A pencil of K3- surfaces related to Apéry's recurrence for $\varsigma(3)$ and Fermi surfaces for potential zero

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Introduction

The affine threefold Z with equation

$$\xi_0 + \xi_0^{-1} + \xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} = 0 \tag{1}$$

fibres via the projection to $\mathbf{P}^1(\xi_0)$ into surfaces which turn out to be open parts of K3-surfaces for $\xi_0 \neq 0, \infty, \pm 1, \pm 3 \pm 2\sqrt{2}$. We examine this in detail in section 2. The relative 2-form

$$\omega = \frac{1}{\xi_3 - \xi_3^{-1}} \frac{d\xi_1}{\xi_1} \wedge \frac{d\xi_2}{\xi_2} \tag{2}$$

gives a nowhere vanishing 2-form on these K3-surfaces . Its periods are multi-valued functions of ξ_0 which satisfy a third order differential equation, the Picard-Fuchs equation. This equation is determined in section 3. It appears to be equivalent with a recurrence Apéry used in his proof for the irrationality of $\varsigma(3)$ (cf. [Po]). The monodromy of the Picard- Fuchs equation is determined in section 4. Here some interesting modular functions and cusp forms come up, which were previously studied in [Be] and [Be2] in a different context.

D. Gieseker, H.Knörrer and E. Trubowitz have for some time been studying Fermi curves. These are certain real curves associated to electrons, moving in a periodic 2-dimensional potential created by a 2-dimensional lattice of positive ions. In fact they consider a discrete approximation which equally well can be formulated in any dimension. But whereas in two dimensions the theory is fairly complete, the technical complications in higher dimensions are so enormous that virtually no result is available. So the interest arose in simple 3-dimensional potentials, e.g potential zero. The corresponding Fermi-variety is now a surface depending on a parameter s (essentially the energy of the electron) and in the coarsest discrete approximation is given by the equation $\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} = s$. Replacing s by $-\xi_0 - \xi_0^{-1}$ has certain advantages , discussed at the end of section 2. For this reason we call (1) the Fermi-threefold and the family of K3-surfaces the Fermi-fibration. The integral of the 2-form ω over the 2-cycle $|\xi_1| = |\xi_2| = |\xi_3| = 1$ (when not empty) has a physical interpretation as density of states function. We recall this background in section 1.

One of the motivations for our work has been to extend the domain of the density of states function to all of $\mathbf{C}$ (minus a few singular points). Since this function is defined as a period integral, we can find such an extension as a solution to the Picard-Fuchs equation for the Fermi-fibration.

1 Fermi-surfaces

In solid state physics the wave function $\psi : \mathbf{R}^3 \to \mathbf{C}$ describing an electron with energy λ moving about a lattice $\Gamma \subset \mathbf{R}^3$ is determined by a potential $V : \mathbf{R}^3 \to \mathbf{R}$ which is assumed to be periodic in Γ. The physical meaning of ψ is given by the probability $\int_B \|\psi\|^2 dx / \int_{\mathbf{R}^3} \|\psi\|^2 dx$ to find an electron in a region $B \subset \mathbf{R}^3$. To find ψ belonging to energy λ one solves the time-independent Schrödinger equation

$$(-\Delta + V)\psi = \lambda\psi \tag{3}$$

and one restricts to solutions ψ for which

$$\psi(x + \gamma) = e^{2\pi i k(\gamma)}\psi(x) \quad \forall \gamma \in \Gamma$$

Here $k : \mathbf{R}^3 \to \mathbf{R}$ is a linear functional and one is led to consider those $k \in (\mathbf{R}^3)^*$ for which non zero solutions ψ for (3) exist. In solid state physics it is assumed that these k form a surface -the *Fermi surface for energy* λ. This surface is periodic in k-space and so one may equally well consider its image in the three- torus obtained by identifying opposite faces of a fundamental domain for the lattice dual to Γ.

Following Gieseker, Knörrer and Trubowitz [G-K-T] we look at a certain discrete analogue. We replace $\mathbf{R}^3$ by $\mathbf{Z}^3$ and Γ by the sublattice with basis $a_1\mathbf{e}_1, a_2\mathbf{e}_2, a_3\mathbf{e}_3$. Here $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standardbasis for $\mathbf{R}^3$. The Laplacian is replaced by the second order difference operator

$$\Delta f(x) := \frac{1}{6}(\sum_n f(x+n) - f(x)), \quad n \in \{\pm\mathbf{e}_1, \pm\mathbf{e}_2 \pm \mathbf{e}_3\}$$

and potentials are allowed to be complex-valued (but still periodic with respect to Γ). The Fermi surface $F_\lambda(\mathbf{R})$ is the intersection with $S^1 \times S^1 \times S^1 \subset \mathbf{C}^* \times \mathbf{C}^* \times \mathbf{C}^*$ of the *complex Fermi surface*

$$\begin{aligned} F_\lambda := \{(\xi_1, \xi_2, \xi_3) \in \mathbf{C}^{*3} \mid \exists \psi : \mathbf{Z}^3 \to \mathbf{C} \quad \text{solving (3)} \\ \text{with } \psi(x + a_j\mathbf{e}_j) = \xi_j\psi(x), j = 1,2,3\} \end{aligned}$$

If we collect these Fermi surfaces for all *complex* values of the energy we obtain a variety B in $\mathbf{C}^{*3} \times \mathbf{C}$. It fibers over $\mathbf{C}$ and we let π be the projection onto $\mathbf{C}$. The relative 2-form ω defined by

$$\frac{d\xi_1}{\xi_1} \wedge \frac{d\xi_2}{\xi_2} \wedge \frac{d\xi_3}{\xi_3}|_B = \omega \wedge \pi^* d\lambda$$

is regular on F_λ. If $\gamma(\lambda) := F_\lambda(\mathbf{R})$ is non empty the integral of the relative two form :

$$\int_{\gamma(\lambda)} \omega|_{F_\lambda} \qquad \text{(the \textit{density of states function})}$$

has a physical interpretation (cf [G-K-T]). In the sequel we study the simplest situation where $V = 0$ and $\Gamma = \mathbf{Z}^3$. It is easy to determine the equations of the corresponding Fermi-surfaces, since there is only one function with $\underline{\xi} = (\xi_1, \xi_2, \xi_3)$ as quasi-periodicity factor, namely

$$e_{\underline{\xi}} : \mathbf{Z}^3 \to \mathbf{C}, \quad e_{\underline{\xi}}(x_1, x_2, x_3) = \xi_1^{x_1} \xi_2^{x_2} \xi_3^{x_3}$$

The relation $\Delta e_{\underline{\xi}} = \frac{1}{6}(\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} - 6)e_{\underline{\xi}}$ implies the following equation for F_λ

$$\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} = 6(\lambda + 1).$$

We end this section by observing that from this equation we can easily deduce the explicit representation of the form ω we gave in the introduction (see (2)).

2 The Fermi threefold

The affine threefold $\mathcal{X}$ given by the equation

$$\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} = s$$

admits a nice compactification $\overline{\mathcal{X}}$ in $\mathbf{P}^6 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$, which in terms of the coordinates $((w, u_1, u_2, u_3, v_1, v_2, v_3), (p_1, q_1), (p_2, q_2), (p_3, q_3), (S, T))$ is given by the equations

$$\begin{aligned} p_i u_i = q_i w, \quad q_i v_i = p_i w \quad (i = 1, 2, 3) \\ T(u_1 + v_1 + u_2 + v_2 + u_3 + v_3) \quad = \quad Sw \end{aligned} \tag{4}$$

The isomorphism $\overline{\mathcal{X}} \setminus \{w = 0\} \longrightarrow \mathcal{X}$ is defined by $s = ST^{-1}$ and $\xi_i = u_i w^{-1}$ for $i = 1, 2, 3$. The threefold $\overline{\mathcal{X}}$ has 48 singular points, all situated on $T = 0$. Projection onto the last $\mathbf{P}^1$-factor defines a fibration $\pi : \overline{\mathcal{X}} \longrightarrow \mathbf{P}^1$. The fibre of π over a point $(S, T) \neq (1, 0)$ will be denoted by $\overline{\mathcal{X}}_s$, where $s = ST^{-1}$. From (4) we see that this fibre is an intersection of three quadrics in $\mathbf{P}^5$ in which 12 rational double points where $w = 0$ have been resolved into a $\mathbf{P}^1$. For $s \notin \{2, -2, 6, -6, \infty\}$ it is smooth and hence a K3-surface (it is simply connected by Lefschetz' theorem and the canonical bundle is trivial by the adjunction formula applied to the embedding of $\overline{\mathcal{X}}_s$ in $\mathbf{P}^6 \times \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$). The fibre has exactly 3 (resp 1) rational double points if $s = \pm 2$ (resp. ± 6). Resolving these, we obtain K3-surfaces too.

Let $G = S_3 \rtimes (\mathbf{Z}/2\mathbf{Z})^3$, where the first factor acts by permutation on the second factor. This group acts on each surface $\overline{\mathcal{X}}_s$; the first factor permutes the indices 1,2,3

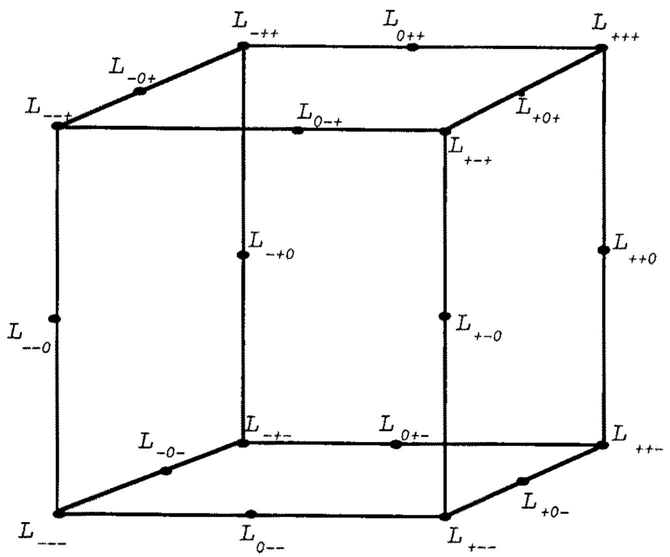


Figure 1: A cube of lines on the Fermi K3

and the i-th generator of the second factor simultaneously interchanges (u_i, v_i) and (p_i, q_i) leaving the the other coordinates fixed.
The zero locus of w on $\overline{\mathcal{X}}_s$ consists of the following twenty lines:
the 12 lines forming the G-orbit of

$$L_{0++} := \{((0,0,1,-1,0,0,0),(p_1,q_1),(0,1),(0,1))|(p_1,q_1) \in \mathbf{P}^1\}$$

and the 8 lines in the G-orbit of

$$L_{+++} = \{((0,u_1,u_2,u_3,0,0,0),(0,1),(0,1),(0,1)|u_1+u_2+u_3=0\}$$

(coordinates as in (4) with s omitted).

The pattern of these 20 lines is presented in Fig. 1, in which the vertices correspond to lines with self-intersection -2 and the edges represent intersections of multiplicity 1. The indexing of the vertices should be obvious. We remark that our group G actually is isomorphic to the group of symmetries of the cube.
Let $\mathbf{L}$ be the sublattice of $H^2(\overline{\mathcal{X}}_s, \mathbf{Z})$ generated by the cycle classes of the 20 lines pictured in Fig. 1.
Notation: Lattices A_i, D_i, E_i denote the usual (positive definite) root lattices, H stands for the hyperbolic plane, i.e the rank two lattice with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and finally $\langle i \rangle$ is the rank one lattice with Gram matrix (i). If Γ is a lattice with form $\langle\, , \rangle$ the lattice $k\Gamma$ is the lattice with the same group as Γ, but with form $k\langle\, , \rangle$. Finally Γ^* is the dual of Γ and if Γ is non degenerate, $\langle\, , \rangle$ embeds it naturally in Γ^*.

Proposition 1 *The lattice $\mathbf{L}$ has rank 17, signature (1,16) and discriminant 16*

Proof: The lattice $\mathbf{L}$ contains the lattice $\mathbf{L}_0$ generated by four $(-D_4)$-lattices centered at lines with indices $+++, +--, -+-, --+$ and the rank 1 lattice $\langle 4\rangle$ with basis

$$
\begin{gathered}
2(L_{---} + L_{0--} + L_{-0-} + L_{--0} + L_{+--} + L_{-+-} + L_{--+})+ \\
+L_{+0-} + L_{0+-} + L_{0-+} + L_{+-0} + L_{-+0} + L_{-0+}.
\end{gathered}
$$

This yields a sublattice of rank 17, signature (1,16) and discriminant 4^5. Relations in $\mathbf{L}$ are found as follows. We observe that projecting the fibre onto one of the (p_i, q_i)-axes gives an (elliptic) pencil with two special fibres at $(0,1)$ and $(1,0)$ composed of the lines in two opposite faces of the cube of Fig. 1. Modulo $\mathbf{L}_0$ this gives the three relations

$$
\begin{aligned}
L_{+-+} + L_{-++} &\equiv L_{---} + L_{++-} \bmod \mathbf{L}_0 \\
L_{+-+} + L_{++-} &\equiv L_{---} + L_{-++} \bmod \mathbf{L}_0 \\
L_{+-+} + L_{---} &\equiv L_{++-} + L_{-++} \bmod \mathbf{L}_0.
\end{aligned}
$$

Since $2L_{---} \in \mathbf{L}_0$, by the construction of $\mathbf{L}_0$ we find

$$
2L_{---} \equiv 2L_{+-+} \equiv 2L_{-++} \equiv 2L_{++-} \equiv 0 \bmod \mathbf{L}_0
$$

and

$$
L_{---} + L_{+-+} + L_{-++} + L_{++-} \equiv 0 \bmod \mathbf{L}_0.
$$

The divisors $T_1 = L_{+-0} + L_{0--}$, $T_2 = L_{-+0} + L_{-0-}$, $T_3 = L_{+0-} + L_{0--}$ intersect the elements of $\mathbf{L}_0$ with even multiplicity, whereas

$$
\begin{aligned}
&L_{+-+}\cdot T_1 = 1, \quad L_{+-+}\cdot T_2 = 0, \quad L_{+-+}\cdot T_3 = 0 \\
&L_{-++}\cdot T_1 = 0, \quad L_{-++}\cdot T_2 = 1, \quad L_{-++}\cdot T_3 = 0 \\
&L_{++-}\cdot T_1 = 0, \quad L_{++-}\cdot T_2 = 0, \quad L_{++-}\cdot T_3 = 1.
\end{aligned}
$$

This shows that $\{L_{+-+},\ L_{++-},\ L_{-++}\}$ reduces modulo $\mathbf{L}_0$ to a basis of $\mathbf{L}/\mathbf{L}_0$. So the index of $\mathbf{L}_0$ in $\mathbf{L}$ is 8. Hence $\mathbf{L}$ has discriminant $4^5/8^2 = 16$. □

The lattice $\mathbf{L}$ is embedded in $H^2(\overline{\mathcal{X}}_s, \mathbf{Z})$ for every $s \neq \infty$ and is independent of s. The monodromy action of $\pi_1(\mathbf{C} \setminus \{\pm 2, \pm 6\}, s)$ on $H^2(\overline{\mathcal{X}}_s, \mathbf{Z})$ $(s \neq \pm 2, \pm 6)$ is trivial on this sublattice. For $s \neq \pm 2, \infty$ we have 12 more lines forming the G-orbit of the line

$$
\begin{aligned}
M_{1++} := \{(w, \sigma w, u_2, -u_2, \sigma^{-1} w, v_2, -v_2), (1,\sigma), (p_2, q_2), (p_2, -q_2) \mid \\
p_2 u_2 = q_2 w, \quad q_2 v_2 = p_2 w\}
\end{aligned}
$$

Here σ is fixed so that $\sigma + \sigma^{-1} = s$ and coordinates are as in (4). We label these twelve lines as $M_{k,\alpha,\beta}$, $k = 1,2,3$, $\alpha, \beta = +, -$ in such a way that in the parameter presentation one has $u_k = \sigma w$ if $\alpha = +$, $u_k = \sigma^{-1} w$ if $\alpha = -$, $u_i = -u_j$ if $i, j \neq k$ and $\beta = +, u_i = -v_j$ if $i, j \neq k$ and $\beta = -$.

Lemma 1 *The intersection products of the M-lines are*

$$M_{k\alpha\beta}\cdot M_{h\gamma\delta} = \begin{cases} -2 & \textit{if} \quad (k,\alpha,\beta)=(h,\gamma,\delta) \\ 2 & \textit{if} \quad (k,\alpha)=(h,\gamma), \quad \beta\neq\delta \\ 1 & \textit{if} \quad (\alpha,\beta)=(\gamma,\delta), \quad k\neq h \\ 1 & \textit{if} \quad \textit{all indices differ} \\ 0 & \quad\quad \textit{otherwise} \end{cases}$$

The intersection products of the L-lines and the M-lines are

$$M_{k\alpha\beta}\cdot L_{e_1e_2e_3} = \begin{cases} 1 & \textit{if} \quad e_k=0 \textit{ and } \beta=e_ie_j, \quad i\neq k\neq j\neq i \\ 0 & \quad\quad \textit{otherwise} \end{cases}$$

□

Let $\mathbf{M}_s$ be the sublattice of $H^2(\overline{\mathcal{X}}_s,\mathbf{Z})$ generated by the cycle classes of the L-lines and the M-lines.

Theorem 1 *The lattice* $\mathbf{M}_s$ *is isometric to*

$$-E_8\perp -E_8\perp H\perp\langle -12\rangle$$

for every $s\neq\pm 2,\infty$.

Proof: Consider the following three sublattices of $\mathbf{M}_s$:

$$\mathbf{K}_1=\langle L_{+-0},L_{--+},L_{0-+},L_{+-+},L_{+0+},L_{+++},L_{0++},M_{1++}\rangle,$$
$$\mathbf{K}_2=\langle L_{-+0},L_{---},L_{-0-},L_{-+-},L_{0+-},L_{++-},L_{+0-},M_{2+-}\rangle,$$

and finally

$$\begin{aligned}\mathbf{K}_3 \;=\; &\langle L_{0--}-L_{--0}+L_{-0+}-M_{3++}-M_{3-+},\\ &L_{+--}+L_{-++}-M_{1++}-M_{2--}-M_{3--},\\ &-L_{-0+}+M_{3-+}+2M_{3++}+2M_{1++}+2L_{0++}+2L_{+++}\\ &\qquad +2L_{+0+}+2L_{+-+}+L_{+-0}+L_{0-+}\rangle.\end{aligned}$$

Using Fig. 1 and Lemma 1 one checks that the preceding three lattices are mutually orthogonal, that the first two are isometric to $-E_8$ and that the intersection matrix for the third sublattice (for the given basis) is

$$\begin{pmatrix} -6 & 4 & 5 \\ 4 & -4 & -6 \\ 5 & -6 & -8 \end{pmatrix}.$$

The matrix relation

$$\begin{pmatrix} 1 & 2 & -1 \\ -2 & -5 & 4 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} -6 & 4 & 5 \\ 4 & -4 & -6 \\ 5 & -6 & -8 \end{pmatrix}\begin{pmatrix} 1 & -2 & 0 \\ 2 & -5 & 1 \\ -1 & 4 & -1 \end{pmatrix}=\begin{pmatrix} 0 & 0 & 1 \\ 0 & -12 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

shows that $\mathbf{K}_3$ is isometric to $H\perp\langle -12\rangle$. An easy argument, using the fact that the generic fibre of a non constant family of K3 surfaces has Picard number at most 19 shows that $\mathbf{K}_1\perp\mathbf{K}_2\perp\mathbf{K}_3$ has finite index in $\mathbf{M}_s$ and as in the proof of [P, prop. 7.1.1] we show that this index must be one . $\square$

As in [P, lemma 7.1.2] we then find:

Corollary 1 *For every $s \neq \pm 2, \pm 6, \infty$ the orthogonal complement of $\mathbf{M}_s$ in $H^2(\overline{\mathcal{X}}_s, \mathbf{Z})$ is isometric with the lattice $H\perp\langle 12\rangle$.*

Remark 1 For the minimal resolution of singularities of the fibres $\overline{\mathcal{X}}_{\pm 6}$ one computes the Néron-Severi lattice in a similar fashion and one finds that it is isometric to $-E_8\perp - E_8\perp H\perp\langle -12\rangle\perp\langle -2\rangle$ and its transcendental lattice is therefore isometric to $\langle 2\rangle\perp\langle 12\rangle$.
For $\overline{\mathcal{X}}_{\pm 2}$ we find $-E_8\perp - E_8\perp H\perp\langle -4\rangle\perp\langle -2\rangle$, resp. $\langle 4\rangle\perp\langle 2\rangle$.

Remark 2 The monodromy action of $\pi_1(\mathbf{C}\setminus\{\pm 2, \pm 6\}, s)$ on $H^2(\overline{\mathcal{X}}_s, \mathbf{Z})$ stabilizes the sublattice $\mathbf{M}_s$, but the monodromy action is not trivial. Positive simple loops around the two points ± 6 act trivially, but for similar such loops around the points ± 2 this only holds for the L-lines. The M-lines are pairwise permuted:

$$M_{k+\beta} \leftrightarrow M_{k-\beta} \text{ for } k = 1, 2, 3, \beta = \pm.$$

The M-lines are obviously fixed on the double cover of $\mathbf{P}^1(s)$ given by $s = \sigma + \sigma^{-1}$ (see the parametrization of these lines as given before). It follows that monodromy on the sublattice $\mathbf{M}_\sigma$ has become trivial on this double cover. For the action on the orthogonal complement this implies that monodromy takes place in the subgroup of isometries of $T = H\perp\langle 12\rangle$ inducing the identity on the discriminant lattice $T^*/T = \mathbf{Z}/12\mathbf{Z}$.

From the preceding remark we see that it is natural to make the substitution $s = -\xi_0 - \xi_0^{-1}$. The relation between the s-parameter and the new parameter is depicted in Fig. 2.

The resulting threefold Z

$$\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} + \xi_0 + \xi_0^{-1} = 0 \tag{5}$$

we call the <u>*Fermi-threefold*</u>. A compactification $\overline{Z}$ of the Fermi-threefold in $\mathbf{P}^8\times\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1\times\mathbf{P}^1$ with coordinates $((w, u_1, u_2, u_0, v_1, v_2, v_0), (s_1, t_1), (s_2, t_2), (s_3, t_3), (s_0, t_0))$ is given by

$$\begin{array}{rcl} s_i u_i = t_i w, \;\; t_i v_i = s_i w & & (i = 1, 2, 3, 0) \\ u_1 + v_1 + u_2 + v_2 + u_3 + v_3 + u_0 + v_0 & = & 0. \end{array}$$

There is an isomorphism

$$\begin{array}{ccc} \overline{Z}\setminus\{w = 0\} & \longrightarrow & Z \\ \xi_i = v_i/w & (i = 1, 2, 3, 0). & \end{array}$$

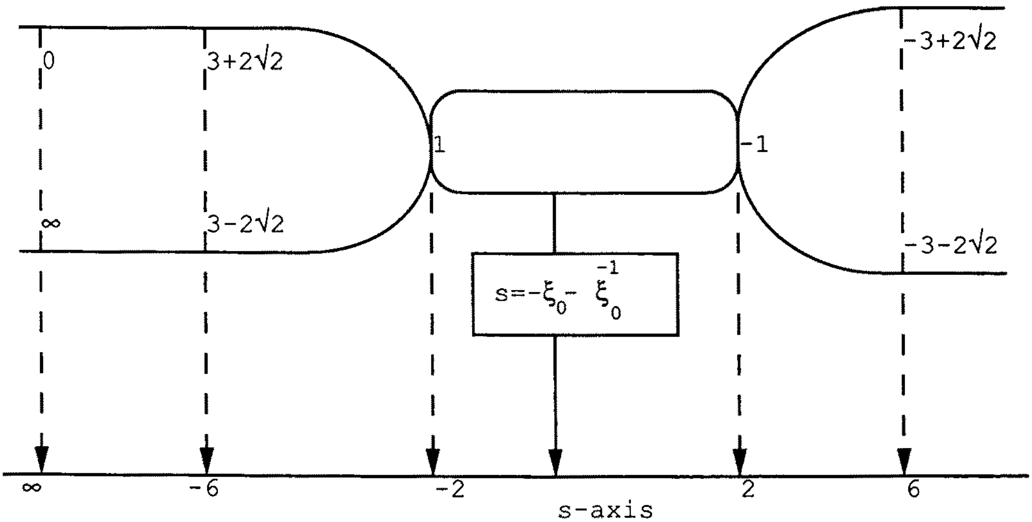


Figure 2: Relation between s and ξ_0.

We call the projection onto the ξ_0-axis

$$\pi_0 : \overline{Z} \longrightarrow \mathbf{P}^1(\xi_0) \tag{6}$$

the Fermi-fibration. An immediate corollary is the following central result.

Theorem 2 *For $\xi_0 \notin S := \{0, \infty, \pm 1, 3 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2}\}$ the fibres of the Fermi-fibration are K3-surfaces. The Néron-Severi lattice of the generic fibre is isometric to $-E_8 \perp -E_8 \perp H \perp \langle -12 \rangle$ and its transcendental lattice is isometric to $T = H \perp \langle 12 \rangle$. The monodromy is trivial on the Néron-Severi lattice and on the transcendental lattice takes place in the subgroup of isometries of T inducing the identity on the discriminant lattice $T^*/T = \mathbf{Z}/12\mathbf{Z}$.*

□

Remark 3 The precise monodromy representaion is given in Theorem 6. The exact meaning of "generic" is explained in Remark 7

3 The Picard-Fuchs equation

Consider, for $|s| > 6$, the integral

$$I(s) = (2\pi i)^{-3} \int_{|\xi_1|=1} \int_{|\xi_2|=1} \int_{|\xi_3|=1} \frac{\xi_1^{-1}\xi_2^{-1}\xi_3^{-1} d\xi_1 \wedge d\xi_2 \wedge d\xi_3}{\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1} - s}$$

By the Poincaré residue theorem one has

$$I(s) = (2\pi i)^{-2} \int_{\Gamma_s} \frac{1}{\xi_3 - \xi_3^{-1}} \frac{d\xi_1}{\xi_1} \wedge \frac{d\xi_2}{\xi_2},$$

where Γ_s is a 2-cycle on the surface $\overline{X}_s$. On the other hand one can expand $I(s)$ as a power series in the variable $t = s^{-1}$:

$$I(s) = -\sum_{n=0}^{\infty} a_n t^{n+1}$$

with

$$a_n = (2\pi i)^{-3} \int_{|\xi_1|=1} \int_{|\xi_2|=1} \int_{|\xi_3|=1} (\xi_1 + \xi_1^{-1} + \xi_2 + \xi_2^{-1} + \xi_3 + \xi_3^{-1})^n \frac{d\xi_1}{\xi_1} \wedge \frac{d\xi_2}{\xi_2} \wedge \frac{d\xi_3}{\xi_3}.$$

So

$$a_{2m+1} = 0$$
$$a_{2m} = \sum_{p+q+r=m} \frac{(2m)!}{(p!q!r!)^2}$$

We shall construct a differential equation, of which $I(s)$ is a solution, from a recurrence relation for the coefficients a_{2m}.

First we note, setting $k = m - p$,

$$a_{2m} = \binom{2m}{m} \sum_{k=0}^{m} \binom{m}{k}^2 \sum_{q+r=k} \binom{k}{q}\binom{k}{r} = \binom{2m}{m} u_m$$

with

$$u_m = \sum_{k=0}^{m} \binom{m}{k}^2 \binom{2k}{k}.$$

In [S-B, p.288] it is shown that these integers u_m satisfy the recurrence relation

$$(m+1)^2 u_{m+1} - (10m^2 + 10m + 3)u_m + 9m^2 u_{m-1} = 0.$$

Multiplying this relation with $8(m+1)\binom{2m+2}{m+1}$ one obtains the following recurrence relation for the integers a_{2m}

$$(2m+3-1)^3 a_{2m+2} - 8(5(2m+1)^2+1)(2m+1)a_{2m} +$$
$$+144(2m-1+1)(2m-1)(2m-1+2)a_{2m-2} = 0$$

From this one immediately sees that the function $I(t^{-1})$, given by the power series

$$-\sum_{m=0}^{\infty} a_{2m} t^{2m+1} \quad \text{for } |t| < \frac{1}{6}$$

is annihilated by the differential operator

$$(\Theta - 1)^3 - 8t^2\Theta(5\Theta^2 + 1) + 144t^4\Theta(\Theta+1)(\Theta+2),$$

where $\Theta = td/dt$. Setting $s = t^{-1}$ so that $\Theta = -sd/ds$, a simple calculation shows that $I(s)$ satisfies the differential equation

$$[(s^2-4)(s^2-36)(d/ds)^3 + 6s(s^2-20)(d/ds)^2 + (7s^2-48)(d/ds) + s]\ I = 0$$

We note that all other periods of the form ω (see (2)) also satisfy this differential equation. The argument is the same as that given in [B-P, p.46-47]. In fact the differential operator as studied in loc. cit. is the same as the preceding one, up to change of parameters as we shall state in a more precise form in Remark 4. So we arrive at

Theorem 3 *The Picard Fuchs equation for the periods of the form $\omega(s)$ is given by*

$$\{(s^2-4)(s^2-36)(d/ds)^3+6s(s^2-20)(d/ds)^2+(7s^2-48)(d/ds)+s\}I=0$$

In particular, the periods of $\omega(s)$ span the solution space.

□

In his proof of the irrationality of $\zeta(3)=\sum_1^\infty n^{-3}$ Apéry considered the recurrence relation

$$(n+1)^3v_{n+1}-(34n^3+51n^2+27n+5)v_n+n^3v_{n-1}=0,$$

of which one solution is given by

$$v_n=\sum_k\binom{n}{k}^2\binom{n+k}{k}^2. \tag{7}$$

(cf.[Po]). Apéry's recurrence is equivalent with

$$(2n+2)^3v_{n+1}-(34(2n+1)^3+6(2n+1))v_n+(2n)^3v_{n-1}=0$$

This form shows immediately that the generating power series

$$F(x)=\sum_{n\geq 0}v_nx^{2n+1}$$

is annihilated by the differential operator

$$\mathcal{L}=(P-1)^3-x^2(34P^3+6P)+x^4(P+1)^3,$$

where $P=x\,d/dx$. One now easily checks the following remarkable identity of differential operators

$$x^{-2}\mathcal{L}=(x-x^{-1})\cdot P^3\cdot(x-x^{-1})-32P^3.$$

The differential operator $(x-x^{-1})^{-1}x^{-2}\mathcal{L}$ is invariant under the involution $x\leftrightarrow x^{-1}$ and is (the pull back of) an operator in the variable $s=x+x^{-1}$. Indeed, from the easy to prove identities $d/ds=(x-x^{-1})^{-1}(x\,d/dx)$ and $(x\,d/dx)\cdot(x-x^{-1})=(s^2-4)(d/ds)+s$, this operator becomes

$$\Big((s^2-4)(d/ds)+s\Big)(d/ds)\Big((s^2-4)(d/ds)+s\Big)-32(d/ds)\Big((s^2-4)(d/ds)+s\Big)(d/ds),$$

which by direct calculation is seen to be equal to

$$(s^2-4)(s^2-36)(d/ds)^3+6s(s^2-20)(d/ds)^2+(7s^2-48)(d/ds)+s.$$

This is precisely the differential operator which we found to annihilate the period integral $I(s)$ (cf. Theorem 3).

Combining the results of this section we arrive at the following geometric picture.

Theorem 4 *The Fermi-fibration (6) is a fibration into K3-surfaces. The Picard Fuchs operator for the transcendental periods of the form ω (see (2)) is*

$$(\xi_0 - \xi_0^{-1})(\xi_0\, d/d\xi_0)^3 \cdot (\xi_0 - \xi_0^{-1}) - 32(\xi_0\, d/d\xi_0)^3,$$

For power series solutions

$$F(\xi_0) = \sum_{n\geq 0} v_n \xi_0^{2n+1},$$

the differential equation is equivalent with Apéry's recurrence relation on the v_n.

Remark 4 In [B-P] Apéry's recurrence is translated into a differential equation for the generating power series $G(u) := \sum_{n\geq 0} v_n u^n$ and it is shown that this is the Picard-Fuchs equation for the transcendental periods of a family of K3-surfaces. We have worked out the coordinate transformations relating the family of K3-surfaces from [B-P] to the Fermi-fibration (6). It turned out that the fibered threefold in [B-P] is the quotient of (1) by the involution $(\xi_0, \xi_1, \xi_2, \xi_3) \leftrightarrow (-\xi_0, -\xi_1, -\xi_2, -\xi_3)$. Since $F(\xi_0) = \xi_0 G(\xi_0^2)$ the Picard-Fuchs equation for $G(u)$ from [B-P] lifts to the Picard-Fuchs equation for $F(\xi_0)$.

4 The monodromy representation

Theorem 2 tells us that the monodromy of the Picard-Fuchs equation in Theorem 4 takes place in the index 4 subgroup of isometries of the lattice $T = H\perp\langle 12\rangle$ inducing the identity on the discriminant lattice. In this section we shall determine the exact monodromy representation. We shall do this by representing ξ_0 as a modular function and use it to pull back the variation of weight two Hodge structure on the smallest irreducible local system containing the transcendental lattices of the Fermi-fibration (6) to the upper half plane

$$\mathbf{H} = \{\tau \in \mathbf{C} \mid \mathrm{Im}\tau > 0\}.$$

To keep the notation simple we write ξ instead of ξ_0. First we recall Dedekind's η-function:

$$\eta(\tau) = e^{\pi i\tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i\tau n}) \quad , \tau \in \mathbf{H}$$

and its basic functional equations (see [R], p.163)

$$\eta(\tau + 1) = e^{\pi i/12}\eta(\tau) \quad , \quad \eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau) \tag{8}$$

Now define

$$\xi(\tau) = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)}\right)^6 = e^{\pi i\tau} \prod_{n=1,(n,6)=1}^{\infty} (1 - e^{2\pi i\tau n})^6 \tag{9}$$

From (8) one obtains the functional equations for $\xi(\tau)$

$$\xi(\tau + 1) = -\xi(\tau), \quad \xi(-1/6\tau) = \xi(\tau), \quad \xi\Big(\frac{2\tau - 1}{6\tau - 2}\Big) = \xi(\tau)^{-1} \tag{10}$$

Table 1:

τ	stabilizer	$\xi(\tau)$	"vanishing cycles"
$i\,\infty$	$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$	0	
$\pm 1/2$	$\begin{pmatrix} 1 \mp 12 & 6 \\ -24 & 1 \pm 12 \end{pmatrix}$	∞	
$i/\sqrt{6}$	$\begin{pmatrix} 0 & 1 \\ -6 & 0 \end{pmatrix}$	$3 - 2\sqrt{2}$	$(1, 0, -1)$
$\pm 1 + i/\sqrt{6}$	$\begin{pmatrix} \mp 6 & 7 \\ -6 & \pm 6 \end{pmatrix}$	$-3 + 2\sqrt{2}$	$(1, \pm 1, -7)$
$\pm 2/5 + i/(5\sqrt{6})$	$\begin{pmatrix} \mp 12 & 5 \\ -30 & \pm 12 \end{pmatrix}$	$3 + 2\sqrt{2}$	$(-5, \pm 2, 5)$
$\pm 3/5 + i/(5\sqrt{6})$	$\begin{pmatrix} \mp 18 & 11 \\ -30 & \pm 18 \end{pmatrix}$	$-3 - 2\sqrt{2}$	$(5, \pm 3, -11)$

Theorem 5 *The space of solutions of the differential equation*

$$[(\xi - \xi^{-1})(\xi\, d/d\xi)^3 \cdot (\xi - \xi^{-1}) - 32(\xi\, d/d\xi)^3]\, g(\xi) = 0$$

has a basis of multiple valued functions on the ξ-line, which on the upper half plane can be given as the triple $\{G(\tau), \tau G(\tau), \tau^2 G(\tau)\}$ of univalent functions, where $G(\tau)$ is a cusp form of weight two for $\Gamma_1(6,2)$ which satisfies $G(-1/6\tau) = -6\tau^2 G(\tau)$.

Proof: We use [Be, sections 1 and 2]. First one observes that any solution $h(t)$ of equation (A) in [Be] yields a solution $g(\xi) := \xi h(\xi^2)$ of the above differential equation (cf. Remark 4). The results of [Be] in combination with the argument of [S-B, p. 296] now show that this differential equation has a basis of solutions $\xi(\tau)F(\tau), \tau\xi(\tau)F(\tau), \tau^2\xi(\tau)F(\tau)$, where $F(\tau)$ is given in [Be, p.205]. From [loc. cit. p.206] one may conclude that $G(\tau) = \xi(\tau)F(\tau)$ has the desired properties . □

Remark 5 The space of cusp forms of weight two for $\Gamma_1(2,6)$ has dimension one, since the canonical morphism of modular curves

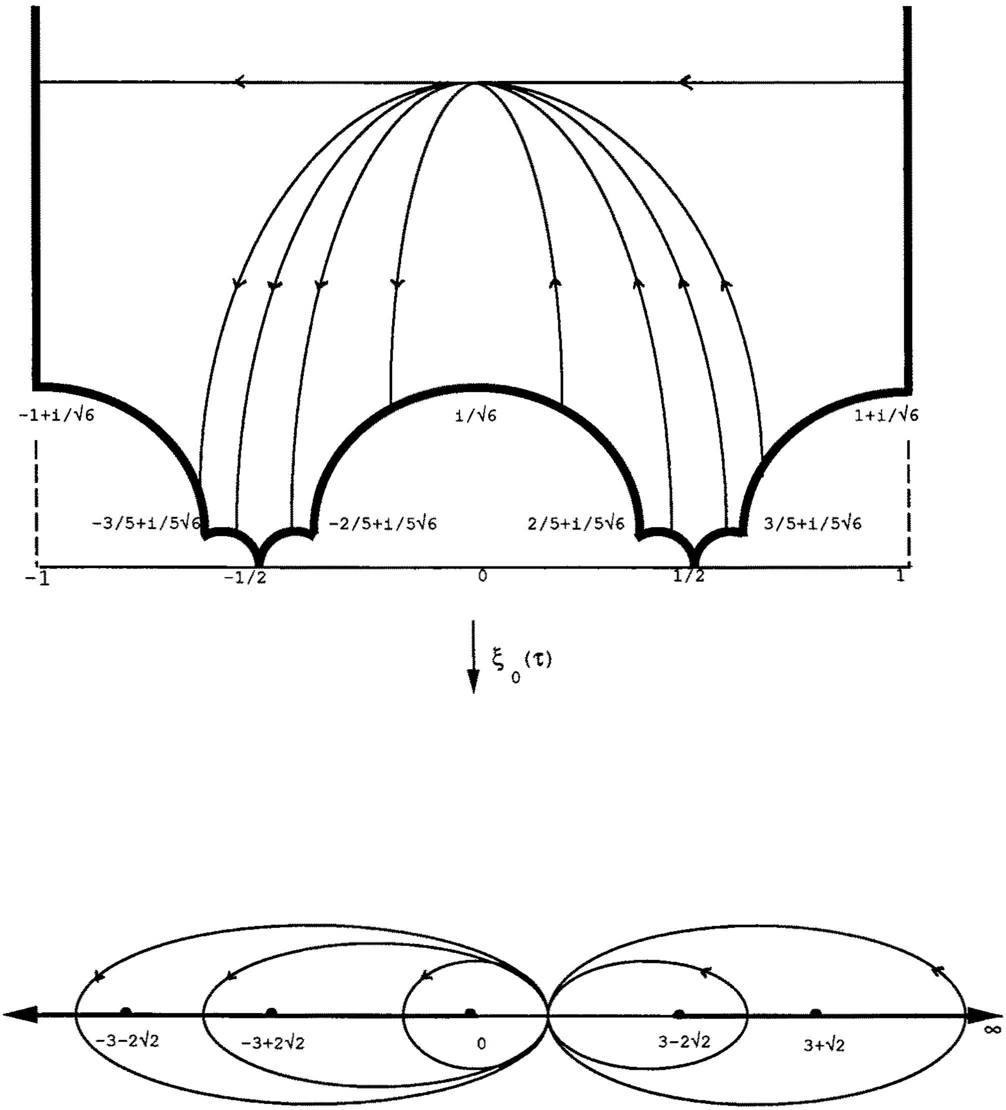
$$\Gamma_1(6,2)\backslash \mathbf{H}^* \longrightarrow \Gamma_1(6,2)^* \backslash \mathbf{H}^*$$

is a double cover of $\mathbf{P}^1$ branched in the four points $3 \pm 2\sqrt{2}, -3 \pm \sqrt{2}$ and hence the source is an elliptic curve. Since $\eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$ is easily seen to be a cusp form of weight two for $\Gamma_1(6,2)$ with the same first Fourier expansion term as $G(\tau)$, we have identified $G(\tau)$ as

$$G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau).$$

We now take a basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$ of the lattice T with Gram matrix $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 12 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ and we define

$$\omega(\tau) := G(\tau)(\mathbf{e}_0 + \tau \mathbf{e}_1 - 6\tau^2 \mathbf{e}_2).$$



-1+i/√6
i/√6
1+i/√6
-3/5+i/5√6
-2/5+i/5√6
2/5+i/5√6
3/5+i/5√6
-1
-1/2
0
1/2
1
ξ 0 (τ)
-3-2√2
-3+2√2
0
3-2√2
3+√2
∞

Figure 3

To describe the transformation behaviour of $\xi(\tau)$ we need the following subgroups of $Gl(2,\mathbf{Q})$:

$$\begin{aligned}
\Gamma_1(6) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2\mathbf{Z} \;\mid a \equiv d \equiv 1 \bmod 6, \;\; c \equiv 0 \bmod 6 \right\}, \\
\Gamma_1(6,2) &= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6) \mid \;\; c \equiv 6b \bmod 12 \right\}, \\
\Gamma_1(6)^* &= \text{group generated by } \Gamma_1(6) \text{ and } w_6, \text{ with } w_6 = \begin{pmatrix} 0 & 1 \\ -6 & 0 \end{pmatrix}, \\
\Gamma_1(6,2)^* &= \text{group generated by } \Gamma_1(6,2) \text{ and } w_6.
\end{aligned}$$

Proposition 2

$$\xi\left(\frac{a\tau+b}{c\tau+d}\right) = \xi(\tau) \quad \text{for} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6,2)^*.$$

Proof: This follows from [R, p.163] and (10) . □

The group $\Gamma_1(6,2)^*$ acts by fractional linear transformations on the extended upper half plane $\mathbf{H}^* := \mathbf{H} \cup \mathbf{Q} \cup \{i\infty\}$. A fundamental domain for this action is shown in Fig. 3.

The function $\xi(\tau)$ extends to a meromorphic function on $\mathbf{H}^*$. Its only zero in the fundamental domain is at $i\infty$ and has order 1. Consequently, $\xi(\tau)$ generates the field of modular functions for the group $\Gamma_1(6,2)^*$ and induces an isomorphism of the modular curve $\Gamma_1(6,2)^*\backslash\mathbf{H}^*$ onto $\mathbf{P}^1$. Since w_6^2 acts trivially, the action on $\mathbf{H}^*$ factors through an effective action of

$$P\Gamma_1(6,2)^* := \Gamma_1(6,2)^*/\langle w_6^2\rangle.$$

The points in the fundamental domain with a non-trivial stabiliser are listed in the first column of Table 1. The second column gives a generator for this stabiliser which is of infinite order for the first two entries and of order two for the remaining ones. The third column gives the ξ-images of these points; these values have been determined up to sign in [Be2, Proposition 2.1], while the signs are found upon using (9) and (10). The last column will be explained below (cf. Remark 6). The group $P\Gamma_1(6,2)^*$ acts freely on

$$\mathbf{H}^0 := \mathbf{H} \setminus \{P\Gamma_1(6,2)^*\text{-orbit of } \frac{i}{\sqrt{6}} \cup 1 + \frac{i}{\sqrt{6}} \cup \frac{2}{5} + \frac{i}{5\sqrt{6}} \cup \frac{3}{5} + \frac{i}{5\sqrt{6}}\}$$

and so ξ induces an unramified covering

$$\xi : \mathbf{H}^0 \longrightarrow \mathbf{P}^1 \setminus \{0, \infty, 3 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2}\} \tag{11}$$

with covering group $P\Gamma_1(6,2)^*$. We pull back the Picard-Fuchs equation from Theorem 4 and obtain a basis of solutions.

The vector valued function $\omega(\tau)$ is a global section of the vector bundle $T \otimes_{\mathbf{Z}} \mathcal{O}_{\mathbf{H}^0}$. The trivial local system T on $\mathbf{H}^0$ carries a variation of Hodge structure of weight two given by

$$\begin{array}{rcl} T \otimes_{\mathbf{Z}} \mathcal{O}_{\mathbf{H}^0} & \supset \mathcal{F}^1 & \supset \mathcal{F}^2, \\ \mathcal{F}^2 & = & \text{line bundle spanned by } \omega(\tau), \\ \mathcal{F}^1 & = & (\mathcal{F}^2)^{\perp}. \end{array} \tag{12}$$

Introduce the three dimensional representation (using the basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2\}$ for T)

$$j : \Gamma_1(6,2)^* \longrightarrow Gl(T)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d^2 & 2cd & -c^2/6 \\ bd & ad+bc & -ac/6 \\ -6b^2 & -12ab & a^2 \end{pmatrix} \cdot (-6)^{-k} \quad \text{if } ad - bc = 6^k, k \in \mathbf{Z},$$

and the groups

$$\begin{array}{rcl} O^0(T) & = & \{\text{isometries of } T \text{ inducing the identity on the discriminant lattice of } T\} \\ O^1(T) & = & \left\{ (a_{ij}) \in O^0(T) \;\middle|\; \begin{array}{ccc} a_{11} \equiv 1 \bmod 12 & \text{and} & a_{13} \equiv a_{31} \bmod 12 \\ & \text{or} & \\ a_{13} \equiv 1 \bmod 12 & \text{and} & a_{11} \equiv a_{33} \bmod 12 \end{array} \right\}. \end{array}$$

Note that $O^0(T)$, resp. $O^1(T)$ has index 4, resp. 16 in the group of all isometries of T.

Proposition 3 *We have for* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6,2)^*$:

$$\omega\left(\frac{a\tau+b}{c\tau+d}\right) = j\begin{pmatrix} a & b \\ c & d \end{pmatrix}\omega(\tau)$$

The map j induces an isomorphism

$$\bar{j} : P\Gamma_1(6,2)^* \xrightarrow{\sim} O^1(T).$$

The action of $\Gamma_1(6,2)^*$ on the variation of Hodge structure (12) induces one on the quotient local system over $\mathbf{P}^1 \setminus \{0, \infty, 3 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2}\}$ (cf. (11)) and the monodromy group of this variation is the group $O^1(T)$.

Proof: The first assertion follows directly from the the transformation properties of the form $G(\tau)$: it is modular with respect to $\Gamma_1(6,2)$ and $G(-1/6\tau) = -6\tau^2 G(\tau)$. One checks injectivity of $\bar{j}$ directly. Surjectivity can be seen in the following way. After multiplying with $j(w_6)$ if necessary we may assume

$$\begin{array}{lcl} a_{11} & \equiv & 1 \bmod 12, \\ a_{13} & \equiv & a_{31} \bmod 12 \\ 2a_{11}a_{31} + 12a_{21}^2 & = & 0 \\ 2a_{13}a_{33} + 12a_{23}^2 & = & 0 \\ a_{11}a_{33} + a_{13}a_{31} + 12a_{21}a_{23} & = & 1 \end{array}$$

The only integral solutions of these equations are

$$\begin{aligned} a_{11} = d^2, a_{33} = a^2, \quad & a_{31} = -6b^2, a_{13} = -c^2/6 \\ a_{21} = bd, a_{23} = -ac/6 & \end{aligned}$$

with

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(6,2).$$

The remaining equations for the a_{ij} then yield

$$a_{12} = 2cd, \; a_{32} = -12ab, \; a_{22} = ad + bc.$$

The last assertion is obvious . □

This proposition translates now immediately into the geometric setting of the Fermi-fibration (6), which is our main result.

Theorem 6 *For the Fermi-fibration (cf (6))*

$$\pi_0 : \overline{Z} \setminus \pi_0^{-1} S \longrightarrow \mathbf{P}^1 \setminus S$$

where

$$S := \{0, \infty, \pm 1, 3 \pm 2\sqrt{2}, -3 \pm 2\sqrt{2}\}.$$

the form (cf. (2))

$$\omega(\xi_0) = \frac{1}{\xi_3 - \xi_3^{-1}} \frac{d\xi_1}{\xi_1} \wedge \frac{d\xi_2}{\xi_2}$$

gives a variation of weight two Hodge structures on the irreducible subsystem of $R^2(\pi_0)_*\mathbf{Z}$ whose fibres contain the transcendental lattices. The function

$$\xi_0 = \left(\frac{\eta(\tau)\eta(6\tau)}{\eta(2\tau)\eta(3\tau)} \right)^6$$

pulls back this variation to the the variation (12) restricted to

$$\mathbf{H} \setminus \{P\Gamma_1(6,2)^*\text{-orbit of } \; \tfrac{1}{3} + \tfrac{i}{\sqrt{18}} \cup \tfrac{2}{3} + \tfrac{i}{\sqrt{18}} \cup \tfrac{i}{\sqrt{6}} \cup 1 + \tfrac{i}{\sqrt{6}} \cup \tfrac{2}{5} + \tfrac{i}{5\sqrt{6}} \cup \tfrac{3}{5} + \tfrac{i}{5\sqrt{6}}\}$$

The monodromy group of this variation is $P\Gamma_1(6,2)^* \cong O^1(T)$ (see Proposition 3). A triple of multivalued functions forming a basis for the solutions of the Picard-Fuchs equation

$$[(\xi_0 - \xi_0^{-1})(\xi_0\, d/d\xi_0)^3 \cdot (\xi_0 - \xi_0^{-1}) - 32(\xi_0\, d/d\xi_0)^3]\, g(\xi_0) = 0$$

associated to this variation lifts to the univalent triple $\{\, G(\tau), \tau G(\tau), \tau^2 G(\tau) \,\}$ on the upper half plane, where $G(\tau) = \eta(\tau)\eta(2\tau)\eta(3\tau)\eta(6\tau)$.

Remark 6 The monodromy group is generated by the j-images of the matrices given in Table 1. The corresponding elements in the monodromy group, which have finite order can be viewed as Picard Lefschetz transformations in certain vanishing cycles, i.e. reflections of the form $x \mapsto x + \langle x, e \rangle e$, where e is the element of norm -2 given in the last column of Table 1. A set of generators for the fundamental group of $\mathbf{P}^1 \setminus \{0, \infty, \pm 3 \pm 2\sqrt{2}\}$ is depicted in Fig. 3: the arcs in the upper half plane give the closed arcs in the punctured projective plane. Here we use the identifications of the boundary arcs connected by each of the dashed arcs given by the matrices (from top to bottom)

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 6 & -5 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} 7 & -4 \\ -12 & 7 \end{pmatrix}, \begin{pmatrix} -5 & 2 \\ 12 & -5 \end{pmatrix} \text{ and } w_6.$$

Remark 7 It follows from section 2 that the Picard Number (= the rank of the Néron- Severi lattice) of a K3-surface in the Fermi-fibration is 19 or 20. For the fibre over $\xi_0(\tau)$ this rank is 20 if and only if there exists a vector $pe_0 + qe_1 + re_2$ in T which is perpendicular to $\omega(\tau)$ i.e. if and only if $-6p\tau^2 + 12q\tau + r = 0$. Thus one sees that this fibre is a K3-surface with Picard number 20 if and only if τ satisfies a quadratic equation over $\mathbf{Q}$. So in Theorem 2 the generic fibres are those over $\xi_0(\tau)$ with τ not in an imaginary quadratic number field. Notice also that by this argument every vector in T with negative norm $12q^2 + 2pr$ becomes an algebraic cycle in some fibre.

Remark 8 The monodromy of the original family of Fermi-surfaces over $\mathbf{P}^1(s)$ is obtained from the monodromy of the Fermi-fibration by adding $w_2 = \begin{pmatrix} 2 & -1 \\ 6 & -2 \end{pmatrix}$ to the generators of $P\Gamma_1(6,2)^*$. This follows directly from (10).

References

[Be] F. Beukers: Another congruence for the Apéry numbers, *Journ. of Number Theory*, **25**, 1987, 201-210.

[Be2] F. Beukers: Irrationality proofs using modular forms. *Journées Arithm. Besançon (1985), Astérisque*, **147**, 1987, 271-283, 345.

[B-P] F. Beukers, C. Peters: A family of K3-surfaces and $\zeta(3)$, *Journ. f. reine u. angew. Math.*, **351**, 1984, 42-54.

[G-K-T] D. Gieseker, H. Knörrer, E. Trubowitz: Fermi curves and density of states, forthcoming.

[P] C. Peters: Monodromy and Picard-Fuchs equations for families of K3-surfaces and elliptic curves, *Ann. Scient. Éc. Norm. Sup. 4.e ser.* **19**, 1986, 583-607.

[Po] A.J. van der Poorten: A proof that Euler missed.. Apéry's proof of the irrationality of $\zeta(3)$, *Math. Intell.*, **1**, 1979, 195-203.

[R] H. Rademacher, *Topics in analytic number theory*, Springer Verlag 1973.

[S-B] J. Stienstra, F. Beukers: On the Picard-Fuchs equation and the formal Brauer group of certain elliptic K3-surfaces, *Math. Ann.*, **271**, 1985, 269-304.

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