II.A Introduction

For a given signal $f$, the power spectrum or power spectral density (PSD) gives a plot of the portion of a signal’s power (energy per unit time) falling within given frequency intervals

$$\omega_i \sim \int_{\omega_{i-1}}^{\omega_i} |\hat{f}(\omega)|^2 \, d\omega,$$

where $\omega_i \equiv i \Delta \omega$ ($i \in \mathbb{Z}$). Matlab represents the energy on dB scale and the energy is given (or, more accurately, estimated) per frequency unit, for instance, per Hertz (dB/Hz). The PSD is a way of measuring the strength of the different frequencies that form the signal. Often, as we will see below, it is not computationally feasible to get access to the spectrum $\hat{f}$ of $f$. However, in many applications, already an estimate of the PSD gives the information that is needed.

The most common way of generating a power spectrum is by using a discrete Fourier transform (DFT), but there are other techniques as well. DFT assumes that the signal is sampled and concentrates on a part of the signal in time domain.

If $f$ is of bounded bandwidth and sampled at sample frequency $1/\Delta t$, $1/\Delta t \geq 2\Omega$, where $\Omega$ is the maximum frequency of $f$ (i.e., $|\hat{f}(\omega)| = 0$ if $|\omega| > \Omega$), then

$$\hat{f}(\omega) = \Delta t \sum_{n=-\infty}^{\infty} f_n e^{-2\pi i n \Delta t \omega} \quad (|\omega| \leq \Omega).$$

(2)

Here, $f_n \equiv f(t_n)$, where $t_n \equiv t_0 + n\Delta t$ for some $t_0$. Here, for notational convenience, we assume that $t_0 = 0$. Why is this result correct? Is the restriction $|\omega| \leq \Omega$ needed?

Unfortunately, in practise, only a finite sequence of $f$-values can be used. We are interested in this exercise in the effect of ‘finitizing’. Consider the sequence $(f_0, \ldots, f_{L-1})$; $L$ is some positive integer. We will approximate $\hat{f}(\omega)$ by

$$F(\omega) \equiv \Delta t \sum_{n=0}^{L-1} f_n e^{-2\pi i n \Delta t \omega} \quad (|\omega| \leq \Omega).$$

(3)

We select the frequency intervals of size $\Delta \omega \equiv \frac{1}{L \Delta t}$ around $\omega_i \equiv i \Delta \omega$ to form the PSD and we approximate the PSD by

$$\omega_i \sim \Delta \omega |F(\omega_i)|^2.$$  

(4)

Why is this a good approach assuming $F(\omega)$ approximates $\hat{f}(\omega)$ well, more specifically, why this $\Delta \omega$?

For computational convenience, we select an $N \geq L$ that is a power of 2 (why?), $N = 2^\ell$ ($\ell \in \mathbb{N}$ is minimal such that $2^\ell \geq L$), and we compute $F(\omega)$ as

$$F(\omega) = \Delta t \sum_{n=0}^{N-1} \phi_n e^{-2\pi i n \Delta t \omega} \quad (|\omega| \leq \Omega),$$

(5)

where $\phi_n \equiv f_n$ for $n = 0, \ldots, L - 1$ and $\phi_n \equiv 0$ elsewhere. Check that the equality in (5) is correct. Instead of (4),

$$\tilde{\omega}_i \equiv \frac{i}{N \Delta t} \sim \Delta \omega |F(\tilde{\omega}_i)|^2.$$  

(6)

\[ \text{i.e., } 10 \log_{10} |\hat{f}(\omega)|^2 = 20 \log_{10} |\hat{f}(\omega)| \]
is plotted. Why? Note that the $\Delta \omega$ in (6) is the same as the one in (4).

Consider the “time window” $W$ given by $W_n \equiv 1$ if $n = 0, \ldots, L-1$ and $W_n \equiv 0$ elsewhere. Then $F = (fW) = \hat{f} \ast \hat{W}$. Here, $f$ is the infinite sequence $(\ldots, f_0, f_1, \ldots)$ of sampled $f$-values.

II.B Exercises

Make notes: insights and results will be used in subsequential exercises.

Computer-exercise II.1.

Consider the function $f$ given by

$$f(t) \equiv \sin(2\pi 150t) + 2\sin(2\pi 140t) \quad (t \in \mathbb{R}).$$

This is not a signal. Why not? Nevertheless, we can compute the Fourier transform $\hat{f}$. How does $\hat{f}$ look like?

The matlab command `periodogram(fn,[],'twosided',N,fs)`, produces the PSD in the way as described above. Here, $fn$ is the sequence $(f_0, \ldots, f_L)$, $N$ is $N$, and $fs$ is the sample frequency $1/\Delta t$. How do you expect that the PSD of the above function will look like? Does the matlab command produces the expected result? First take $L = N = 2^{10}$ and $1/\Delta t = L$. What is the difference between ‘twosided’ and ‘onesided’? Can we safely use ‘onesided’ here? Can you explain the shape of the PSD at $-300\text{dB/Hz}$? The spikes have a certain width. Why is that? Is the height at $\omega = 140$ and 150 as expected?

What is the effect of taking an $L$ that is somewhat smaller than $N$, say, $L = 1000$. Are the effects less with a slightly larger $L$, say $L = 1023$. Explain why we have such a nice picture with $L = 1024$? Is this because $N = L$? (Suggestion: change the frequency 140 into 140.5. What is the effect of shifting the original signal in time?)

What is the effect of increasing $N$ ($N = 2^{10}$, $N = 2^{11}$, $N = 2^{12}$, $N = 2^{13}$, $N = 2^{14}$)? Explain your observations. Why is the PSD for smaller $N$ ‘part’ of the graph for larger $N$?

What is the effect of increasing $T$? Explain your observations.

---

2Use Ex1a.m.
3Use Ex1b.m.
4Use Ex1c.m.
5Use Ex1d.m.