Computer session I: Convergence of Fourier series

I.A Introduction

In this exercise, $f : \mathcal{I} \to \mathbb{R}$ is bounded and integrable on the bounded interval $\mathcal{I} \equiv [a, b]$. We assume f to be periodically extended to $\mathbb{R}$ with period $T \equiv b - a$:

$$f(t) = f(t + kT) \qquad (k \in \mathbb{Z}, t \in \mathbb{R}).$$

For $k \in \mathbb{Z}$, $\gamma_k(f)$ is the kth Fourier coefficient of f, and $S_n(f)$ is the nth partial Fourier series ($n \in \mathbb{N}_0$) as defined in §2.2. We are interested in the approximation quality of $S_n(f)$ (cf. §2.4).

We will also consider the best sup-norm approximation $P_n(f)$ of f of the form

$$P_n(f)(t) = \sum_{k=-n}^{n} \mu_k \, e^{2\pi i t \frac{k}{T}} \qquad (t \in \mathbb{R}),$$

with $\mu_k \in \mathbb{C}$. $P_n(f)$ is a so-called *trigonometric polynomial* of degree n.

We put

$$F_n(f) \equiv \|f - S_n(f)\|_\infty \quad \text{and} \quad E_n(f) \equiv \|f - P_n(f)\|_\infty \qquad (n \in \mathbb{N}_0).$$

For some $N \in \mathbb{N}$, we will also consider the *discrete Fourier coefficients* $\widetilde{\gamma}_k(f)$,[1] for $|k| \leq N$, and the nth partial discrete Fourier series $\widetilde{S}_n(f)$, for $n \in \mathbb{N}, n \leq N$. These quantities are given by (cf. §4)

$$\widetilde{\gamma}_k(f) \equiv \frac{1}{N} \sum_{n=0}^{N-1} f(t_n) \, e^{-2\pi i \frac{nk}{N}} \quad \text{and} \quad \widetilde{S}_n(f)(t_m) \equiv \sum_{k=-n}^{n} \widetilde{\gamma}_k(f) \, e^{2\pi i t_m \frac{k}{T}}. \tag{1}$$

Notation. For $\mu, \nu : \mathbb{Z} \to \mathbb{R}$ we write

$$\mu \sim \nu \quad \text{if} \quad \lim_{|k| \to \infty} \frac{\mu(k)}{\nu(k)} \quad \text{exists and is non-zero:}$$

then (μ_k) and (ν_k) have the same asymptotic behaviour for $|k| \to \infty$.

In this exercise, we study the relation between the 'speed' by which the sequence $(\gamma_k(f))$ of Fourier coefficients approaches 0 if $|k| \to \infty$ and the smoothness of f. For instance, we will try to find out when

$|\gamma_k(f)| \sim k^a$ for some $a < 0$,
$|\gamma_k(f)| \sim \lambda^k$ for some $\lambda \in [0, 1)$, or
$|\gamma_k(f)| \sim \frac{1}{k!}$.

For this purpose, we try to find a function $\mu : \mathbb{C} \times \mathbb{Z} \to \mathbb{R}$ such that $\lim_{|k| \to \infty} \mu(\gamma_k(f), k)$ exists and is non-zero: if, for instance, $|\gamma_k(f)| \sim k^{-2}$, then the choice $\mu(\gamma, k) = k^2 \gamma$, implies that $\mu(\gamma_k(f), k) \sim 1$, with $\mu(\gamma, k) = \log(\gamma)/k$ we have that $\mu(\gamma_k(f), k) \sim 1$ if $\gamma_k(f) \sim \lambda^k$. We employ scaling functions μ to make the asymptotic behaviour of $(\gamma_k(f))$ better visible in graphical displays.

[1] Actually the **Matlab** code that we use does not compute the Fourier coefficients $\gamma_k(f)$ but a discrete variant: it computes the discrete Fourier coefficients for $N = 2^l > 8n + 100$, where n is the largest value of $|k|$ that is of interest to us. These values can be computed with the so-called Fast Fourier Transform (FFT) technique. The technique is fast if N is a power of 2. The program uses the defining formulas if N is not a power of 2.

I.B Exercises

The **Matlab** *program* `FSstart.m` *can be used for all exercises in this session. This program has to be edited to change parameters. To run* `FSstart.m`*, type* `FSstart` *in* **Matlab***'s execution screen. Instructions that are program-dependent are in footnotes in order to keep the main text of the exercises independent of a specific program. Since* **Matlab** *does not accept Greek letters as input, we spelled out the names of the Greek letters: for instance, we used* `mu` *for μ and* `gamma` *for γ. You may have to hit the 'space' bar once in a while to get the next picture; the* `pause` *statement has been included in the* **Matlab** *programs between pictures. Further references in this text to the program* `FSstart.m` *will be in footnotes.*

Make notes: insights and results will be used in subsequential exercises.

Computer-exercise I.1. Take for f the function $f(t) \equiv |t|$ for $t \in \mathcal{I} \equiv [-2,+2]$. Plot the Fourier coefficients.[2] The Fourier coefficients appear to be real and symmetric ($k \rightsquigarrow \gamma_k(f)$ is even). Could this be anticipated? Find a function μ such that $\mu_k \equiv \mu(\gamma_k(f),k)$ is more or less constant for large k.[3] The speed with which the Fourier coefficients of the function $f(t) = |t-1|$ on $[-2,+2]$ converge to 0 is slower (the absolute value of the Fourier coefficients are larger for large $|k|$).[4] Why is the asymptotic behaviour for $|k| \to \infty$ of the Fourier coefficients of $|t|$ and of $|t-1|$ on $[-2,+2]$ so different? (A heuristic argument suffices here; a detailed explanation will be derived in the exercises below.) The Fourier coefficients of $f = |t-1|$ on $[-1,3]$ appear to be purely imaginary, but their asymptotic behaviour is the same as for $f = |t|$ on $[-2,+2]$.[5] Can you explain this?

Computer-exercise I.2. In this exercise, $\mathcal{I} = [-2,+2]$. The functions are defined for $t \in \mathcal{I}$.

Take for f the function $f(t) = |t|$. Design a function μ such that $\mu_k = \mu(\gamma_k(f),k)$ is more or less constant for large k, k odd. Design also a function μ such that $\mu(F_k(f),k)$ is more or less constant for larger k.[6]

Investigate the asymptotic behaviour of $(\gamma_k(f))$ and of $(F_k(f))$ for the function f defined by $f(t) = |\sin(\frac{1}{4}\pi t)|$.[7] Does the asymptotic behaviour depend on the number of singularities in one period?

Find the asymptotic behaviour of $(\gamma_k(f))$ and of $(F_k(f))$ for the function $f(t) = \sqrt{|\sin(\frac{1}{4}\pi t)|}$.[8] What do you expect for the speed by which $(\gamma_k(f))$ an $(F_k(f))$ converge towards 0 in case $f(t) = |\sin(\frac{1}{4}\pi t)|^a$ for some $a > 0$? Check your conjectures for several $a > 0$. Why are $a \in \mathbb{N}_0$, a even, exceptional values?

[2] Check whether the values in the executable lines of this program are correct: `f='abs(t)'`, `I=[-2,2]`, and, for the moment, take `ok=0`, `delta=0`, `mu='1*gamma'`, and `k=-50:50`. 'Run' the program. First, the graph of f will be displayed. Then, if you hit the space bar, the graph of the Fourier coefficients will be plotted in the next figure window. To be more precise, you will see the graph of $k \rightsquigarrow \mu_k \equiv \mu(\gamma_k(f),k)$, with $\mu(\gamma,k) = \gamma$, for $k \in \mathbb{Z}$, $|k| \leq 50$: the line `mu='1*gamma'` sets the choice for μ and the line `k=-50:50` defines the collection of ks for which μ_k will be shown.

[3] Try a μ of the form $\mu(\gamma,k) = \gamma\, k^i$ for $i = 1,2,\ldots$: `mu='gamma.*(k.∧1)'`, `mu='gamma.*(k.∧2)'`, ...

[4] Take `f='abs(t-1)'`. Try another i.

[5] `I=[-1,3]`.

[6] Now take `ok=1;`. As with `ok=0`, the graph f and of $k \rightsquigarrow \mu(\gamma_k(f),k)$ will be displayed. Another hit of the space bar will show the graph $k \rightsquigarrow \mu(F_k(f),k)$ in the same figure as $k \rightsquigarrow \mu(\gamma_k(f),k)$. Adjust `mu` to capture the correct asymptotic behaviour.

[7] `f='abs(sin(pi*t/4))'`.

[8] Take `f='abs(sin(pi*t/4)).∧0.5';`.

Consider the functions $f = |\sin(\cos(\frac{1}{2}\pi t))|$ and $f = \sin(\cos(\frac{1}{2}\pi t))$ as well: if f' has a singularity, then the asymptotic behaviour of the sequences $(\gamma_k(f))$ and $(F_k(f))$ seems to depend only on the type of the singularity and not on the specific values of f away from the singular point (can you explain this? We will return to this issue in Computer-exercise I.6 below).

Can you predict the asymptotic behaviour of $(\gamma_k(f))$ and $(F_k(f))$ for $f(t) = |t|^{\frac{1}{2}}$ and for $f(t) = |t|^{\frac{3}{2}}$?

Computer-exercise I.3. Consider the convergence behaviour of $(\gamma_k(f))$ and $(F_k(f))$ for $f(t) \equiv \exp(\cos(t))$ $(t \in [-\pi, \pi])$.[9] The function f is smooth and fast convergence could have been anticipated. Why? However, the fast convergence for the $(\gamma_k(f))$ seems to get lost for $k \geq 13$. Can you explain the behaviour of $(\gamma_k(f))$ and of $(F_k(f))$ for $k \geq 13$?

From the graphs of $k \rightsquigarrow \mu(\gamma_k(f), k)$ and $k \rightsquigarrow \mu(F_k(f), k)$ it appears that, for $k \leq 12$, $|F_k(f)| \leq 2\,|\gamma_k(f)|$ and $|F_k(f)| \approx 2\,|\gamma_{k+1}(f)|$. Can you explain this?

Computer-exercise I.4. For compression purposes, one may consider to approximate f by $P_n(f)$ rather than by $S_n(f)$. In this exercise, we investigate whether this is an attractive idea.

Consider $f = \exp(\cos(t))$ for $t \in [-\pi, \pi]$. Compare the approximation quality of $S_n(f)$ and $P_n(f)$ for $n = 10$.[10] Is it worthwhile to use $P_n(f)$ instead of $S_n(f)$?

Investigate the same questions for $f(t) = |t|$ $(t \in [-2, 2])$. How much can we gain with $P_n(f)$ for, say, $n = 10$? What about $f(t) = |t - 1|$ on $[-2, 2]$? What is your conclusion?

Computer-exercise I.5. We investigate the effect of random perturbations on f in this exercise: we compute the partial Fourier series of $f^\star \equiv f + \varepsilon$, where, for some $\delta > 0$, the "white" noise $\varepsilon : \mathcal{I} \to \mathbb{R}$ takes values that are uniformly randomly distributed between $-\delta$ and δ.

First take $f(t) = 0$ $(t \in \mathcal{I} = [-1, +1])$ and $\delta = 1$.[11]

Consider the Fourier coefficients $\gamma_k(f^\star)$ and $F_k^\star(f) \equiv \|f - S_k(f^\star)\|_\infty$.[12] (These Fourier coefficients have not been formally defined. Why not? However, we use discrete Fourier coefficients in our program $\widetilde{\gamma}_k(f^\star)$ for $M = 2^l > 8N + 200$; see (1).). The $\gamma_k(f^\star)$ seem to be randomly distributed between $[-\gamma, \gamma]$ for some $\gamma > 0$ (with $\gamma \approx \frac{\delta}{2\sqrt{M}}$?).

It can be show that $F_k^\star(f) \leq \lambda_k \delta$ with λ_k the so-called Lebesgue's constant: $\lambda_k \approx 0.1 + \frac{4}{\pi^2}\log(2|k| + 1)$. Can you see this upper bound in the graph of $k \rightsquigarrow F_k^\star(f)$? The upper bound for $F_k^\star(f)$ by a function proportional to $\log(2|k| + 1)$ appears to be an over estimate for smaller values of k ($k \leq \sqrt{M}$?): a linear growth seems to be more realistic here. Can you derive such a linear estimate?

Now, take $f(t) = \exp(\cos(\pi t))$ $(t \in [-1, +1])$ and $\delta = 0.01$.
Consider the Fourier coefficients $\gamma_k(f^\star)$ and $F_k^\star(f)$.[13] Initially $k \rightsquigarrow F_k^\star(f)$ decreases, but for larger k it slowly increases. For what value of k do you get the best approximation of f? Since the sequence $(\gamma_k(f))$ converges rapidly to 0, the approximation error decreases

[9] `ok=1, f='exp(cos(t))', I=[-pi,pi]`, and `k=-30:30`. Take `mu='log10(abs(gamma))'` or another suitable function.

[10] Take `ok=3` to get a graph of $P_n(f)$ and $f - P_n(f)$. The value of n will be equal to the maximum absolute value of the values appearing in the sequence `k`.

[11] `f='0.*t'` and `delta=1.0`.

[12] Take `ok=1, mu='gamma'`, and `k=0:100`.

[13] Take `mu='log10(abs(gamma))'`, etc..

rapidly if k increases. The effects of the noise increase with increasing k. Therefore, the total error $F_k^\star(f)$ takes a minimal value for some $k = k_0$. From the figure, it appears that, by working with some $S_{k_0}(f^\star)$, the effect of the noise can be reduced by, at least, a factor $\approx \frac{k_0}{\sqrt{M}}$. Such a reduction of the noise can be proved for smooth functions f.

(This property is exploited in practice: if, from a smooth function f, only N function values are known from measurements, then the effect of the errors in the measurements can be diminished by working only with an nth partial Fourier series —or, to be more precise, an nth discrete partial Fourier series— with, typically, $n \leq \sqrt{N}$: the complete (discrete) Fourier series is not used.)

For a function $f : \mathcal{I} \to \mathbb{R}$ en $t \in \mathcal{I}$ we use the notation $f(t+)$ if the right-limit $\lim_{\varepsilon>0,\varepsilon\to 0} f(t+\varepsilon)$ of f at t exists. Then, $f(t+)$ is the value of this limit. Similarly, with $f(t-)$ we refer to the left limit.

Computer-exercise I.6. In this exercise, we investigate how accurately the Fourier series converge in the neighborhood of discontinuities of f. (The insights that we gain here are also helpful to explain the convergence behaviour of Fourier series close to discontinuities of f', f'': differentiation brings the discontinuities in lower derivatives.)

Take for f on $\mathcal{I} = [-2, 2]$ the top-hat function Π_1 defined by

$$\Pi_1(t) = \begin{cases} 1 & \text{voor} \quad |t| < 1 \\ 0 & \text{voor} \quad |t| \geq 1. \end{cases}$$

The sequence $(\gamma_k(f))$ converges to 0 (why?). Check graphically that $|\gamma_{2k+1}(f)| \sim \frac{1}{2k+1}$ and $F_n(f) \sim 1$ (why is $F_k(f) \geq \frac{1}{2}$ for all k?).[14]

Pay careful attention to meaning of the various notions of convergence and conclude from the graphs that [15]

(i) $(S_k(f))$ converges point-wise to $\frac{1}{2}[f(\cdot+)+f(\cdot-)]$ (note that this function coincides with f itself in each $t \in [-2, 2]$, $|t| \neq 1$),

(ii) for each $\varepsilon > 0$, $(S_k(f))$ converges uniformly to f on $\{t \in [-2, 2] \,\big|\, ||t|-1| \geq \varepsilon\}$,

(iii) the sequence $(\mathcal{G}_n)$, with $\mathcal{G}_n \equiv \{(t, S_n(f)(t)) \,\big|\, t \in \mathcal{I}\}$ the graph of $S_n(f)$, does not converge to the subset $\mathcal{G}$ of $[-2, 2] \times \mathbb{R}$ in the left picture in Fig. 9, but it does converge to the subset $\mathcal{G}$ in the right picture (here $(t, y) \in \mathbb{R}^2$ is a limit point of $(\mathcal{G}_n)$ if each disk with center (t, y) in $[-2, 2] \times \mathbb{R}$ intersects the graph of $\mathcal{G}_n$ for all sufficiently large n. Why is this type of convergence not in conflict with the one in (i)?).

Comment to (iii). The graphs $\mathcal{G}_n$ exhibit more wiggles with increasing n. For each n, the tops and the bottoms of these wiggles appear to have exactly the same heights; the wiggles are only compressed in the t-direction towards the points of discontinuity if n increases. There is no change in the vertical direction. This effect is known as the *Gibb's phenomenon.*

Do the same for $f = g\Pi_1$, where g is a smooth 4-periodic function with $g(1) \neq 0$.[16] Note that in the neighbourhood of the discontinuity ($t = 1$) only the scaling of the

[14] Take `ok=0;`, `delta=0;`, `f='(abs(t)<1)'`, and `I=[-2,2]`.

[15] Take `ok=2;` and subsequentially for `k` `k=0:10;`, `k=0:20;`, `k=0:40;`, `k=0:80;`. Pay special attention to the height of the first top of $S_k(f)$ in the neighbourhood of the discontinuity of f.

[16] For instance, `f='exp(-cos(4*pi*t)).*(abs(t<1)'`.

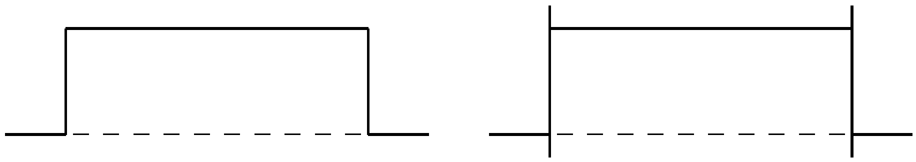

FIGURE 9. Limit functions?

phenomenon changes (with a factor $g(1)$). Can you explain this? (Hint: $f = h + \kappa\Pi_1$ for some $\kappa \in \mathbb{R}$ and a function h that is smoother than f. Note that, now, differentiation leads to an explanation of the phenomenon as observed in the last part of Computer-exercise I.2)

Computer-exercise I.7. Let $\widetilde{\gamma}_k(f)$ be the kth discrete Fourier coefficient of f ($|k| \leq N$). For each $k \in \mathbb{Z}$, $|k| < N$, we have that

$$\widetilde{\gamma}_k(f) = \gamma_k(f) + \sum_{j\in\mathbb{Z}, j\neq 0} \gamma_{k+jN}(f),$$

assuming f is sufficiently smooth: if the Fourier coefficients of f rapidly converge to 0, then the $\widetilde{\gamma}_k(f)$ form a good approximation to $\gamma_k(f)$ (for $|k| \ll N$).

Check this claim experimentally and observe that the approximation is also good if k is not much smaller than $\frac{1}{2}N$.[17] Check the quality of the approximation also in case the Fourier coefficients decrease slowly.[18]

Compare the efficiency of FFT and the naive approach to evaluate the discrete Fourier transform.[19] How is the time needed to computed the Fourier coefficients related to the number of coefficients in case of FFT and in case of the naive approach?

Computer-exercise I.8.

For $a > 0$, the function $f \in C(\mathcal{I})$ is defined by

$$f(t) := \begin{cases} |t|^a \cos\frac{\pi}{|t|} & \text{for} \quad t \in [-1, +1], t \neq 0 \\ 0 & \text{als} \quad t = 0. \end{cases}$$

The Fourier series converges uniformly if $a > 1$. Why? There is no theorem on uniform convergence that can be applied in case $a \in (0, 1)$. Check graphically whether the Fourier series converges for, for instance, $a = 1$ and $a = 0.2$.[20] The sequence $(\gamma_k(f))$ of Fourier coefficients should converge for each choice of $a \in (0, 1)$. Why? Can you observe this in the graphs?

Consider the case where the Fourier series does not seem to converge uniformly. Display the graph of $f - S_k(f)$ for several values of k and observe that the Fourier series

[17] Take `ok=4`, for f, say, `f='1./(25*cos(pi*t).∧2+1)'`, `mu='log10(abs(gamma))'`, and `k=-100:100`. If you run the program, the graphs of $k \rightsquigarrow \mu(k, \gamma_k(f))$ and, after hitting the space bar, the graphs of $k \rightsquigarrow \mu(k, \widetilde{\gamma}_k(f))$ and of $k \rightsquigarrow \mu(k, \gamma_k(f) - \widetilde{\gamma}_k(f))$ will be displayed. The discrete Fourier coefficients are computed for $N = n$, where n is the maximum absolute value in the sequence `k`.

[18] Take, for instance, `f='(1-t.*t).*cos(10*pi*t).∧2'`, etc..

[19] The program does not compute the exact Fourier coefficients, but for $\gamma_k(f)$, it takes discrete Fourier coefficients for $N = 2^\ell > 8n + 100$. These coefficients are evaluated with FFT. The timings are shown in the **Matlab** window. Take for `k=-1000:1000`, `k=-2000:2000`, ...

[20] take `ok=1`, `k=0:100`, `mu='log10(abs(gamma))'` and, for instance, `f='(abs(t)+10∧(-12)).∧0.2.* cos(pi./(abs(t)+10∧(-12))'` (to avoid "overflow" by division by 0, we suggest to add 10^{-12} to `abs(t)`).

converges at t, for each t ($\lim_{n\to\infty} S_n(f)(t) = f(t)$) and even uniformly on $[-1,-\varepsilon]\cup[\varepsilon,1]$,[21] (cf., Computer-exercise I.6). In what sense does the 'uniform convergence' fail?

I.C Best approximations

I.1 Trigonometric polynomials. A function p on $\mathbb{R}$ is a *trigonometric polynomial* of degree $\leq n$ with period T, if, for some scalars α_k, we have that

$$p(t) = \sum_{k=-n}^{n} \alpha_k\, e^{2\pi i t \frac{k}{T}} = \sum_{k=-n}^{n} \alpha_k\, \zeta^k \quad \text{where } \zeta \equiv \zeta_t \equiv e^{2\pi i \frac{t}{T}} \qquad (t \in \mathbb{R}).$$

Let $\mathcal{T}_T^n$ be the space of all T-periodic trigonometric polynomials of degree $\leq n$.

Note that $S_n(f) \in \mathcal{T}_T^n$ for each $f \in L_T^1(\mathbb{R})$.

I.2 The best trigonometric approximation. Consider a real-valued T-periodic function $f \in C(\mathbb{R})$. We call p the *best approximation* of f in $\mathcal{T}_T^n$ with respect to the sup-norm if $p \in \mathcal{T}_T^n$ and $\|f-p\|_\infty \leq \|f-q\|_\infty$ for all $q \in \mathcal{T}_T^n$. Without proof we mention that the best approximation exists and is unique. This best approximation is denoted by $P_n(f)$. We put $E_n(f) \equiv \|f - P_n(f)\|_\infty$: $E_n(f)$ is the sup-norm *distance* from f to $\mathcal{T}_T^n$.

If f is real-valued, then the best approximation in $\mathcal{T}_T^n$ is real-valued as well.

In order to achieve better compression, one may consider to approximate a T- periodic function f by its best approximation $P_n(f)$ in $\mathcal{T}_T^n$: instead of storing a collection of sampled function values $f(n\Delta t)$, one can store the largest coefficients α_k of $P_n(f)$. Standardly, a partial Fourier series of f is used. This polynomial can relatively easily by computed using FFT. Sup-norm best approximations are hard to compute. But, one may hope that $\|f - P_n(f)\|_\infty \leq \|f - S_m(f)\|_\infty$ for some n that is (much) smaller than m.

One can show the following. We leave the proof as an exercise to the reader (see Exercise I.3).

I.3 Theorem

$$\|f - P_n(f)\|_\infty \leq \|f - S_n(f)\|_\infty \leq \|f - P_n(f)\|_\infty (1 + T\lambda_n),$$

where λ_n is Lebesgue's constant:

$$\lambda_n \equiv \int_0^1 \frac{\sin \pi(2n+1)t}{\sin \pi t}\, \mathrm{d}t \approx 0.1 + \frac{4}{\pi^2}\log(2|n|+1). \qquad \square$$

For $n \to \infty$, Lebesgue's constant tends to ∞. However, the convergence is very slow and, for moderate values of n, Lebesgue constant is rather small. Nevertheless, the theorem is rather pessimistic in case of rapid convergence of the Fourier series. Then the following theorem can be useful.

[21] Take `ok=2` and subsequentially, for instance, `k=0:10`, `k=0:20`, `k=0:40`, `k=0:80`.

I.4 The Theorem of de la Vallée–Poussin.
Let f be a real-valued function in $L^1_T(\mathbb{R})$. Consider some real-valued $p \in \mathcal{T}^n_T$. If there is an increasing sequence $t_0, \ldots, t_{2n+1}$ such that $t_{2n+1} - t_0 < T$ and

$$G_k \equiv (-1)^k [f(t_k) - p(t_k)] > 0 \quad \textit{for} \quad k = 0, \ldots, 2n+1,$$

then

$$\min_k G_k \leq E_n(f) \leq \|f - p\|_\infty.$$

Proof. The last equality is trivially true by definition of $E_n(f)$.

Now, assume that $E_n(f) < G_k$ for all $k = 0, \ldots, 2n+1$.
Then, for all $k = 0, \ldots, 2n+1$, we have that

$$(-1)^k (P_n(f) - p)(t_k) = (-1)^k (f - p)(t_k) - (-1)^k (P_n(f) - f)(t_k) \geq G_k - E_n(f) > 0.$$

Put $q \equiv P_n(f) - p$. q is real-valued and in $\mathcal{T}^n_T$. If we put $t_{2n+2} \equiv t_0 + T$, then, by T-periodicity of q, we have that $(-1)^k q(t_k) > 0$ for all $k = 0, \ldots, 2n+2$. Since q is real-valued and continuous, q has at least one zero between each pair of consecutive t_k's. In other words, there are $2n+2$ different zeros $s_0, \ldots, s_{2n+1}$ of q in $(t_0, t_0 + T)$. q is of the form $Q(\zeta_t)$ with $Q(\zeta) \equiv \sum_{k=-n}^{n} \beta_k \zeta^k \quad (\zeta \in \mathbb{C})$ and $\zeta_t \equiv \exp(2\pi i t / T)$. Hence, Q as well as the polynomial $\zeta^n Q$ of degree at most $2n$ has $2n+2$ zeros at the ζ_{s_k}. A corollary of the main theorem of the algebra states that non-trivial polynomials of degree N can have at most N zeros in the complex plain. Consequently, Q and, therefore, q must be zero. Apparently $P_n(f) = p$. But this violates the assumption that $E_n(f) < G_k$ for all k. We can conclude that $E_n(f) \geq G_k$ for some k. □

Note that the number of points t_k that are required by the above theorem is one more than the number of terms involved in the trigonometric polynomial p.

In our applications, $S_n(f)$ will be the approximating trigonometric polynomial p. If the $\gamma_k(f)$ converge quickly towards 0 if $|k| \to \infty$, then we will have that

$$f - S_n(f) \approx 2\,\mathrm{Re}(\gamma_{n+1}(f)\, e^{2\pi i \frac{(n+1)t}{T}}) = 2\,|\gamma_{n+1}(f)|\, \cos(2\pi \left(\frac{(n+1)t}{T} + \phi\right)),$$

where ϕ is such that $\gamma_{n+1}(f) = |\gamma_{n+1}(f)|\, e^{2\pi i \phi}$. Since $\cos(k\pi) = (-1)^k$, we see that $f(t_k) - S_n(f)(t_k) \approx (-1)^k |\gamma_k(f))|$ at $t_k = (\frac{1}{2}k - \phi)\frac{T}{n+1}$ for $k = 0, \ldots, 2n+1$. Since we will also have that $\|f - S_n(f)\|_\infty \approx 2\,|\gamma_{n+1}(f)|$, it follows that in this situation of quickly decreasing $|\gamma_k(f)|$ the best approximation $P_n(f)$ will not provide a much better approximation than $S_n(f)$ (see Exercise I.4).

As a corollary we have that $p \in \mathcal{T}^n_T$ is the best approximation of f if there is a increasing sequence $t_0, \ldots, t_{2n+1}$ in $[t_0, t_0 + T)$ such that $(-1)^k [f(t_k) - p(t_k)] = \|f - p\|_\infty$ for all $k = 0, \ldots, 2n+1$; we say that $f - p$ has the *alternation property* at $2n+2$ points. The converse is also correct if f is continuous. For a proof we refer to the literature.

I.5 Alternation theorem. *Let f be a real-valued continuous function in $L^1_T(\mathbb{R})$, and let $p \in \mathcal{T}^n_T$. Then, p is the best sup-norm approximation of f in $\mathcal{T}^n_T$ if and only if there is an increasing sequence $t_0, \ldots, t_{2n+1}$ in $[t_0, t_0 + T)$ such that*

$$(-1)^k [f(t_k) - p(t_k)] = \|f - p\|_\infty \quad \textit{for all } k = 0, \ldots, 2n+1. \quad \square$$

Exercises

Exercise I.1. Consider a T-periodic function f. Suppose that p is the best approximation of f in $\mathcal{T}_T^n$.
(a) If $q \in \mathcal{T}_T^n$, then $\frac{1}{2}(q+\overline{q}) \in \mathcal{T}_T^n$.
(b) If f is real-valued then p is real valued. Prove this.
(Hint: If $a \in \mathbb{R}$ and $b \in \mathbb{C}$, then $|a - \mathrm{Real}(b)| \leq |a-b|$.)

Exercise I.2.
(a) Show that

$$S_n(f)(s) = \sum_{|k|\leq n} \gamma_k(f)\, e^{2\pi i s \frac{k}{T}} = \int_0^T f(t) \sum_{|k|\leq n} e^{2\pi i (s-t)\frac{k}{T}}\,\mathrm{d}t.$$

Apparently, with $D_n(t) \equiv \sum_{|k|\leq n} \exp(2\pi i t k)$, we have that $S_n(f)(s) = \int_0^T f(t) D_n(\frac{s-t}{T})\,\mathrm{d}t$. D_n is the so-called *Dirichlet kernel*. Note that D_n is real-valued, even and 1-periodic.
(b) Prove that

$$\|S_n(f)\|_\infty \leq \|f\|_\infty\, T\lambda_n \quad \text{where} \quad \lambda_n \equiv \frac{1}{T}\int_0^T |D_n(\frac{t}{T})|\,\mathrm{d}t = 2\int_0^{\frac{1}{2}} |D_n(t)|\,\mathrm{d}t.$$

λ_k is *Lebesgue's constant.*
(c) Show that, with $\zeta \equiv \exp(2\pi t)$, we have that

$$D_n(t) = \sum_{|k|\leq n} \zeta^k = \zeta^{-n}\sum_{k=0}^{2n}\zeta^k = \zeta^{-n}\frac{\zeta^{2n+1}-1}{\zeta-1} = \frac{\zeta^{n+\frac{1}{2}} - \zeta^{-n-\frac{1}{2}}}{\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}} = \frac{\sin(\pi(2n+1)t)}{\sin(\pi t)}.$$

(d) Show that, $|D_n(t)| \leq \pi(n+\frac{1}{2})$ if $t \in [0, \frac{1}{2n+1}]$, and $|D_n(t)| \leq \frac{1}{2t}$ if $t \in [0, \frac{1}{2}]$. (Hint: $\sin(\pi t) \geq 2t$ for $|t| \leq \frac{1}{2}$). Now, show that

$$\lambda_n \leq \pi + \log(n + \tfrac{1}{2}).$$

The sharper upper bound in Theorem I.3 requires more careful estimates. However, note that this simple derivation already shows the logarithmical dependence on n.

Exercise I.3. Consider a $f \in L_T^1(\mathbb{R})$.
(a) Prove that $\|f - P_n(f)\|_\infty \leq \|f - S_n(f)\|_\infty$.
(b) Prove that $S_n(p) = p$ for all $p \in \mathcal{T}_T^n$.
(c) Prove that $\|f - S_n(f)\|_\infty \leq \|f-p\|_\infty + \|S_n(f-p)\|_\infty$ for all $p \in \mathcal{T}_T^n$.
(d) Prove that $\|f - S_n(f)\|_\infty \leq \|f - P_n(f)\|_\infty(1 + T\lambda_n)$, where λ_n is as defined in Exercise I.2.

Exercise I.4. Let f be real-valued and in $L_T^1(\mathbb{R})$. Suppose there is some $\vartheta \in [0,1)$ such that

$$2(1-\vartheta)\,|\gamma_{n+1}(f)| \leq \sum_{|k|>n} |\gamma_k(f)| \leq 2(1+\vartheta)\,|\gamma_{n+1}(f)|.$$

(a) Prove that $2(1-\vartheta)\,|\gamma_{n+1}(f)| \leq \|f - S_n(f)\|_\infty \leq 2(1+\vartheta)\,|\gamma_{n+1}(f)|$.

(b) Show that there is an increasing sequence $t_0, \ldots, t_{2n+1}$ in $[t_0, t_0 + T)$ such that $(-1)^k [f(t_k) - S_n(f)(t_k)] \geq 2\,(1 - \vartheta)|\gamma_{n+1}(f)|$ for all $k = 0, \ldots, 2n+1$.
(c) Prove that $E_n(f) \leq \|f - S_n(f)\|_\infty \leq \frac{1+\vartheta}{1-\vartheta} E_n(f)$.

Exercise I.5. Use de la Vallée–Poussin's theorem to prove that a $p \in \mathcal{T}_T^n$ is the best sup-norm approximation of a real-valued function $f \in L_T^1(\mathbb{R})$ if the 'error' function $f - p$ has the alternation property at at least $2n + 2$ points (that is, prove the 'if' part of the alternation theorem).

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