ for $t \geq t_0$.

Stability of dynamical system

We have to solve a **control system** (**dynamical system**)

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t) \end{cases}$$

Consider an **eigenpair** $(\lambda, \mathbf{v})$ of the matrix pair $(\mathbf{A}, \mathbf{B})$:

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{B}\mathbf{v}.$$

The system is **stable** if all eigenvalues of $(\mathbf{A}, \mathbf{B})$ are in $\mathbb{C}^- \equiv \{\lambda \in \mathbb{C} \mid \mathrm{Re}(\lambda) < 0\}$, the left half of the complex plane.

Then, all singularities of $\lambda \rightsquigarrow \mathbf{c}^{\mathsf{T}}(\mathbf{A} - \lambda\mathbf{B})^{-1}\mathbf{b}$ are in $\mathbb{C}^-$.

The transfer function

The transfer function of the dynamical system

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

is given by $\quad H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b} \qquad (\omega \in \mathbb{R}).$

Properties.

- $k + 2n$ is huge
- $\mathbf{A}$ and $\mathbf{B}$ are sparse (only a few non-zeros in all rows).
- $\mathbf{A}$ and $\mathbf{B}$ are general matrices (not symmetric, ...).
- The differences in the coefficients R_i, C_i and L_i can be many order of magnitudes.

The transfer function

The transfer function of the dynamical system

$$\begin{cases} \mathbf{B}\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\,u(t), \\ V_{\text{out}}(t) = y(t) \equiv \mathbf{c}^{\mathsf{T}}\mathbf{x}(t). \end{cases}$$

is given by $\quad H(\omega) \equiv \mathbf{c}^{\mathsf{T}}(\mathbf{A} - 2\pi i\omega\mathbf{B})^{-1}\mathbf{b} \qquad (\omega \in \mathbb{R}).$

Computational challenges • $N \equiv k + 2n$ is huge ($\approx 10^9$).
• $H(\omega)$ has to be computed for a large range of ω.
• The transfer function has to be computed for several (related) matrices $(\mathbf{A}, \mathbf{B})$ (in the design stage).
• Practical systems contain not only **passive** elements, like resistors, capacitors, and inductors, but also many **active** components (doides), which turn the problem into a non-linear one.
• Practical system do not have only one **S**ingle **I**nput vector and a **S**ingle **O**utput vector (SISO system), but they have multiple inputs and multiple outputs (MIMO):
$\mathbf{b}$ is $N \times \ell$, $\mathbf{c}$ is $N \times \ell'$.

Computerised Tomography

X-rays are transmitted from a straight line (the red beam in the picture) through an object, a slab of material (the yellow and black figure). The material partly 'absorbs' the x-rays. The intensity of the x-rays is measured at the detector (the green line parallel to the red line).

The detector is constructed to measure the intensity of those beams that pass straight through the object (scattered beams will not be detected).

The absorption depends on the kind of material and on the thickness of the slab of material.

If a x-ray with initial intensity I_0 travels through d cm of homogeneous material with absorption coefficient κ, then the measured intensity I equals

$$I = I_0\, e^{-\kappa d}.$$

Use Cartesian coordinates (x, y) to describe the scanner.

Suppose the absorption coefficient at point (x, y) of the object to be scanned is $f(x, y)$. The value of f at (x, y) depends on the (concentration of the) material at (x, y) of which the object is composed.

Consider an x-ray that travels along a line orthogonal to the detector: this is a line of points (x, y) with

$$x = x(\eta) = \xi \cos(\phi) - \eta \sin(\phi), \quad y = y(\eta) = \xi \sin(\phi) + \eta \cos(\phi)$$

with ξ fixed and ϕ the angle of the detector with x-axis (the dashed line in the picture).

We therefore, can measure

$$p_\phi(\xi) \equiv \int f(x(\eta), y(\eta))\, d\eta.$$

Put $c_\phi \equiv \cos(\phi)$ and $s_\phi = \sin(\phi)$. With

$$x(\eta) \equiv \xi\, c_\phi - \eta\, s_\phi, \quad y(\eta) \equiv \xi\, s_\phi + \eta\, c_\phi$$

we obtain the value $p_\phi(\xi)$ from measurements, where

$$p_\phi(\xi) \equiv \int \quad f(x(\eta), y(\eta))\, d\eta \qquad .$$

Assignment.
Given $p_\phi(\xi)$ for all $\xi \in \mathbb{R}$ and all $\phi \in [0, 2\pi)$, compute f.

With $p(\xi, \phi) \equiv p_\phi(\xi)$,

the map $f \rightsquigarrow p$ is the **Radon transformation** of f,

the graph of p as a 2-d picture is **the sinogram** of f.

To obtain a sharp reconstruction,
we use Fourier transforms.

$$\widehat{f}(\omega_1, \omega_2) = \iint f(x, y)\, e^{-2\pi i (x\omega_1 + y\omega_2)}\, dx\, dy.$$

Rotate the coordinates in both (x, y)-plane as well as in (ω_1, ω_2)-plane:

$$\begin{cases} x = \xi\, c_\phi - \eta\, s_\phi, & y = \xi\, s_\phi + \eta\, c_\phi \\ \omega_1 = \rho_1 c_\phi - \rho_2 s_\phi, & \omega_2 = \rho_1 s_\phi + \rho_2 c_\phi. \end{cases}$$

Then

$$\widehat{f}(\rho_1 c_\phi - \rho_2 s_\phi, \rho_1 s_\phi + \rho_2 c_\phi) = \iint f(\xi c_\phi - \eta s_\phi, \xi s_\phi + \eta c_\phi)\, e^{-2\pi i (\xi\rho_1 + \eta\rho_2)}\, d\eta\, d\xi.$$

In particular, if $\rho_2 = 0$ and putting $\rho \equiv \rho_1$

$$\begin{aligned} \widehat{f}(\rho c_\phi, \rho s_\phi) &= \textstyle\iint f(\xi c_\phi - \eta s_\phi, \xi s_\phi + \eta c_\phi)\, e^{-2\pi i \xi\rho}\, d\eta\, d\xi \\ &= \textstyle\int \left(\int f(\xi c_\phi - \eta s_\phi, \xi s_\phi + \eta c_\phi)\, d\eta \right) e^{-2\pi i \xi\rho}\, d\xi \\ &= \textstyle\int p_\phi(\xi)\, e^{-2\pi i \xi\rho}\, d\xi = \widehat{p}_\phi(\rho). \end{aligned}$$

Theorem. $\widehat{f}(\rho c_\phi, \rho s_\phi) = \widehat{p}_\phi(\rho) \quad (\rho \in \mathbb{R}, \phi \in [0, \pi))$.

Note. The point $(\rho c_\phi, \rho s_\phi)$ represents an arbitrary point in (ω_1, ω_2)-plane in **polar coordinates**.

We therefore express the Fourier back transform

$$f(x,y) = \iint \widehat{f}(\omega_1, \omega_2)\, e^{2\pi i (x\omega_1 + y\omega_2)}\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2.$$

into polar coordinates:

$$\begin{cases} (x,y) = (rc_\theta, rs_\theta) \\ (\omega_1, \omega_2) = (\rho c_\phi, \rho s_\phi) \end{cases}$$

Then

$$\begin{aligned} f(rc_\theta, rs_\theta) &= \int_0^\pi \int_{-\infty}^{+\infty} \widehat{f}(\rho c_\phi, \rho s_\phi)\, e^{2\pi i \rho (rc_{\theta-\phi})} |\rho|\, \mathrm{d}\rho\, \mathrm{d}\phi \\ &= \int_0^\pi \int_{-\infty}^{+\infty} \widehat{p}_\phi(\rho)\, e^{2\pi i \rho (rc_{\theta-\phi})} |\rho|\, \mathrm{d}\rho\, \mathrm{d}\phi \end{aligned}$$

With

$$\begin{cases} (x,y) = (rc_\theta, rs_\theta) \\ (\omega_1, \omega_2) = (\rho c_\phi, \rho s_\phi) \end{cases}$$

we have

Theorem. With $\widetilde{p}_\phi(\xi) \equiv \int |\rho|\, \widehat{p}_\phi(\rho)\, e^{2\pi i \rho \xi}\, \mathrm{d}\rho$, we have that

$$f(rc_\theta, rs_\theta) = \int_0^\pi \widetilde{p}_\phi(rc_{\theta-\phi})\, \mathrm{d}\phi$$

Interpretation. The multiplication of $\widehat{p}_\phi(\rho)$ by $|\rho|$ act as a **filter**, damping low frequency components ($\rho \approx 0$) and amplifying high frequency ones.

f is obtained as a **filtered back-projection**, i.e., the **BP** of the filtered Fourier transform of the Radon transformed p_ϕ.

Recall that the **BP** without filtering (i.e., **BP** of p_ϕ, rather than of $\widetilde{p}_\phi$) leads to a blurred version of f. This can be viewed as an over estimation of low frequency components. The filtering by $|\rho|$ seems to correct this.

CT and Fourier transforms

Theorem. With $\widetilde{p}_\phi(\xi) \equiv \int |\rho|\, \widehat{p}_\phi(\rho)\, e^{2\pi i \rho \xi}\, \mathrm{d}\rho$, we have that

$$f(rc_\theta, rs_\theta) = \int_0^\pi \widetilde{p}_\phi(rc_{\theta-\phi})\, \mathrm{d}\phi$$

Summary.

The statement in the theorem involves
1) a 1-dimensional Fourier transform (FT) (to make $\widehat{p}_\phi$),
2) a filter operation in frequency space,
3) a 1-d inverse FT and
4) BP.

The proof exploits 2-d FT, switching between
- Cartesian coordinates,
- rotated Cartesian coordinates, and
- polar coordinates.