Dimer Models and Hypergeometric Systems.

Jan Stienstra

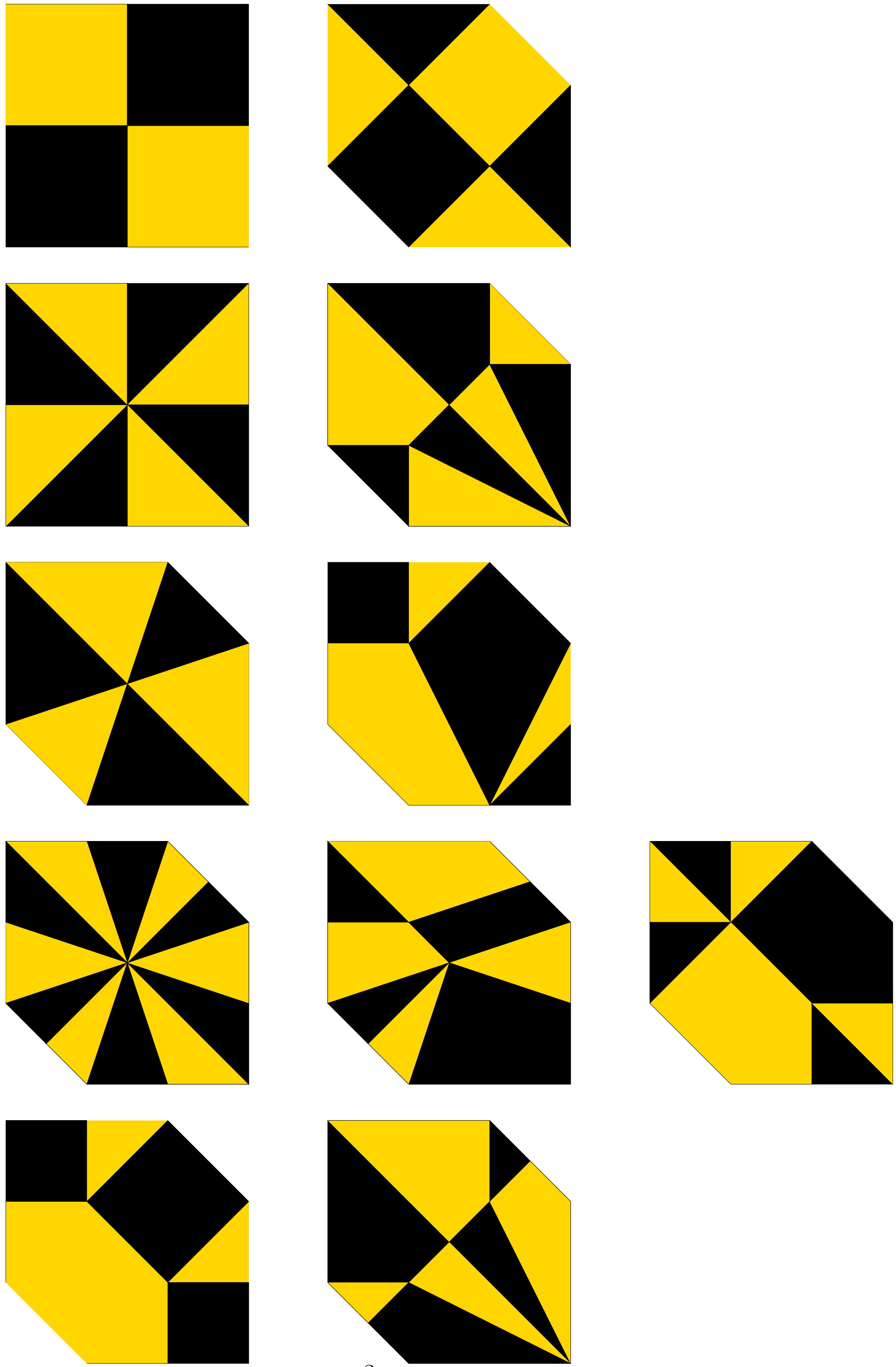
talk at BIRS May 13, 2011

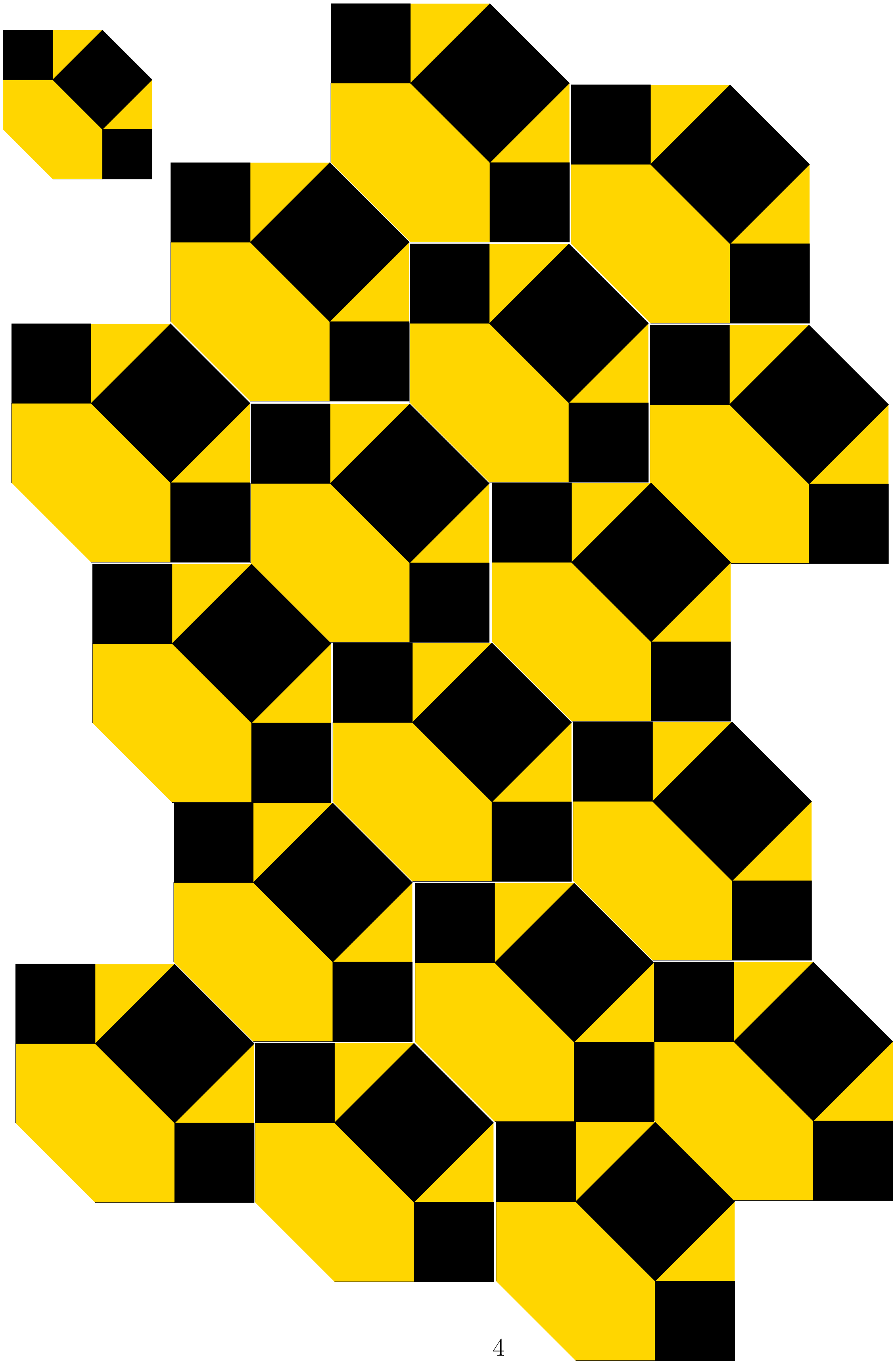
Dimer Model is:

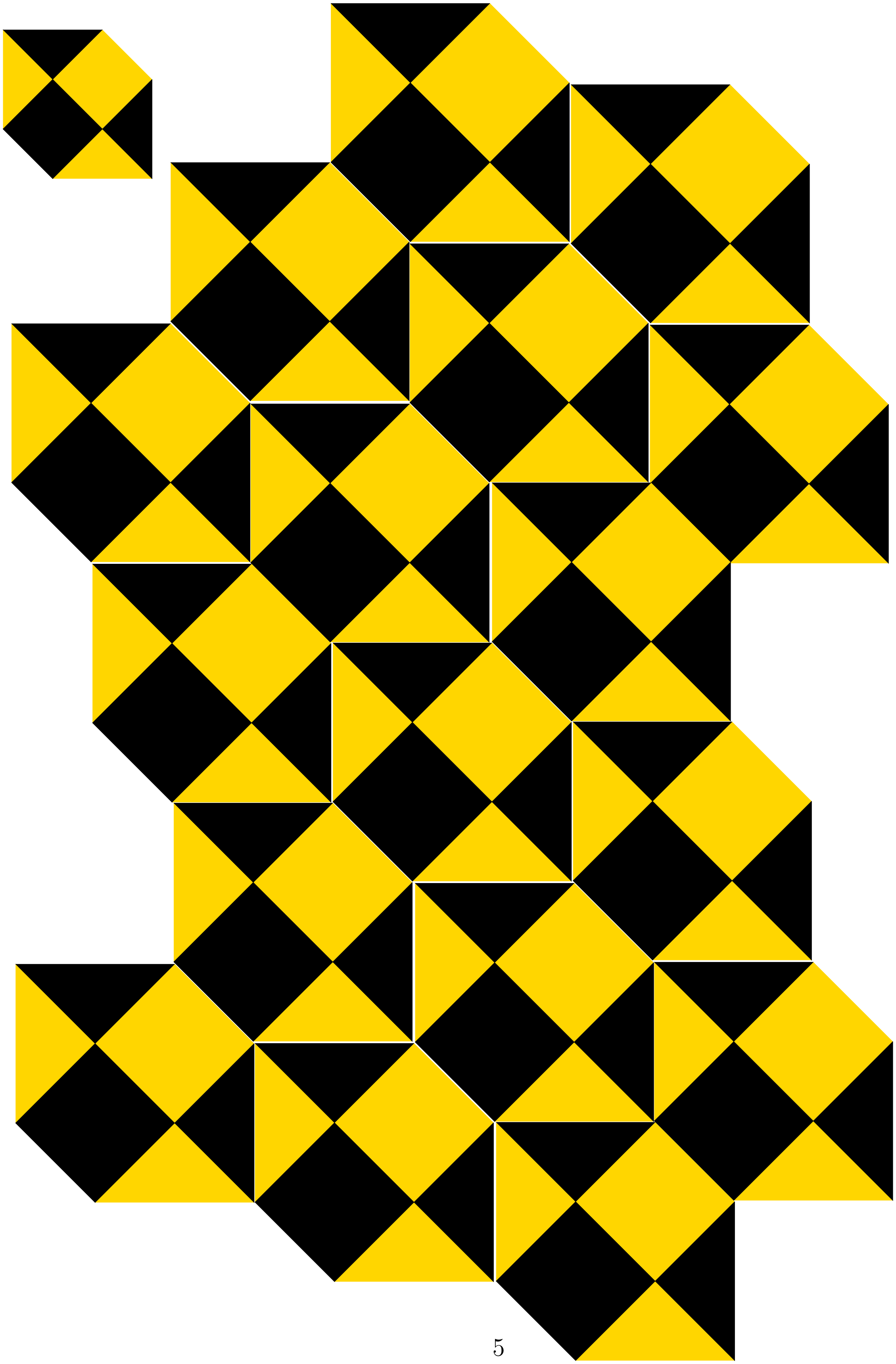
finite bipartite tiling of an
oriented surface without boundary

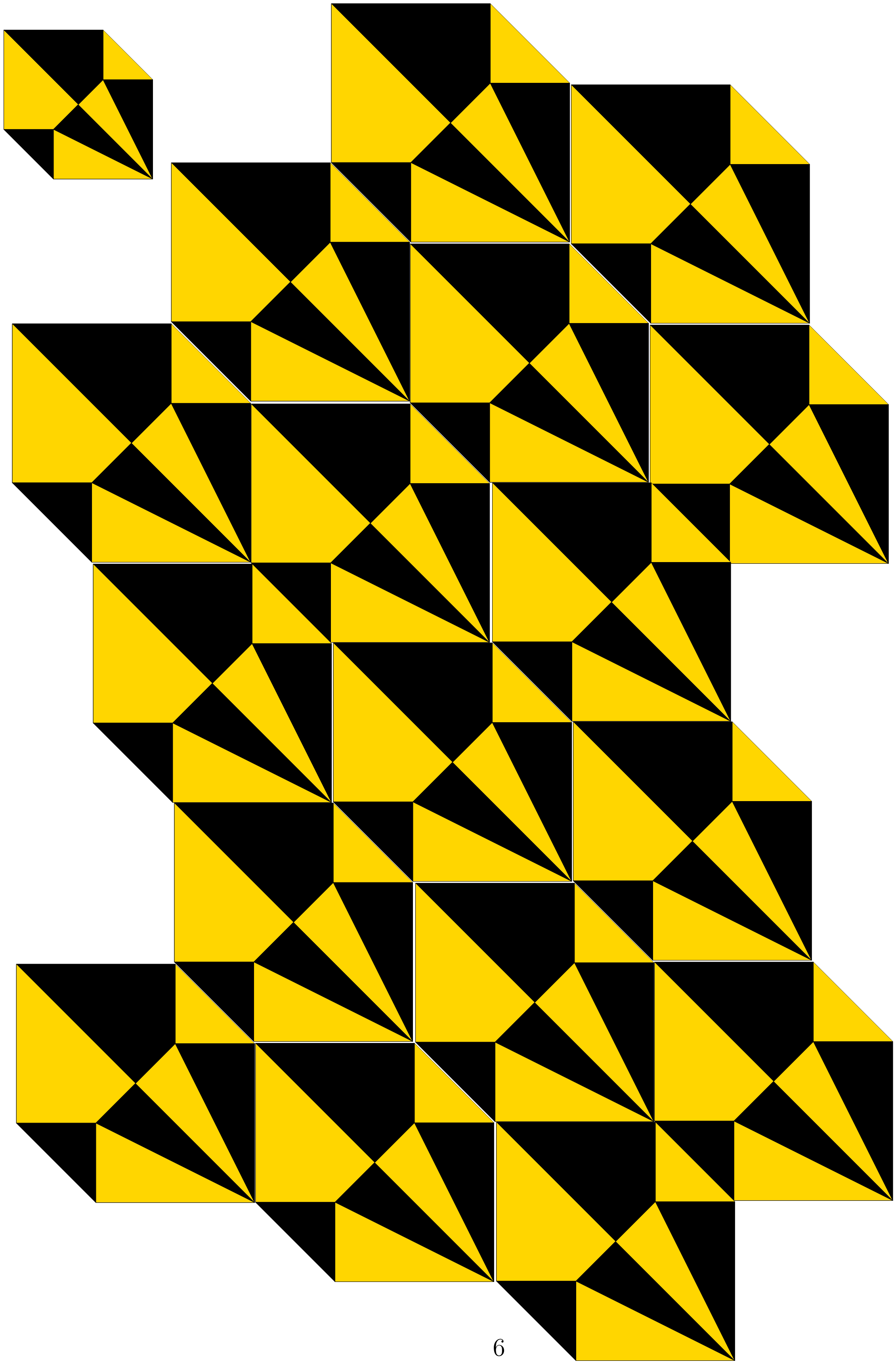
Balanced Dimer Model:

$\sharp$ black cells $=$ $\sharp$ white cells









Dimer Model

$$Q = (Q_0, Q_1, Q_2^\bullet, Q_2^\circ, b, w, s, t)$$

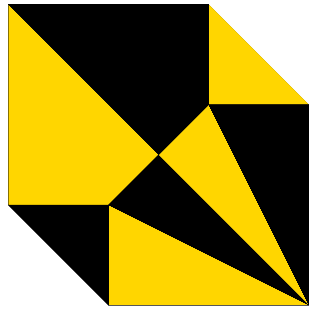
$$\begin{aligned} Q_0 &= \text{set of vertices} \\ Q_1 &= \text{set of edges} \\ Q_2^\bullet &= \text{set of black cells} \\ Q_2^\circ &= \text{set of white cells} \end{aligned}$$

$$(b,\ w,\ s,\ t) :\ Q_1 \longrightarrow Q_2^\bullet \sqcup Q_2^\circ \sqcup Q_0 \sqcup Q_0$$

$$(b,\ w,\ s{+}t) :\ \mathbb{Z}Q_1 \longrightarrow \mathbb{Z}Q_2^\bullet \oplus \mathbb{Z}Q_2^\circ \oplus \mathbb{Z}Q_0$$

lattice $\mathbf{M}_Q$ =

$$\ker((b,\ w,\ s{+}t)^\vee : \mathbb{Z}^{Q_2^\bullet} \oplus \mathbb{Z}^{Q_2^\circ} \oplus \mathbb{Z}^{Q_0} \longrightarrow \mathbb{Z}^{Q_1})$$



Q_1	$\xrightarrow{b,w,s,t}$	$Q_2^\bullet$	Q_2°	Q_0	Q_0
1		1	1	2	4
2		1	2	1	2
3		1	3	4	1
4		2	1	4	5
5		2	2	5	1
6		2	3	1	3
7		2	4	3	4
8		3	1	5	2
9		3	2	2	4
10		3	4	4	5
11		4	2	4	5
12		4	3	3	4
13		4	4	5	3

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 \\
0 & -1 & -1 & -2 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 1
\end{bmatrix}^t
$$

$$N = \sharp Q_0$$

Grassmannian of lines in $\mathbb{P}^{N-1}$:

$$\mathbf{G}(2, N) = {\mathrm{Gl}(2)}\backslash^{\mathrm{M}(2, N)}$$

Plücker coordinates on $\mathbf{G}(2, N)$:

$$[k, m] \qquad \text{for} \qquad k, m = 1, \ldots, N$$

$$[k, m]((y_{ij})) = y_{1k}y_{2m} - y_{2k}y_{1m}$$

$$N' = \sharp Q_2^{\bullet} = \sharp Q_2^{\circ}$$

$$E_{pq} = N' \times N'\text{-matrix with}$$
$$\quad 1 \text{ in position } p, q$$
$$\quad 0 \text{ elsewhere}$$

Definition:
The **Grassmann-Kasteleyn matrix** of dimer model Q is:

$$\mathbf{K}_Q = \sum_{e \in Q_1} [s(e)\, t(e)]\, E_{b(e)w(e)}$$

Q_1	$\xrightarrow{b,w,s,t}$	$Q_2^{\bullet}$	Q_2°	Q_0	Q_0
1		1	1	2	4
2		1	2	1	2
3		1	3	4	1
4		2	1	4	5
5		2	2	5	1
6		2	3	1	3
7		2	4	3	4
8		3	1	5	2
9		3	2	2	4
10		3	4	4	5
11		4	2	4	5
12		4	3	3	4
13		4	4	5	3

$$\mathbf{K}_Q = \begin{pmatrix} [2\ 4] & [1\ 2] & [4\ 1] & 0 \\ [4\ 5] & [5\ 1] & [1\ 3] & [3\ 4] \\ [5\ 2] & [2\ 4] & 0 & [4\ 5] \\ 0 & [4\ 5] & [3\ 4] & [5\ 3] \end{pmatrix}$$

Torus ${\mathbb{C}^*}^{Q_0}$ acts on $\mathbb{P}^{N-1}$
by coordinatewise multiplication.

$$\rightsquigarrow$$

${\mathbb{C}^*}^{Q_0}$ acts on coordinate ring of $\mathbf{G}(2, N)$

Torus ${\mathbb{C}^*}^{Q_2^\bullet}$ (resp. ${\mathbb{C}^*}^{Q_2^\circ}$)
acts on $N' \times N'$-matrices by
left (resp. right) multiplication with
diagonal matrices

$$\rightsquigarrow\rightsquigarrow$$

Torus $\mathbb{C}^{*Q_2^\bullet} \times \mathbb{C}^{*Q_2^\circ} \times \mathbb{C}^{*Q_0}$ acts on Grassmann-Kasteleyn matrix $\mathbf{K}_Q$:

$$(\alpha, \beta, \gamma) * \mathbf{K}_Q \ =$$

$$\sum_{e \in Q_1} \alpha(b(e))\beta(w(e))\gamma(s(e))\gamma(t(e)) \\ [s(e)\ t(e)]\, E_{b(e)w(e)}$$

Note:

$$(\alpha, \beta, \gamma) * \mathbf{K}_Q \ = \ \mathbf{K}_Q$$

$$\Leftrightarrow$$

$$(\alpha, \beta, \gamma) \ \in \ \mathbf{M}_Q \otimes \mathbb{C}^*$$

Definition

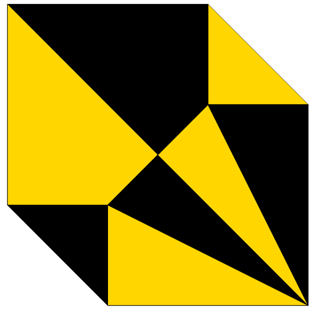
The **Chow form** of the dimer model Q is the determinant of the Grassmann-Kasteleyn matrix of Q,

$$\mathbf{Chow}_Q = \det \mathbf{K}_Q \,,$$

considered as a homogeneous element of degree N' in the coordinate ring of $\mathbf{G}(2, N)$

Definition

The **Spectral hypersurface** $\mathbf{Spec}_Q$ of the dimer model Q is the closed subscheme of the Grassmannian $\mathbf{G}(2, N)$ defined by the Chow form $\mathbf{Chow}_Q$



$$\mathbf{K}_Q \;=\; \begin{pmatrix} [2\;4] & [1\;2] & [4\;1] & 0 \\ [4\;5] & [5\;1] & [1\;3] & [3\;4] \\ [5\;2] & [2\;4] & 0 & [4\;5] \\ 0 & [4\;5] & [3\;4] & [5\;3] \end{pmatrix}$$

$$\begin{aligned} \mathbf{Chow}_Q \;=\; & [41][34][52][45] - [41][45][45][45] \\ & + [24][13][45][45] - [12][34][52][34] \\ & + [24][34][24][34] + [12][45][45][34] \\ & - [24][51][45][34] - [41][51][52][53] \\ & + [12][13][52][53] + [41][45][24][53] \\ & - [24][13][24][53] \end{aligned}$$

Definition

A **dimer** is a pair of adjacent cells.
A **dimer configuration** (a.k.a. perfect matching) is a set of dimers such that every cell belongs to exactly one dimer in that set.

The monomials in the Chow form correspond bijectively with dimer configurations

Torus ${\mathbb{C}^*}^{Q_2^\bullet} \times {\mathbb{C}^*}^{Q_2^\circ} \times {\mathbb{C}^*}^{Q_0}$ acts on $\mathbb{P}^{N-1}$ via the factor ${\mathbb{C}^*}^{Q_0}$;
${\mathbb{C}^*}^{Q_0}$ acts by coordinatewise multiplication.

$$\mathbf{M}_Q \otimes \mathbb{C}^* \quad \subset \quad {\mathbb{C}^*}^{Q_2^\bullet} \times {\mathbb{C}^*}^{Q_2^\circ} \times {\mathbb{C}^*}^{Q_0}$$

$$\rightsquigarrow\rightsquigarrow$$

Torus $\mathbf{M}_Q \otimes \mathbb{C}^*$ acts on $\mathbb{P}^{N-1}$

Theorem

If $\quad \text{rank}\,\mathbf{M}_Q = N-1 \quad$ then

1. point $\mathbf{y}$ in $\mathbf{G}(2,N)$ is on the spectral hypersurface $\mathbf{Spec}_Q$

$$\Leftrightarrow$$

   line $\mathbf{y}$ in $\mathbb{P}^{N-1}$ intersects the closure of orbit $(\mathbf{M}_Q \otimes \mathbb{C}^*)[1 : \ldots : 1]$

2. point $\mathbf{x}$ in $\mathbb{P}^{N-1}$ is in the closure of orbit $(\mathbf{M}_Q \otimes \mathbb{C}^*)[1 : \ldots : 1]$

$$\Leftrightarrow$$

   every line in $\mathbb{P}^{N-1}$ through $\mathbf{x}$ "is" a point on $\mathbf{Spec}_Q$

Consequence:

new equations for $\overline{(\mathbf{M}_Q \otimes \mathbb{C}^*)[1 : \ldots : 1]}$, namely:

$$\mathbf{x} \in \overline{(\mathbf{M}_Q \otimes \mathbb{C}^*)[1 : \ldots : 1]}$$

$$\Leftrightarrow$$

$$\mathbf{Chow}_Q(<\mathbf{xz}>) = 0\,, \qquad \forall \mathbf{z} \in \mathbb{P}^{N-1}$$

Another consequence:

For a point $\mathbf{y}$ on the spectral hypersurface $\mathbf{Spec}_Q$ the intersection points of the corresponding line in $\mathbb{P}^{N-1}$ with $\overline{(\mathbf{M}_Q \otimes \mathbb{C}^*)[1 : \ldots : 1]}$ are found as follows:

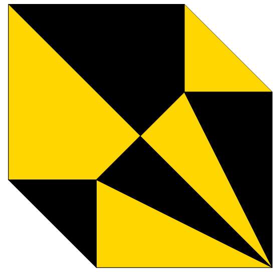
Take points $\mathbf{x} \neq \mathbf{x}'$ on $\mathbf{y}$ with $\mathbf{x}'$ not in $\overline{(\mathbf{M}_Q \otimes \mathbb{C}^*)[1 : \ldots : 1]}$. Points on line $\mathbf{y}$ are $\mathbf{x} + t\mathbf{x}'$ with $t \in \mathbb{C}$.

Then $\mathbf{x} + t\mathbf{x}'$ is an intersection point iff $\det(\mathbf{K}_Q(<\mathbf{xz}>) + t\,\mathbf{K}_Q(<\mathbf{x}'\mathbf{z}>)) = 0$ for every $\mathbf{z} \in \mathbb{P}^{N-1}$.

In particular, if $\mathbf{Chow}_Q(<\mathbf{x}'\mathbf{z}>) \neq 0$, then

t is an eigenvalue of the matrix

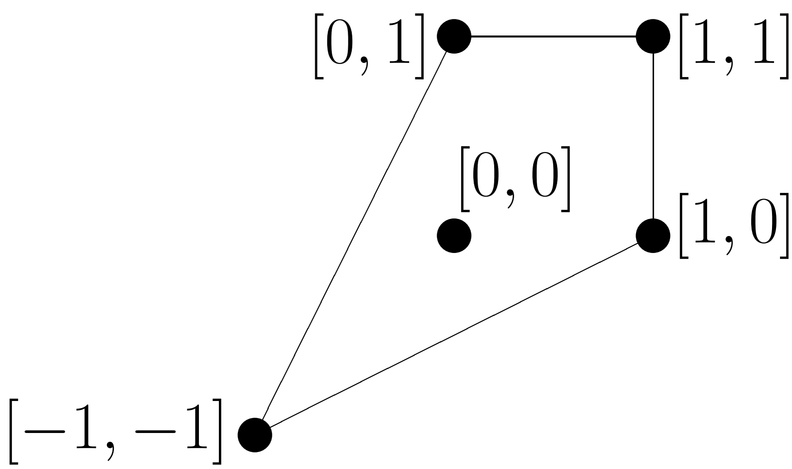
$$\mathbf{K}_Q(<\mathbf{z}\mathbf{x}'>)^{-1}\mathbf{K}_Q(<\mathbf{x}\mathbf{z}>)$$

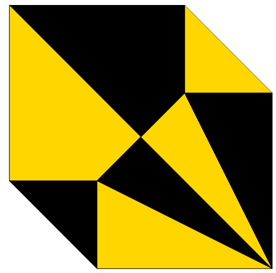


$$\begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -2 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\rightsquigarrow \qquad \begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

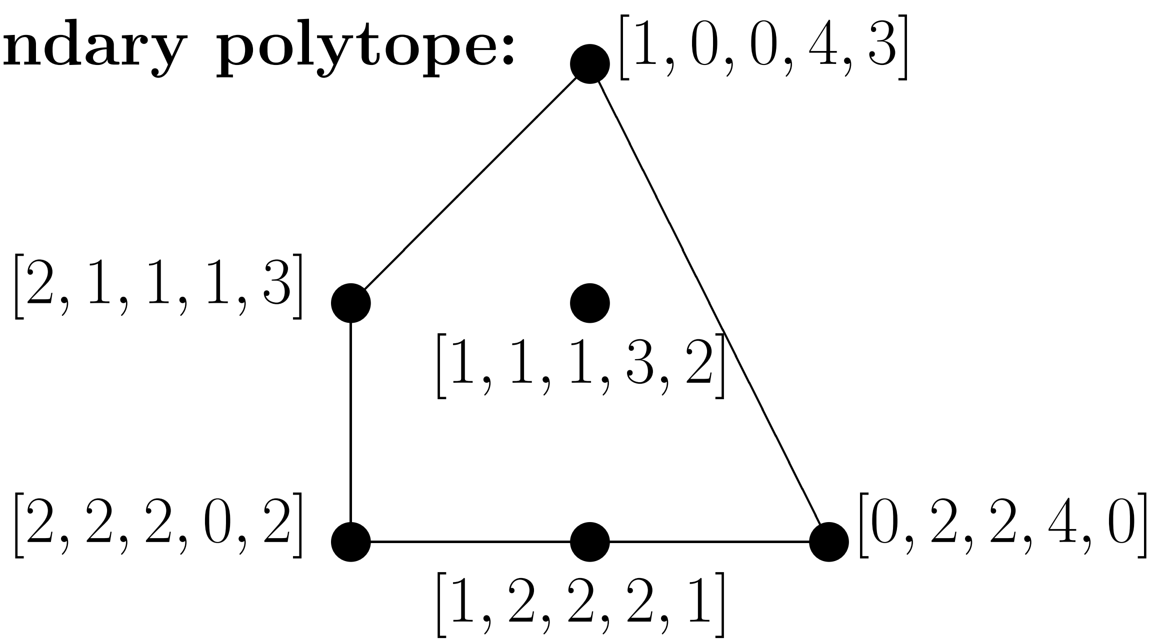
Primary polytope:





$$\mathbf{Chow}_Q = [41][34][52][45] - [41][45][45][45]$$
$$+ [24][13][45][45] - [12][34][52][34] + [24][34][24][34]$$
$$+ [12][45][45][34] - [24][51][45][34] - [41][51][52][53]$$
$$+ [12][13][52][53] + [41][45][24][53] - [24][13][24][53]$$

Secondary polytope:



$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 & 1 \end{bmatrix} \qquad \rightsquigarrow$$

Laurent polynomial

$$F_{\mathbf{u}}(\xi) = u_1 \xi_1^{-1} \xi_2^{-1} + u_2 \xi_1 + u_3 \xi_2 + u_4 + u_5 \xi_1 \xi_2$$

Zero locus of $F_{\mathbf{u}}(\xi)$ is intersection of orbit $(\mathbf{M}_Q \otimes \mathbb{C}^*)[1 : \ldots : 1]$ with hyperplane $\mathbf{u}^\perp$ in $\mathbb{P}^{N-1}$.

Hypergeometric structures

• PDEsystems..... • power series ....

• integrals

$$I(\mathbf{u}, z, \mathbf{s}) = \int_{\square} (F_{\mathbf{u}}(\xi))^z \xi^{\mathbf{s}-\mathbf{1}} d\xi$$

with

$$\mathbf{u} \in \mathbb{C}^N, \qquad z \in \mathbb{C}, \qquad \mathbf{s} \in \mathbb{C}^{\text{rank } \mathbf{M}_Q - 2}$$

$$\square = \square_\theta = \{\xi \in \mathbb{C}^{*\text{rank } \mathbf{M}_Q - 2} \,|\, \text{Arg}\xi = \theta\}$$

or

$$\square = \square_{\mathbf{r}} = \{\xi \in \mathbb{C}^{*\text{rank } \mathbf{M}_Q - 2} \,|\, |\xi| = \mathbf{r}\}$$

for fixed $\theta, \mathbf{r}$.

Domain of integration should avoid zero locus of $F_{\mathbf{u}}(\xi)$; i.e.
θ not in **co-amoeba**, resp.
$\text{Log}\, \mathbf{r}$ not in **amoeba** of zero locus

Fourier-Laplace integrals

$$\int c(\lambda) e^{\lambda \cdot \mathbf{u}} d\lambda$$

This satisfies GKZ hypergeometric PDEs iff
c vanishes outside $\overline{(\mathbf{M}_Q \otimes \mathbb{C}^*)(1,\ldots,1)}$
and
transforms under the action of $\mathbf{M}_Q \otimes \mathbb{C}^*$ according to some character.

The dimer model approach attempts to reconstruct c from its behavior on lines in $\mathbb{P}^{N-1}$ which intersect $\overline{(\mathbf{M}_Q \otimes \mathbb{C}^*)(1,\ldots,1)}$; i.e.
reconstruct c from its Radon transform, which is a function on the spectral hypersurface $\mathbf{Spec}_Q$.

For a point $\mathbf{y}$ in $\mathbf{Spec}_Q$ we have the following function $\Phi_{\mathbf{y}}(\mathbf{u})$ of $\mathbf{u} \in \mathbb{C}^N$, which satisfies the GKZ hypergeometric PDE's of order > 1.

Take points $\mathbf{x} \neq \mathbf{x}'$ on line $\mathbf{y}$ with $\mathbf{x}'$ not in $\overline{(\mathbf{M}_Q \otimes \mathbb{C}^*)[1 : \ldots : 1]}$. Then

$$\Phi_{\mathbf{y}}(\mathbf{u}) = \sum e^{\mathbf{x}\cdot\mathbf{u} + t\mathbf{x}'\cdot\mathbf{u}},$$

the sum is over the eigenvalues of the matrix $\mathbf{K}_Q(<\mathbf{z}\mathbf{x}'>)^{-1}\mathbf{K}_Q(<\mathbf{x}\mathbf{z}>)$ for some $\mathbf{z} \in \mathbb{P}^{N-1}$ with $<\mathbf{x}'\mathbf{z}> \notin \mathbf{Spec}_Q$.

This can be written more elegantly as:

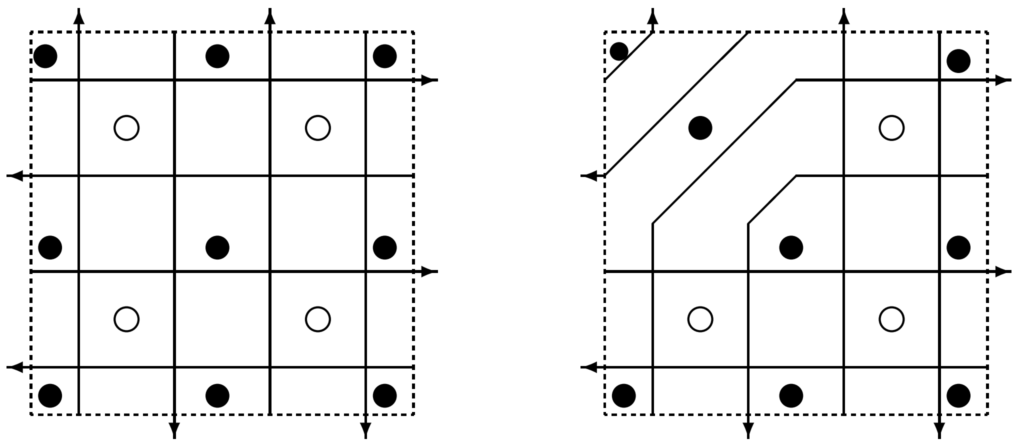
$$\Phi_{\mathbf{y}}(\mathbf{u}) =$$

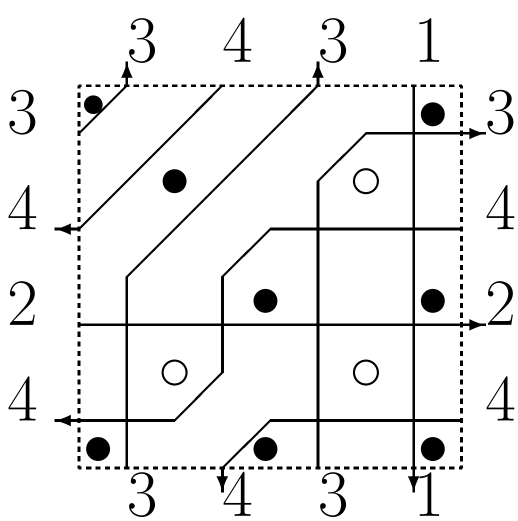
$$e^{\mathbf{x}\cdot\mathbf{u}}\mathrm{Trace}\left(e^{(\mathbf{x}'\cdot\mathbf{u})\mathbf{K}_Q(<\mathbf{z}\mathbf{x}'>)^{-1}\mathbf{K}_Q(<\mathbf{x}\mathbf{z}>)}\right)$$

Gulotta's algorithm for

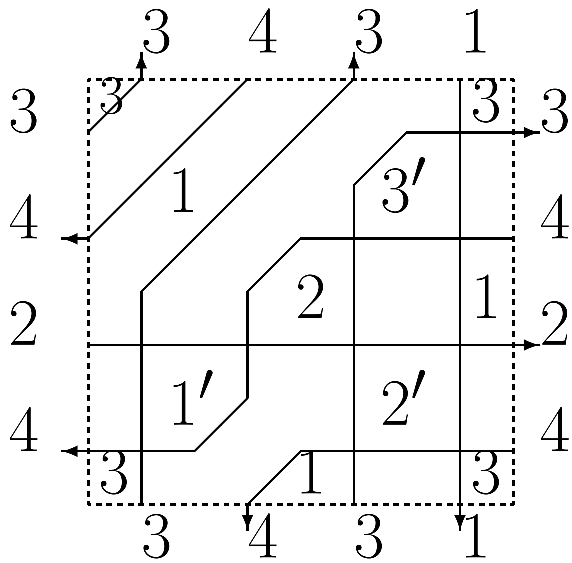
$$F_{\mathbf{u}}(\xi) = u_1 + u_3\xi + u_4\xi^2 + u_2\xi^3$$

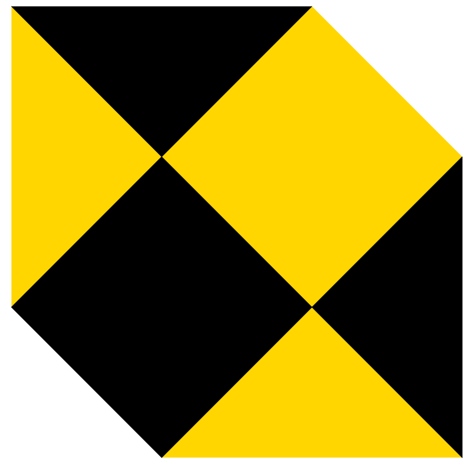
$$B_2 = \begin{bmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 2 & -1 \end{bmatrix}$$





winding numbers of zig-zag paths given by columns of B_2





In the left-hand picture each intersection point is incident to two zigzag-paths and two numbered 2-cells as indicated in the matrix

$$\mathbf{K}_Q = \begin{pmatrix} [2\ 3] & [4\ 2] & [3\ 4] \\ [1\ 2] + [3\ 4] & [2\ 3] & [4\ 1] \\ [4\ 1] & [3\ 4] & [1\ 3] \end{pmatrix}$$

which is equal to the Grassmann-Kasteleyn matrix of the dimer model on the right.