September 13, 2017

Matlab Assignments – Lecture 14, Fall 2016

In this assignment you will develop a simple multigrid code for solving the 1-dimensional Poisson equation.

Assignment 14.1. Generate a 1D Poisson system using the following commands:

```
level = input('Level = ')
n = 2∧level-1
h = 1/(n+1);
e = ones(n,1);
A = (1/h∧2)*spdiags([-e 2*e -e], -1:1, n, n);
b = ones(n,1);
```

The parameter `Level` determines the size of the system. Take `Level = 10`. Write a code that performs k steps of Gauss–Seidel iteration on the system, starting with a random initial guess. Plot the residual and the error after every iteration (until $k = 10$) and verify that the Gauss–Seidel iterations also smooth the residual. Why is the error smoother than the corresponding residual?

Assignment 14.2. Write subroutines for the prolongation and the restriction operation. The restriction operation is such that a vector $\mathbf{x}_c$ on the courser level takes as values in the gridpoints

$$\mathbf{x}_c(i) = 0.25\,\mathbf{x}_f(2i-1) + 0.5\,\mathbf{x}_f(2i) + 0.25\,\mathbf{x}_f(2i+1),$$

where $\mathbf{x}_f$ is the vector on the finer grid. The prolongation operation is such that

$$\mathbf{x}_f(2i) = \mathbf{x}_c(i)$$

and

$$\mathbf{x}_f(2i+1) = 0.5\,(\mathbf{x}_c(i) + \mathbf{x}_c(i+1)).$$

Note that $\mathbf{x}(0) = \mathbf{x}(n+1) = 0$.
Test your subroutines, for example on the solution of the system.

Assignment 14.3. Write a two grid method.
A cycle must consist of the following steps:

- Perform k steps of Gauss-Seidel iteration on the approximate solution $\mathbf{x}_f$ (pre-smoothing);
- Compute the residual $\mathbf{r}_f$ (stop if the norm of the residual is small enough).
- Transfer the residual to the courser grid (one level courser), using your restriction routine.
- Solve the system $\mathbf{A}_c\mathbf{u}_c = \mathbf{r}_c$, where all vectors are defined at the courser level.
- Prolong $\mathbf{u}_c$ to the finer level, add the resulting $\mathbf{u}_f$ to $\mathbf{x}_f$.
- Perform one Gauss–Seidel iteration (post smoothing).
- Repeat the above steps until convergence.

Test your program for $k = 1$ and different problem sizes. How does the number of iterations depends on the problem size? What is the effect of increasing k?

Assignment 14.4. Make a recursive version of your program such that your program performs a complete V-cycle. Take `level` $= 2$ the lowest, on which you solve the system with a direct solver. How does the number of iterations depend on the problem size? How does the (maximal) residual reduction factor per cycle depend on the grid size?