Matlab Assignments – Lecture 8, Fall 2016

Assignment 8.1. CG has been designed for symmetric definite systems. MINRES and SYMMLQ can handle symmetric non-definite systems. Nevertheless we can try to apply CG also to non-definite systems. In this assignment, we investigate how this works in practice.

Consider the diagonal matrix $\mathbf{D}$ with eigenvalues $1:1000$ shifted by σ.

(a) Solve the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ with $\mathbf{A} = \mathbf{D}$ and $\mathbf{b}$ random and normalised with CG, MINRES and SYMMLQ. Select the shift close to a small eigenvalue.

What is the effect of rotating the matrix, i.e., replace $\mathbf{A}$ by $(\mathbf{I}-2\mathbf{V}\mathbf{V}^*)\mathbf{D}(\mathbf{I}-2\mathbf{V}\mathbf{V}^*)$ with $\mathbf{V}$ a low dimensional (dimension 1?) orthonormal matrix?

What is the effect of replacing $\mathbf{b}$ by a vector of all ones and then normalised?

What is the effect of selecting $\mathbf{x}$ randomly and setting $\mathbf{b}$ equal to $\mathbf{A}\mathbf{x}$?

(b) Select the shift close to a small Ritz value (if you did not code the extraction of Ritz values from CG, then you can use `minres0.m` to compute the Ritz values).

Assignment 8.2. The expansion vector for the search subspace in GCR at step k is selected to be equal $\mathbf{r}_k$: $\mathbf{u}_k = \mathbf{r}_k$, whereas the expansion vector in GMRES is equal to $\mathbf{v}_k$.

(a) What is the effect of perturbing the expansion vector at, say step $k = 10$, by a relatively small perturbation, i.e., use $\mathbf{u}_k = \mathbf{r}_k + \delta\Delta$ in GCR and $\mathbf{v}_k \leftarrow \mathbf{v}_k + \delta\Delta_k$ in GMRES with $\|\Delta\| \approx \|\mathbf{r}_k\|$, and $\|\Delta\| \approx \|\mathbf{v}_k\|$ respectively, and $\delta = 10^{-3}$. Also take $\delta = 1$.

(b) Perturb the first step.

Explain why GCR seems to be insensitive to perturbations.

(c) Consider the system $\mathbf{A}\mathbf{x} = \mathbf{b} + \delta\Delta$ and use $\mathbf{v}_0 = \mathbf{b}/\|\mathbf{b}\|_2$ to generate the Krylov basis used in GMRES. Explain why GMRES is sensitive to perturbations.

Conclusion. GCR is a *subspace method*: any vector can be used to effectively expand the search subspace. As a consequence any additional information available on the solution (singularities, ...) cab be exploited in GCR.

GMRES is *Krylov subspace method*: the algorithm heavily exploits the Krylov structure (Hessenberg matrix, etc.) to have an efficient algorithm (efficient as compared to GCR). However, perturbations spoil the Krylov structure and, consequently, the effectivity of the method.

(d) GCR may stagnate (if $\mathbf{c}_k^*\mathbf{r}_k = \mathbf{r}_k^*\mathbf{A}\mathbf{r}_k = 0$) and breakdown in the next step (why?). The choice $\mathbf{u}_k = \mathbf{c}_{k-1}$ overcomes this breakdown (see Exercise 8.5).

Write an ORTHODIR code (i.e., use the choice $\mathbf{u}_k = \mathbf{c}_{k-1}$) and check that ORTHODIR and GCR have the same residuals if GCR does not break down. How sensitive is ORTHODIR to perturbations on the vectors that generate the search subspace.