September 13, 2017

Matlab Assignments – Lecture 11, Fall 2016

In these assignments you will investigate the iterative solution of the real linear system $\mathbf{Ax} = \mathbf{b}$, in which $\mathbf{A}$ is a block matrix of the following form

$$\mathbf{A} = \begin{bmatrix} \mathbf{F} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & -\mathbf{C} \end{bmatrix}. \tag{11.1}$$

Here, $\mathbf{F}$ is symmetric and positive definite, and $\mathbf{C}$ is symmetric positive semi-definite. $\mathbf{C}$ may be zero. Such block matrices arise in many applications. Systems with a block matrix (11.1) are known as **KKT systems** or **saddle-point systems**.

In this assignment both $\mathbf{F}$ and $\mathbf{C}$ are diagonal matrices.

Saddle-point systems are notoriously difficult to solve by iterative methods. One approach is to precondition them using a preconditioner based on the block LU-factorization

$$\begin{bmatrix} \mathbf{F} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & -\mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{B}^{\mathrm{T}}\mathbf{F}^{-1} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{F} & \mathbf{B} \\ \mathbf{O}^{\mathrm{T}} & -\mathbf{M}_S \end{bmatrix}.$$

Here, $\mathbf{M}_S \equiv \mathbf{B}\mathbf{F}^{-1}\mathbf{B}^{\mathrm{T}} + \mathbf{C}$ and $-\mathbf{M}_S$ is the **Schur complement**. Take

$$\mathbf{P} = \begin{bmatrix} \mathbf{F} & \mathbf{B} \\ \mathbf{O} & -\mathbf{M}_S \end{bmatrix} \tag{11.2}$$

as preconditioner. This leads to the right-preconditioned matrix

$$\mathbf{A}\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{B}^{\mathrm{T}}\mathbf{F}^{-1} & \mathbf{I} \end{bmatrix}.$$

Explain why this implies that GMRES applied to a right-preconditioned system saddle point system with (11.2) as preconditioner must find the exact solution in at most two iterations (see Exercise 11.9).

The preconditioner that we have defined above is *nonsymmetric* while the system matrix (11.1) is symmetric. This may seem somewhat unnatural. A *symmetric* preconditioner for saddle point systems is the block diagonal matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{F} & \mathbf{O} \\ \mathbf{O}^{\mathrm{T}} & \mathbf{M}_S \end{bmatrix}. \tag{11.3}$$

Note that this preconditioner is also positive definite. Therefore, (implicitly preconditioned) MINRES is applicable.

The preconditioned matrix $\mathbf{P}^{-1}\mathbf{A}$ has three distinct non-zero eigenvalues in case $\mathbf{C} = \mathbf{0}$ (see Exercise 11.10) and MINRES find the exact solution then in at most three iterations.

In the next assignments you have to perform several numerical experiments on saddle-point systems from the test-set `Schenk_IBMNA`, which is part of the Tim Davis' matrix collection at the University of Florida (http://www.cise.ufl.edu/research/sparse/matrices/). In all the assignments you should use the MATLAB-script `kkt.m` that you can download form the course web page. It is not necessary to program yourself. In the experiments you also need the script `idrs.m` which you can download from *http://ta.twi.tudelft.nl/nw/users/gijzen/IDR.html*.

Assignment 11.1. Download, in MATLAB format, the problem `c-18` from the Tim Davis' collection (the MATLAB command `load c-18`, loads the problem $\mathbf{Ax} = \mathbf{b}$ into a `struct` named '`Problem`' that contains the matrix [problem.A] as well as the right-hand-side vector [Problem.b]. With the MATLAB command `Problem`, the other quantities in the struct are displayed as well). Check that the matrix `c-18` has the above block structure. What is the size of the matrix? Explain why MINRES is a natural choice for solving the system. Try to solve this system with MINRES without preconditioning. Conclusion?

Now we are going to combine MINRES with preconditioning.

Assignment 11.2. Solve the problem with preconditioned MINRES, taking (11.3) as the preconditioner. Discuss the convergence. (Warning: MATLAB uses left preconditioning in MINRES, but gives the residual norms for the *unpreconditioned* system. Therefore, residual norms may go up in your convergence curve.)

For large systems it is not possible to compute the Schur-complement matrix. A simple idea is to approximate the Schur complement by its main diagonal. The resulting preconditioning matrix then becomes

$$\mathbf{P} = \begin{bmatrix} \mathbf{F} & \mathbf{O} \\ \mathbf{O}^{\mathrm{T}} & \mathrm{diag}(\mathbf{M}_S) \end{bmatrix}, \tag{11.4}$$

which is a diagonal matrix.

Assignment 11.3. Solve the problem with preconditioned MINRES, taking (11.4) as the preconditioner. Also solve the system with the following methods (preconditioned with (11.4)): Bi-CGSTAB, IDR(1) and IDR(4). Plot the convergence curves of the four methods in one figure. Put on the x-axis the number of matrix-vector multiplications and on the y-axis the residual norm divided by the norm of the right-hand side vector. Which method is fastest? Is this what you would expect?

Following the same reasoning as above we can also use

$$\mathbf{P} = \begin{bmatrix} \mathbf{F} & \mathbf{B} \\ \mathbf{O}^{\mathrm{T}} & -\mathrm{diag}(\mathbf{M}_S) \end{bmatrix} \tag{11.5}$$

as preconditioner. This matrix is nonsymmetric, but a Krylov solver for nonsymmetric systems combined with this preconditioner might be more efficient than MINRES combined with the symmetric preconditioner. We will investigate this in the next assignment.

Assignment 11.4. Solve the problem preconditioned with (11.5) with the following methods: Bi-CGSTAB, IDR(1), IDR(4), Bi-CG and QMR. Plot the convergence curves of the five methods in one figure. Put on the x-axis the number of matrix-vector multiplications and on the y-axis the residual norm divided by the norm of the right-hand side vector. Note that Bi-CG and QMR need two matrix-vector multiplications per iteration. Discuss the convergence. Which method is fastest?

Assignment 11.5. Select at least two other test problems from the same test set. Repeat Assignment 11.3 and Assignment 11.4. Discuss and draw conclusions.