Computer session II: Digital Spectral Analysis

II.A Introduction

For a given signal f, the *power spectrum* or *power spectral density* (PSD) gives a plot of the portion of a signal's power (energy per unit time) falling within given frequency intervals

$$\omega_i \rightsquigarrow \int_{\omega_{i-1}\leq|\omega|\leq\omega_i} |\widehat{f}(\omega)|^2 \, \mathrm{d}\omega, \tag{1}$$

where $\omega_i \equiv i\Delta\omega \ \ (i \in \mathbb{Z})$. Matlab represents the energy on dB scale[1] and the enerergy is given (or, more accurately, estimated) per frequency unit, for instance, per Hertz (dB/Hz). The PSD is a way of measuring the strength of the different frequencies that form the signal. Often, as we will see below, it is not computationally feasible to get access to the spectrum $\widehat{f}$ of f. However, in many applications, already an estimate of the PSD gives the information that is needed.

The most common way of generating a power spectrum is by using a discrete Fourier transform (DFT), but there are other techniques as well. DFT assumes that the signal is sampled and concentrates on a part of the signal in time domain.

If f is of bounded bandwidth and sampled at sample frequency $1/\Delta t$, $1/\Delta t \geq 2\Omega$, where Ω is the maximum frequency of f (i.e., $|\widehat{f}(\omega)| = 0$ if $|\omega| > \Omega$), then

$$\widehat{f}(\omega) = \Delta t \sum_{n=-\infty}^{\infty} f_n \, e^{-2\pi i\, n\, \Delta t\, \omega} \qquad (|\omega| \leq \Omega). \tag{2}$$

Here, $f_n \equiv f(t_n)$, where $t_n \equiv t_0 + n\Delta t$ for some t_0. Here, for notational convenience, we assume that $t_0 = 0$. Why is this result correct? Is the restriction $|\omega| \leq \Omega$ needed?

Unfortunately, in practise, only a finite sequence of f-values can be used. We are interested in this exercise in the effect of 'finitizing'. Consider the sequence $(f_0, \ldots, f_{L-1})$; L is some positive integer. We will approximate $\widehat{f}(\omega)$ by

$$F(\omega) \equiv \Delta t \sum_{n=0}^{L-1} f_n \, e^{-2\pi i\, n\, \Delta t\, \omega} \qquad (|\omega| \leq \Omega). \tag{3}$$

We select the frequency intervals of size $\Delta\omega \equiv \frac{1}{L\Delta t}$ around $\omega_i \equiv i\Delta\omega$ to form the PSD and we approximate the PSD by

$$\omega_i \rightsquigarrow \Delta\omega \, |F(\omega_i)|^2. \tag{4}$$

Why is this a good approach assuming $F(\omega)$ approximates $\widehat{f}(\omega)$ well, more specifically, why this $\Delta\omega$?
For computational convenience, we select an $N \geq L$ that is a power of 2 (why?), $N = 2^\ell$ ($\ell \in \mathbb{N}$ is minimal such that $2^\ell \geq L$), and we compute $F(\omega)$ as

$$F(\omega) = \Delta t \sum_{n=0}^{N-1} \phi_n \, e^{-2\pi i\, n\, \Delta t\, \omega} \qquad (|\omega| \leq \Omega), \tag{5}$$

where $\phi_n \equiv f_n$ for $n = 0, \ldots, L-1$ and $\phi_n \equiv 0$ elsewhere. Check that the equality in (5) is correct. Instead of (4),

$$\widetilde{\omega}_i \equiv \frac{i}{N\Delta t} \rightsquigarrow \Delta\omega \, |F(\widetilde{\omega}_i)|^2. \tag{6}$$

[1] i.e., $10\log_{10}|\widehat{f}(\omega)|^2 = 20\log_{10}|\widehat{f}(\omega)|$

is plotted. Why? Note that the $\Delta\omega$ in (6) is the same as the one in (4).

Consider the "*time window*" W given by $W_n \equiv 1$ if $n = 0, \ldots, L-1$ and $W_n \equiv 0$ elsewhere. Then $F = \widehat{(\mathbf{f}W)} = \widehat{\mathbf{f}} * \widehat{W}$. Here, $\mathbf{f}$ is the infinite sequence $(\ldots, f_0, f_1, \ldots)$ of sampled f-values.

II.B Exercises

Make notes: insights and results will be used in subsequential exercises.

Computer-exercise II.1.

Consider the function f given by

$$f(t) \equiv \sin(2\pi 150 t) + 2\sin(2\pi 140 t) \qquad (t \in \mathbb{R}).$$

This is not a signal. Why not? Nevertheless, we can compute the Fourier transform $\widehat{f}$. How does $\widehat{f}$ look like?

The matlab command `periodegram(fn,[],'twosided',N,fs)`, produces the PSD in the way as described above. Here, `fn` is the sequence $(f_0, \ldots, f_L)$, `N` is N, and `fs` is the sample frequency $1/\Delta t$. How do you expect that the PSD of the above function will look like? Does the matlab command produces the expected result? First take $L = N = 2^{10}$ and $1/\Delta t = L$.[2] What is the difference between `'twosided'` and `'onesided'`? can we safely use `'onesided'` here? Can you explain the shape of the PSD at -300dB/Hz? The spikes have a certain width. Why is that? Is the height at $\omega = 140$ and 150 as expected?

What is the effect of taking an L that is somewhat smaller than N, say, $L = 1000$. Are the effects less with a slightly larger L, say $L = 1023$.[3] Explain why we have such a nice picture with $L = 1024$? Is this because $N = L$? (Suggestion: change the frequency 140 into 140.5. What is the effect of shifting the original signal in time?)

What is the effect of increasing N ($N = 2^{10}$, $N = 2^{11}$, $N = 2^{12}$, $\ldots$)?[4] Explain your observations. Why is the PSD for smaller N 'part' of the graph for larger N?

What is the effect of increasing T?[5] Explain your observations.

[2] Use `Ex1a.m`.
[3] Use `Ex1b.m`.
[4] Use `Ex1c.m`.
[5] Use `Ex1d.m`.