September 13, 2017

Matlab Assignments – Lecture 4, Fall 2016

In these assignments we will implement the SOR method for solving $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$ and investigate its convergence.

Download the file `sor.m` from the course webpage. This file contains code to generate a 2D-Poisson matrix $\mathbf{A}$ on a square equidistant $n \times n$ grid; $\mathbf{A}$ is $n^2 \times n^2$. The right-hand-side vector $\mathbf{b}$ is such that the solution $\mathbf{x}$ equals one. The code splits the matrix $\mathbf{A}$ into a strictly lower triangular part $\mathbf{L}$, a main diagonal $\mathbf{D}$, and a strictly upper triangular part $\mathbf{U}$.

Assignment 4.1. Implement the SOR method to solve this system. Use the most efficient variant of the algorithm. Use the relative residual norm to test for convergence. Test your algorithm with $n = 10$, maximum number of iterations 1000, tolerance 10^{-6}, and omega = 1.

Assignment 4.2. Determine numerically a near-optimal value for ω (Hint: ω should be between 1 and 2).

Assignment 4.3. Investigate how the number of iterations depends on the gridsize n. Do this by determining the number of iterations for $n = 4$, $n = 8$, $n = 16$ and $n = 32$. Take $\omega = 1$. Repeat this assignment but now for near optimal values of ω.

Assignment 4.4. The calculation of the residual is an expensive operation. We would therefore like to use a cheaper termination criterion, preferably one on basis of the true error instead of on the residual. In this assignment we derive such a criterion.

(a) Show that the spectral radius of $\mathbf{G} \equiv \mathbf{M}^{-1}\mathbf{R}$ approximately satisfies

$$\rho(\mathbf{G}) \approx \frac{\|\mathbf{x}_{k+1} - \mathbf{x}_k\|_2}{\|\mathbf{x}_k - \mathbf{x}_{k-1}\|_2}.$$

(b) Show that if $\rho(\mathbf{M}^{-1}\mathbf{R})$ is known, an estimate for the error is given by

$$\|\mathbf{x} - \mathbf{x}_k\|_2 \le \frac{\rho(\mathbf{G})}{1 - \rho(\mathbf{G})} \|\mathbf{x}_k - \mathbf{x}_{k-1}\|_2.$$

(c) Implement this criterion in your code, and test it for $n = 10$.