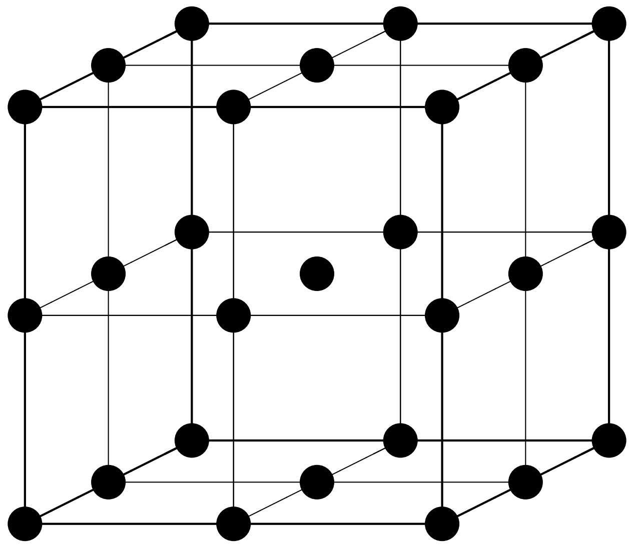
Mirror Symmetry, Hypergeometric Systems and the transcendental part of K3 surfaces associated with 3D Fano Polytopes

Jan Stienstra



Kolloquium Geometrie und Arithmetik **JGU Mainz** **5 December 2024**

a 3D Fano Polytope is a convex polytope P in $\mathbb{R}^3$ such that

- all faces of P are triangles
- $P \cap \mathbb{Z}^3$ consists of the vertices of P and one point in its interior

Up to affine transformations of $\mathbb{Z}^3$ there are 18 3D Fano Polytopes.

$$\sharp\{\text{vertices}\} = N\,, \qquad \sharp\{\text{edges}\} = 3N-6\,, \qquad \sharp\{\text{faces}\} = 2N-4$$

$N = 4, 5, 6, 7, 8$

case	N	$\mathcal{V} = [\mathsf{v}_1, \dots, \mathsf{v}_N]$ (vertices)	$\mathcal{F}$ (faces)
1	4	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 2 & 3 \\ 3 & 4 & 4 & 4 \end{bmatrix}$
2	5	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 3 & 4 & 4 \\ 3 & 4 & 4 & 5 & 5 & 5 \end{bmatrix}$
3	5	$\begin{bmatrix} 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 3 & 4 & 4 \\ 3 & 4 & 4 & 5 & 5 & 5 \end{bmatrix}$
4	5	$\begin{bmatrix} 1 & -1 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 3 & 4 & 4 \\ 3 & 4 & 4 & 5 & 5 & 5 \end{bmatrix}$
5	5	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 2 \\ 2 & 3 & 2 & 3 & 4 & 4 \\ 3 & 4 & 4 & 5 & 5 & 5 \end{bmatrix}$
6	6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 & 3 & 4 & 5 & 5 \\ 3 & 4 & 5 & 5 & 6 & 6 & 6 & 6 \end{bmatrix}$
7	6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 & 3 & 4 & 5 & 5 \\ 3 & 4 & 5 & 5 & 6 & 6 & 6 & 6 \end{bmatrix}$
8	6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 & 3 & 4 & 5 & 5 \\ 3 & 4 & 5 & 5 & 6 & 6 & 6 & 6 \end{bmatrix}$
9	6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 & 3 & 4 & 5 & 5 \\ 3 & 4 & 5 & 5 & 6 & 6 & 6 & 6 \end{bmatrix}$
10	6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 3 & 4 & 2 \\ 2 & 3 & 4 & 2 & 3 & 4 & 5 & 5 \\ 3 & 4 & 5 & 5 & 6 & 6 & 6 & 6 \end{bmatrix}$
11	6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 4 \\ 2 & 2 & 3 & 4 & 4 & 3 & 5 & 5 \\ 3 & 5 & 6 & 5 & 6 & 6 & 6 & 6 \end{bmatrix}$
12	6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 2 & 4 \\ 2 & 2 & 3 & 4 & 4 & 3 & 5 & 5 \\ 3 & 5 & 6 & 5 & 6 & 6 & 6 & 6 \end{bmatrix}$
13	7	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 5 & 2 & 3 & 4 & 5 & 6 & 6 \\ 3 & 4 & 5 & 6 & 6 & 7 & 7 & 7 & 7 & 7 \end{bmatrix}$
14	7	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 5 & 2 & 3 & 4 & 5 & 6 & 6 \\ 3 & 4 & 5 & 6 & 6 & 7 & 7 & 7 & 7 & 7 \end{bmatrix}$
15	7	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 5 & 2 & 3 & 4 & 5 & 6 & 6 \\ 3 & 4 & 5 & 6 & 6 & 7 & 7 & 7 & 7 & 7 \end{bmatrix}$
16	7	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 2 \\ 2 & 3 & 4 & 5 & 2 & 3 & 4 & 5 & 6 & 6 \\ 3 & 4 & 5 & 6 & 6 & 7 & 7 & 7 & 7 & 7 \end{bmatrix}$
17	8	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 2 \\ 2 & 3 & 4 & 5 & 6 & 2 & 3 & 4 & 5 & 6 & 7 & 7 \\ 3 & 4 & 5 & 6 & 7 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \end{bmatrix}$
18	8	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 2 & 3 & 4 & 5 & 6 & 2 \\ 2 & 3 & 4 & 5 & 6 & 2 & 3 & 4 & 5 & 6 & 7 & 7 \\ 3 & 4 & 5 & 6 & 7 & 7 & 8 & 8 & 8 & 8 & 8 & 8 \end{bmatrix}$

case	polytope	Fano threefold
1	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$	$\mathbb{P}^3$
2	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathbb{P}^2$
6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
9	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathsf{dP}_1$
13	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathsf{dP}_2$
17	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathsf{dP}_3$

Fano threefold

$$\mathsf{Fano}(\mathcal{V}) \xrightarrow{\simeq} \mathbb{C}^N_{\mathcal{F}}/\mathbb{L}\otimes\mathbb{C}^*$$

$\mathbb{C}^N_{\mathcal{F}} = \bigcup_{[h,i,j]\in\mathcal{F}} \mathbb{C}^N_{[h,i,j]}$

$\mathbb{C}^N_{[h,i,j]} = \left\{(z_1,\ldots,z_N)\in\mathbb{C}^N \;\middle|\; z_m\neq 0 \text{ if } m\neq h,i,j\right\}$

$\mathbb{L} = \left\{(\ell_1,\ldots,\ell_N)\in\mathbb{Z}^N \;\middle|\; \ell_1\mathsf{v}_1+\ldots+\ell_N\mathsf{v}_N=\mathsf{0}\right\}, \qquad \mathsf{rank}\,\mathbb{L} = N-3$

$\mathbb{L}\otimes\mathbb{C}^* \quad \hookrightarrow \quad {\mathbb{C}^*}^N$

${\mathbb{C}^*}^N$ acts on $\mathbb{C}^N$ and $\mathbb{C}^N_{\mathcal{F}}$ by componentwise multiplication

$\mathbb{C}^3 \xrightarrow{\simeq} \mathbb{C}^N_{[h,i,j]}/\mathbb{L}\otimes\mathbb{C}^* \qquad (y_1,y_2,y_3)\mapsto(z_1,\ldots,z_N)$

with $z_h=y_1,\ z_i=y_2,\ z_j=y_3$ and $z_m=1$ for $m\neq h,i,j$

The coordinates $(z_1, \ldots, z_N)$ on $\mathbb{C}^N$ function as **homogeneous coordinates** on $\mathsf{Fano}(\mathcal{V})$.

- For a vertex v_i of the Fano polytope $\mathsf{conv}(\mathcal{V})$

  $z_i = 0$ defines a divisor D_i on $\mathsf{Fano}(\mathcal{V})$.

- For an edge $\mathsf{conv}(\mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V})$

  $z_i = z_j = 0$ defines the intersection $\mathsf{D}_i \cap \mathsf{D}_j$

- For a face $\mathsf{conv}(\mathsf{v}_h, \mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V})$

  $z_h = z_i = z_j = 0$ defines the intersection $\mathsf{D}_h \cap \mathsf{D}_i \cap \mathsf{D}_j$

The Chow ring $\mathsf{CH}^*\big(\mathsf{Fano}(\mathcal{V})\big) \;=\; \bigoplus_{m=0}^{3} \mathsf{CH}^m\big(\mathsf{Fano}(\mathcal{V})\big)$

and cohomology ring $\mathsf{H}^*\big(\mathsf{Fano}(\mathcal{V}), \mathbb{Z}\big) \;=\; \bigoplus_{m=0}^{3} \mathsf{H}^{2m}\big(\mathsf{Fano}(\mathcal{V}), \mathbb{Z}\big)$

are isomorphic as graded rings with

$$\mathcal{R}^*(\mathcal{V}) \;=\; {}^{\mathbb{Z}\left[\mathsf{D}_1, \ldots, \mathsf{D}_N\right]}\!\big/_{\left(\mathcal{I} + \mathcal{J}\right)}$$

- ideal $\mathcal{I}$ generated by products $\mathsf{D}_a \cdot \mathsf{D}_b$ and $\mathsf{D}_c \cdot \mathsf{D}_d \cdot \mathsf{D}_e$ with $\{a, b\}$ and $\{c, d, e\}$ not contained in any $[h, i, j] \in \mathcal{F}$
- ideal $\mathcal{J}$ generated by the three linear forms in the system $\mathsf{v}_1\mathsf{D}_1 + \ldots + \mathsf{v}_N\mathsf{D}_N$

$\mathsf{rank}\mathcal{R}^0(\mathcal{V}) = \mathsf{rank}\mathcal{R}^3(\mathcal{V}) = 1, \qquad \mathsf{rank}\mathcal{R}^1(\mathcal{V}) = \mathsf{rank}\mathcal{R}^2(\mathcal{V}) = N - 3,$

$\mathsf{rank}\mathcal{R}^*(\mathcal{V}) \;=\; 2N - 4$

Let $\overline{\mathsf{D}}_1, \ldots, \overline{\mathsf{D}}_N \in \mathcal{R}^1(\mathcal{V})$ denote the residue classes $\mathsf{D}_1, \ldots, \mathsf{D}_N$ mod $\mathcal{I} + \mathcal{J}$. Then

- $1 \in \mathcal{R}^0(\mathcal{V})$ $\quad \Leftrightarrow$ equivalence class of $\mathsf{Fano}(\mathcal{V})$ in $\mathsf{CH}^0\big(\mathsf{Fano}(\mathcal{V})\big)$.
- $\overline{\mathsf{D}}_i \in \mathcal{R}^1(\mathcal{V})$ $\quad \Leftrightarrow$ equivalence class of the divisor D_i in $\mathsf{CH}^1\big(\mathsf{Fano}(\mathcal{V})\big) = \mathsf{Pic}\big(\mathsf{Fano}(\mathcal{V})\big)$
- $\overline{\mathsf{D}}_i \cdot \overline{\mathsf{D}}_j \in \mathcal{R}^2(\mathcal{V})$ for an edge $\mathsf{conv}(\mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V})$ $\quad \Leftrightarrow$ equivalence class of the curve $\mathsf{D}_i \cap \mathsf{D}_j$ in $\mathsf{CH}^2\big(\mathsf{Fano}(\mathcal{V})\big)$
- $\overline{\mathsf{D}}_h \cdot \overline{\mathsf{D}}_i \cdot \overline{\mathsf{D}}_j \in \mathcal{R}^3(\mathcal{V})$ for a face $\mathsf{conv}(\mathsf{v}_h, \mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V})$ $\quad \Leftrightarrow$ equivalence class of the point $\mathsf{D}_h \cap \mathsf{D}_i \cap \mathsf{D}_j$ in $\mathsf{CH}^3\big(\mathsf{Fano}(\mathcal{V})\big)$
- All points $\mathsf{D}_h \cap \mathsf{D}_i \cap \mathsf{D}_j$ for $[h, i, j] \in \mathcal{F}$ are rationally equivalent,

i.e. $\overline{\mathsf{D}}_h \cdot \overline{\mathsf{D}}_i \cdot \overline{\mathsf{D}}_j = \overline{\mathsf{D}}_1 \cdot \overline{\mathsf{D}}_2 \cdot \overline{\mathsf{D}}_3$ in $\mathsf{CH}^3\big(\mathsf{Fano}(\mathcal{V})\big)$

bilinear form $\langle , \rangle$ on $\mathsf{Pic}\big(\mathsf{Fano}(\mathcal{V})\big) = \mathcal{R}^1(\mathcal{V})$:

for c, $\mathsf{c}' \in \mathcal{R}^1(\mathcal{V})$

$$\mathsf{c} \cdot \mathsf{c}' \cdot \overline{\mathsf{D}}_0 \;=\; \big\langle \mathsf{c}, \mathsf{c}' \big\rangle \cdot \overline{\mathsf{D}}_1 \cdot \overline{\mathsf{D}}_2 \cdot \overline{\mathsf{D}}_3.$$

where $\overline{\mathsf{D}}_0 = \overline{\mathsf{D}}_1 + \ldots + \overline{\mathsf{D}}_N$ is the *anti-canonical class.*

case	$\big(\langle \overline{\mathsf{D}}_i, \overline{\mathsf{D}}_j \rangle\big)_{1 \le i,j \le N}$
1 $\mathbb{P}^3$	$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$
3	$\begin{bmatrix} 4 & 4 & 4 & 4 & 0 \\ 4 & 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 \\ 4 & 2 & 2 & 2 & 2 \\ 0 & 2 & 2 & 2 & -2 \end{bmatrix}$
5	$\begin{bmatrix} 0 & 3 & 3 & 3 & 0 \\ 3 & 4 & 4 & 1 & 3 \\ 3 & 4 & 4 & 1 & 3 \\ 3 & 1 & 1 & -2 & 3 \\ 0 & 3 & 3 & 3 & 0 \end{bmatrix}$
2 $\mathbb{P}^1 \times \mathbb{P}^2$	$\begin{bmatrix} 0 & 3 & 3 & 3 & 0 \\ 3 & 2 & 2 & 2 & 3 \\ 3 & 2 & 2 & 2 & 3 \\ 3 & 2 & 2 & 2 & 3 \\ 0 & 3 & 3 & 3 & 0 \end{bmatrix}$
4	$\begin{bmatrix} 10 & 5 & 5 & 5 & 0 \\ 5 & 2 & 2 & 2 & 1 \\ 5 & 2 & 2 & 2 & 1 \\ 5 & 2 & 2 & 2 & 1 \\ 0 & 1 & 1 & 1 & -2 \end{bmatrix}$
6 $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$\begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}$

case	$\left(\langle \overline{\mathsf{D}}_i, \overline{\mathsf{D}}_j \rangle\right)_{1\le i,j\le N}$	case	$\left(\langle \overline{\mathsf{D}}_i, \overline{\mathsf{D}}_j \rangle\right)_{1\le i,j\le N}$
7	$\begin{bmatrix} 0 & 2 & 3 & 2 & 1 & 0 \\ 2 & 0 & 3 & 0 & 1 & 2 \\ 3 & 3 & 6 & 3 & 0 & 3 \\ 2 & 0 & 3 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 & -2 & 1 \\ 0 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$	8	$\begin{bmatrix} 0 & 2 & 1 & 2 & 3 & 0 \\ 2 & 0 & 3 & 0 & 1 & 2 \\ 1 & 3 & -2 & 3 & 0 & 1 \\ 2 & 0 & 3 & 0 & 1 & 2 \\ 3 & 1 & 0 & 1 & -2 & 3 \\ 0 & 2 & 1 & 2 & 3 & 0 \end{bmatrix}$
9	$\begin{bmatrix} 0 & 2 & 3 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 & 2 & 2 \\ 3 & 2 & 2 & 2 & 0 & 3 \\ 2 & 0 & 2 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 & -2 & 1 \\ 0 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$	10	$\begin{bmatrix} 0 & 2 & 3 & 2 & 1 & 0 \\ 2 & 2 & 4 & 0 & 2 & 2 \\ 3 & 4 & 4 & 1 & 0 & 3 \\ 2 & 0 & 1 & -2 & 1 & 2 \\ 1 & 2 & 0 & 1 & -2 & 1 \\ 0 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$
11	$\begin{bmatrix} 2 & 1 & 4 & 1 & 1 & 2 \\ 1 & -2 & 4 & 0 & 3 & 1 \\ 4 & 4 & 4 & 0 & 0 & 4 \\ 1 & 0 & 0 & -2 & 1 & 1 \\ 1 & 3 & 0 & 1 & -2 & 1 \\ 2 & 1 & 4 & 1 & 1 & 2 \end{bmatrix}$	12	$\begin{bmatrix} 2 & 1 & 3 & 2 & 1 & 2 \\ 1 & -2 & 3 & 0 & 3 & 1 \\ 3 & 3 & 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & -2 & 2 & 2 \\ 1 & 3 & 0 & 2 & -2 & 1 \\ 2 & 1 & 3 & 2 & 1 & 2 \end{bmatrix}$
13	$\begin{bmatrix} 0 & 2 & 2 & 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 0 & 0 & 2 \\ 1 & 0 & 2 & -2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 & -2 & 2 & 1 \\ 1 & 2 & 0 & 0 & 2 & -2 & 1 \\ 0 & 2 & 2 & 1 & 1 & 1 & 0 \end{bmatrix}$	14	$\begin{bmatrix} 0 & 2 & 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 3 & 2 \\ 2 & 3 & 0 & 1 & 0 & 0 & 2 \\ 1 & 0 & 1 & -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & -2 & 2 & 1 \\ 1 & 3 & 0 & 0 & 2 & -2 & 1 \\ 0 & 2 & 2 & 1 & 1 & 1 & 0 \end{bmatrix}$
15	$\begin{bmatrix} 0 & 2 & 2 & 1 & 1 & 1 & 0 \\ 2 & -2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 1 & 0 & 3 & 0 & 0 & 2 \\ 1 & 0 & 3 & -2 & 3 & 0 & 1 \\ 1 & 0 & 0 & 3 & -2 & 2 & 1 \\ 1 & 1 & 0 & 0 & 2 & -2 & 1 \\ 0 & 2 & 2 & 1 & 1 & 1 & 0 \end{bmatrix}$	16	$\begin{bmatrix} 0 & 2 & 2 & 1 & 1 & 1 & 0 \\ 2 & 2 & 4 & 0 & 0 & 2 & 2 \\ 2 & 4 & 2 & 2 & 0 & 0 & 2 \\ 1 & 0 & 2 & -2 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & -2 & 1 & 1 \\ 1 & 2 & 0 & 0 & 1 & -2 & 1 \\ 0 & 2 & 2 & 1 & 1 & 1 & 0 \end{bmatrix}$
17	$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 & 2 & 1 \\ 1 & 2 & -2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & -2 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & -2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2 & -2 & 2 & 1 \\ 1 & 2 & 0 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$	18	$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 2 & -2 & 3 & 0 & 0 & 0 & 1 \\ 1 & 0 & 3 & -2 & 3 & 0 & 0 & 1 \\ 1 & 0 & 0 & 3 & -2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2 & -2 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

Global sections of the anti-canonical bundle on $\mathsf{Fano}(\mathcal{V})$ are homogeneous polynomials

$$\clubsuit = \sum_{\mu \in \mathsf{conv}(\mathcal{V})^\vee \cap \mathbb{Z}^3} c_\mu \, z_1^{1+\mu\cdot \mathsf{v}_1} \cdot \ldots \cdot z_N^{1+\mu\cdot \mathsf{v}_N}$$

in $\mathbb{C}[z_1, \ldots, z_N]$ where

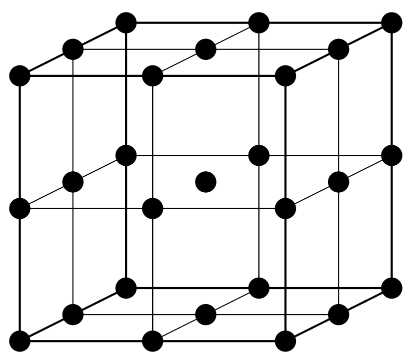
$$\mathsf{conv}(\mathcal{V})^\vee \cap \mathbb{Z}^3 = \left\{ \mu \in \mathbb{Z}^3 \;\middle|\; \mu \cdot \mathsf{v}_j \geq -1 \quad \text{for} \quad j = 1, \ldots, N \right\}.$$

The zero set of $\clubsuit$ is an *anti-canonical* **K3 surface** in $\mathsf{Fano}(\mathcal{V})$.

anti-canonical K3 in $\mathbb{P}^3 \Leftrightarrow$ surface of degree 4

anti-canonical K3 in $\mathbb{P}^1 \times \mathbb{P}^2 \Leftrightarrow$ surface of degree $(2,3)$

anti-canonical K3 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \Leftrightarrow$ surface of degree $(2,2,2)$



Define $$\overline{\mathcal{R}}^*(\mathcal{V}) \,=\, \overline{\mathcal{R}}^0(\mathcal{V}) \oplus \overline{\mathcal{R}}^1(\mathcal{V}) \oplus \overline{\mathcal{R}}^2(\mathcal{V})$$

as the graded ring with homogeneous components

$\overline{\mathcal{R}}^0(\mathcal{V}) = \mathbb{Z}\mathsf{1}$,

$\overline{\mathcal{R}}^1(\mathcal{V})$ is additively generated by $\delta_1, \ldots, \delta_N$ subject to

the linear relations $\mathsf{v}_1\delta_1 + \ldots + \mathsf{v}_N\delta_N = \mathsf{0}$,

$\overline{\mathcal{R}}^2(\mathcal{V}) = \mathbb{Z}\delta_\infty$

The multiplication is given by: for $i, j = 1, \ldots, N$:

$$\begin{aligned}
&\mathsf{1}\cdot\mathsf{1} = \mathsf{1}\,, \quad \mathsf{1}\cdot\delta_j = \delta_j\,, \quad \mathsf{1}\cdot\delta_\infty = \delta_\infty\,,\\
&\delta_\infty\cdot\delta_j = \delta_\infty\cdot\delta_\infty = 0,\\
&\delta_i\cdot\delta_j = \left\langle \delta_i, \delta_j \right\rangle \delta_\infty
\end{aligned}$$

The number $\left\langle \delta_i, \delta_j \right\rangle$ is equal to the number $\left\langle \overline{\mathsf{D}}_i, \overline{\mathsf{D}}_j \right\rangle$

The homomorphism of graded rings

$$f : \mathcal{R}^*(\mathcal{V}) \longrightarrow \overline{\mathcal{R}}^*(\mathcal{V})\,, \qquad f(\overline{\mathsf{D}}_j) = \delta_j \quad \text{for } j = 0, \ldots, N$$

induces isomorphisms $\quad \mathcal{R}^0(\mathcal{V}) \simeq \overline{\mathcal{R}}^0(\mathcal{V})\,, \quad \mathcal{R}^1(\mathcal{V}) \simeq \overline{\mathcal{R}}^1(\mathcal{V})$.

In degree 2 the image of f in $\overline{\mathcal{R}}^2(\mathcal{V}) = \mathbb{Z}\delta_\infty$ is

$$f\big(\mathcal{R}^2(\mathcal{V})\big) = \begin{cases} 4\mathbb{Z}\delta_\infty & \text{in Case 1} \\ 2\mathbb{Z}\delta_\infty & \text{in Cases 3 and 6} \\ \mathbb{Z}\delta_\infty & \text{in all other cases} \end{cases}$$

and

$$\ker(f) \cap \mathcal{R}^2(\mathcal{V}) \;=\; \left\{ \sum_{i,j} n_{ij} \overline{\mathsf{D}}_i \cdot \overline{\mathsf{D}}_j \;\middle|\; \sum_{i,j} n_{ij} \big\langle \overline{\mathsf{D}}_i, \overline{\mathsf{D}}_j \big\rangle = 0 \right\}$$

In degree 3: $\qquad \mathcal{R}^3(\mathcal{V}) \;\subset\; \ker(f)$

$$\ker(f) \;=\; \mathsf{Ann}(\overline{\mathsf{D}}_0) = \big\{ \mathsf{c} \in \mathcal{R}^*(\mathcal{V}) \;\big|\; \mathsf{c} \cdot \overline{\mathsf{D}}_0 = 0 \big\},$$

the annihilator ideal of $\overline{\mathsf{D}}_0 = \overline{\mathsf{D}}_1 + \ldots + \overline{\mathsf{D}}_N$.

The **Beauville-Voisin ring of a K3 surface** $\mathcal{Y}$

Recall the *Chow ring* and the *cohomology ring* of $\mathcal{Y}$

$$\begin{aligned} \mathsf{CH}^*(\mathcal{Y}) &= \mathsf{CH}^0(\mathcal{Y}) \oplus \mathsf{CH}^1(\mathcal{Y}) \oplus \mathsf{CH}^2(\mathcal{Y}) \\ \mathsf{H}^*(\mathcal{Y},\mathbb{Z}) &= \mathsf{H}^0(\mathcal{Y},\mathbb{Z}) \oplus \mathsf{H}^2(\mathcal{Y},\mathbb{Z}) \oplus \mathsf{H}^4(\mathcal{Y},\mathbb{Z}) \end{aligned}$$

The cycle class map $\mathsf{CH}^*(\mathcal{Y}) \longrightarrow \mathsf{H}^*(\mathcal{Y},\mathbb{Z})$
is a homomorphism of graded rings such that:

$$\begin{aligned} \mathsf{CH}^0(\mathcal{Y}) &= \mathsf{H}^0(\mathcal{Y},\mathbb{Z}) = \mathbb{Z} \\ \mathsf{Pic}(\mathcal{Y}) = \mathsf{CH}^1(\mathcal{Y}) &\hookrightarrow \mathsf{H}^2(\mathcal{Y},\mathbb{Z}) \\ \mathsf{CH}^2(\mathcal{Y}) &\twoheadrightarrow \mathsf{H}^4(\mathcal{Y},\mathbb{Z}) = \mathbb{Z} \end{aligned}$$

According to a theorem of Beauville and Voisin the cycle class map restricts to a ring isomorphism between on the one hand a subring of $\mathsf{CH}^*(\mathcal{Y})$ – known as the **Beauville-Voisin ring** of $\mathcal{Y}$ – and on the other hand the subring

$$\mathsf{H}^0(\mathcal{Y},\mathbb{Z}) \oplus \mathsf{Pic}(\mathcal{Y}) \oplus \mathsf{H}^4(\mathcal{Y},\mathbb{Z})$$

of $\mathsf{H}^*(\mathcal{Y},\mathbb{Z})$

For a general anti-canonical K3 surface $\mathcal{Y}$ in $\mathsf{Fano}(\mathcal{V})$

$\mathsf{Pic}(\mathcal{Y}) = \mathsf{Pic}(\mathsf{Fano}(\mathcal{V}))$

and the **Beauville-Voisin ring** of $\mathcal{Y}$ is isomorphic with the ring $\overline{\mathcal{R}}^*(\mathcal{V})$

$\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim$

For any K3 surface $\mathcal{Y}$

- the cokernel of the inclusion $\mathsf{Pic}(\mathcal{Y}) = \mathsf{CH}^1(\mathcal{Y}) \hookrightarrow \mathsf{H}^2(\mathcal{Y}, \mathbb{Z})$ is isomorphic with the **transcendental lattice** $\mathsf{Tr}(\mathcal{Y})$ of $\mathcal{Y}$

- the kernel of the surjection $\mathsf{CH}^2(\mathcal{Y}) \twoheadrightarrow \mathsf{H}^4(\mathcal{Y}, \mathbb{Z}) = \mathbb{Z}$ is the so-called **Albanese kernel** of $\mathcal{Y}$

The Albanese kernel is a fairly mysterious "infinite dimensional" object

THEOREM**(Beauville-Voisin)**:

All points on $\mathcal{Y}$ which lie on some rational curve have the same class in $\mathsf{CH}^2(\mathcal{Y})$.

$\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim$

case	$\mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1,x_2,x_3)$
1	$u_1x_1+u_2x_2+u_3x_3+u_4x_1^{-1}x_2^{-1}x_3^{-1}$
2	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}x_3^{-1}+u_5x_1^{-1}$
3	$u_1x_1+u_2x_2+u_3x_3+u_4x_1^{-1}x_2^{-1}x_3^{-1}+u_5x_1^{-1}$
4	$u_1x_1+u_2x_1^{-1}x_2+u_3x_1^{-1}x_3+u_4x_2^{-1}x_3^{-1}+u_5x_1^{-1}$
5	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}x_3^{-1}+u_5x_1^{-1}x_2^{-1}x_3^{-1}$
6	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}+u_5x_3^{-1}+u_6x_1^{-1}$
7	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}x_3^{-1}+u_5x_3^{-1}+u_6x_1^{-1}x_3^{-1}$
8	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}x_3+u_5x_3^{-1}+u_6x_1^{-1}x_3^{-1}$
9	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}x_3^{-1}+u_5x_3^{-1}+u_6x_1^{-1}$
10	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}x_3^{-1}+u_5x_3^{-1}+u_6x_1^{-1}x_2^{-1}x_3^{-1}$
11	$u_1x_1+u_2x_2+u_3x_3+u_4x_3^{-1}+u_5x_2x_3^{-1}+u_6x_1^{-1}x_2^{-1}x_3^{-1}$
12	$u_1x_1+u_2x_2+u_3x_3+u_4x_3^{-1}+u_5x_2x_3^{-1}+u_6x_1^{-1}x_2^{-1}$
13	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}+u_5x_2^{-1}x_3^{-1}+u_6x_3^{-1}+u_7x_1^{-1}$
14	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}+u_5x_2^{-1}x_3^{-1}+u_6x_3^{-1}+u_7x_1^{-1}x_2^{-1}$
15	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}+u_5x_2^{-1}x_3^{-1}+u_6x_3^{-1}+u_7x_1^{-1}x_2$
16	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}+u_5x_2^{-1}x_3^{-1}+u_6x_3^{-1}+u_7x_1^{-1}x_2^{-1}x_3^{-1}$
17	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}x_3+u_5x_2^{-1}+u_6x_3^{-1}+u_7x_2x_3^{-1}+u_8x_1^{-1}$
18	$u_1x_1+u_2x_2+u_3x_3+u_4x_2^{-1}x_3+u_5x_2^{-1}+u_6x_3^{-1}+u_7x_2x_3^{-1}+u_8x_1^{-1}x_2x_3^{-1}$

The Newton polytope of the Laurent polynomial

$$\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3) \;=\; u_0 + \mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1,x_2,x_3)$$

with $\mathsf{u} = (u_0, u_1, \ldots\, u_N) \,\in\, \mathbb{C}\times\mathbb{C}^{*3}$ is equal to the Fano polytope $\mathsf{conv}(\mathcal{V})$.

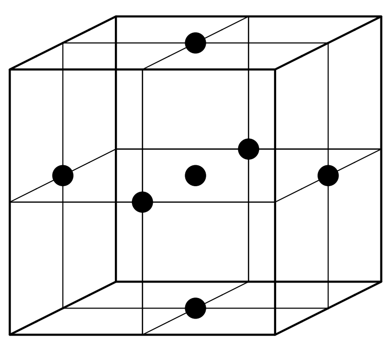
The polynomial $\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)$ can be homogenized to

$$\widetilde{\mathcal{P}}_{\mathcal{V},\mathsf{u}}(X_1,Y_1,X_2,Y_2,X_3,Y_3) \;=\; X_1\cdot Y_1\cdot X_2\cdot Y_2\cdot X_3\cdot Y_3\cdot \mathcal{P}_{\mathcal{V},\mathsf{u}}\left(\frac{X_1}{Y_1},\,\frac{X_2}{Y_2},\,\frac{X_3}{Y_3}\right)$$

Let $\mathcal{X}_{\mathcal{V},\mathsf{u}}$ denote the **K3 surface** *of degree* $(2,2,2)$ *in* $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$ defined by

$$\widetilde{\mathcal{P}}_{\mathcal{V},\mathsf{u}}(X_1,Y_1,X_2,Y_2,X_3,Y_3) = 0$$

Case 6:



Matsumura-Nagano (arXiv:2208.01465):
For all $\mathcal{V}$ and general coefficients $\mathsf{u} = (u_0, u_1, \ldots, u_N)$
there is an isomorphism of lattices

$$\mathsf{Tr}(\mathcal{X}_{\mathcal{V},\mathsf{u}}) \simeq \mathsf{Pic}\big(\mathsf{Fano}(\mathcal{V})\big) \oplus \mathsf{U}$$

where U is the hyperbolic lattice of rank 2 and
$\mathsf{Tr}(\mathcal{X}_{\mathcal{V},\mathsf{u}})$ is the **transcendental lattice** of $\mathcal{X}_{\mathcal{V},\mathsf{u}}$:

$$\mathsf{Tr}(\mathcal{X}_{\mathcal{V},\mathsf{u}}) \;=\; \mathsf{Pic}(\mathcal{X}_{\mathcal{V},\mathsf{u}})^{\perp}$$

There is an isomorphism of lattices

$$\mathsf{Pic}\big(\mathsf{Fano}(\mathcal{V})\big) \oplus \mathsf{U} \;\simeq\; \overline{\mathcal{R}}^{*}(\mathcal{V})$$

where the additive group underlying $\overline{\mathcal{R}}^{*}(\mathcal{V})$ is equipped with
the bilinear form $\big\langle , \big\rangle$ defined by $(\xi \cdot \eta)^{\deg 2} \;=\; \big\langle \xi, \eta \big\rangle \, \delta_\infty$

This means that the two families

K3 *surfaces* $\mathcal{Y}$ *in* $\mathsf{Fano}(\mathcal{V})$ *defined by sections of the anti-canonical bundle*

K3 *surfaces* $\mathcal{X}_{\mathcal{V},\mathsf{u}}$ *of degree* $(2,2,2)$ *in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ *defined by polynomials with Newton polytope equal to* $\mathsf{conv}(\mathcal{V})$

form a

- **Mirror Pair** in the sense of **Batyrev**

- **Mirror Pair of lattice polarized K3 surfaces** in the sense of **Dolgachev**

- The map

$$\mathcal{P}_{\mathcal{V},\mathsf{u}^\star} : {\mathbb{C}^*}^3 \to \mathbb{C}$$

  which is defined by the polynomial $\mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1, x_2, x_3)$ is called (possibly under restrictive conditions on the coefficients $u_1, \ldots, u_N$) a **Landau-Ginzburg Mirror of the Fano threefold** $\mathsf{Fano}(\mathcal{V})$

The variation of Hodge structure on $\mathsf{H}^2(\mathcal{X}_{\mathcal{V},\mathsf{u}},\mathbb{C})$ describes the position of the cohomology class $[\omega_{\mathcal{V},\mathsf{u}}]$ of the differential 2-form

$$\omega_{\mathcal{V},\mathsf{u}} = \frac{u_0}{d\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}$$

w.r.t. the lattice $\mathsf{H}^2(\mathcal{X}_{\mathcal{V},\mathsf{u}},\mathbb{Z})$ in $\mathsf{H}^2(\mathcal{X}_{\mathcal{V},\mathsf{u}},\mathbb{C})$.

Since $[\omega_{\mathcal{V},\mathsf{u}}] \perp \mathsf{Pic}(\mathcal{X}_{\mathcal{V},\mathsf{u}})$ we actually have $[\omega_{\mathcal{V},\mathsf{u}}] \in \mathsf{Tr}(\mathcal{X}_{\mathcal{V},\mathsf{u}}) \otimes \mathbb{C}$

For <u>general</u> $\mathsf{u} = (u_0, u_1, \ldots, u_N)$:

$$\mathsf{Tr}(\mathcal{X}_{\mathcal{V},\mathsf{u}}) \otimes \mathbb{C} \simeq \overline{\mathcal{R}}^*(\mathcal{V}) \otimes \mathbb{C}.$$

THEOREM:

The expression

$$\Phi^\flat(u_0,\ldots,u_N) = \sum_{\ell\in\mathbb{L}} (-1)^{\ell_0} \prod_{k=1}^{-\ell_0} (\delta_0 + k) \cdot \prod_{j=1}^{N} \frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j + 1 + \delta_j)} u_j^{\ell_j+\delta_j} \cdot u_0^{\ell_0 - \delta_0}$$

gives the position of $[\omega_{\mathcal{V},\mathsf{u}}]$ in $\overline{\mathcal{R}}^*(\mathcal{V}) \otimes \mathbb{C}$

as a function of the <u>variables</u> $u_0, u_1, \ldots, u_N$

$$\Phi^\flat(u_0,\ldots,u_N) = \sum_{\ell\in\mathbb{L}}(-1)^{\ell_0}\prod_{k=1}^{-\ell_0}(\delta_0+k)\cdot\prod_{j=1}^{N}\frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j+1+\delta_j)}\,u_j^{\ell_j+\delta_j}\cdot u_0^{\ell_0-\delta_0}$$

LEGEND:

- for $\ell\in\mathbb{L}$: $\ell=(\ell_1,\ldots,\ell_N)\,,\qquad \ell_0=-(\ell_1+\ldots+\ell_N)$
- $\delta_1,\ldots,\delta_N,\delta_\infty\in\overline{\mathcal{R}}^*(\mathcal{V})\,,\qquad \delta_0=\delta_1+\ldots+\delta_N$
- $u_j^{\ell_j+\delta_j}=u_j^{\ell_j}\left(1+\delta_j\,\log(u_j)+\frac{1}{2}\langle\delta_j,\delta_j\rangle\,\delta_\infty\,(\log(u_j))^2\right)$
- Gamma-function: $\Gamma(s)\;=\;\int_0^\infty t^{s-1}e^{-t}dt$

  Pochammer symbol:

  $(s)_0=1,\qquad (s)_n=s\cdot(s+1)\cdot\ldots\cdot(s+n-1)\quad\text{if } n>0$

  $$\frac{\Gamma(1+s)}{\Gamma(n+1+s)}=\begin{cases}\dfrac{1}{(1+s)_n} & \text{if}\;\; n\in\mathbb{Z}_{\geq 0},\\ (-1)^n(-s)_{-n} & \text{if}\;\; n\in\mathbb{Z}_{\leq 0}.\end{cases}$$

  RHS also makes sense if s is a nilpotent element in a $\mathbb{Q}$ algebra.

- If one works with complex variables $u_0,\ldots,u_N$ the logarithms lead to multi-valuedness and monodromy.

  Alternatively, one may restrict to positive real values of $u_0,\ldots,u_N$

- $\Phi^\flat(u_0,\ldots,u_N)$ equals $u_0^{-\delta_0}\prod_{j=1}^N u_j^{\delta_j}$ times a power series with non-zero terms only for ℓ in some pointed convex cone in $\mathbb{L}$. This series converges for $|u_0|$ sufficiently much larger than $|u_1|,\ldots,|u_N|$

$$\Phi^\flat(u_0,\ldots,u_N) = \sum_{\ell\in\mathbb{L}} (-1)^{\ell_0} \prod_{k=1}^{-\ell_0}(\delta_0+k)\cdot \prod_{j=1}^{N} \frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j+1+\delta_j)}\, u_j^{\ell_j+\delta_j}\cdot u_0^{\ell_0-\delta_0}$$

INVARIANCE PROPERTIES:

Define for $t_0 \in \mathbb{C}^*$, $\mathsf{t} = (t_1,t_2,t_3) \in \mathbb{C}^{*3}$ and $\mathsf{u} = (u_0,u_1,\ldots,u_N) \in \mathbb{C}^N$

$$t_0\cdot \mathsf{u} = (t_0u_0, t_0u_1,\ldots,t_0u_N), \qquad \mathsf{t}\star\mathsf{u} = (u_0, \mathsf{t}^{\mathsf{v}_1}u_1,\ldots,\mathsf{t}^{\mathsf{v}_N}u_N) \qquad \blacklozenge$$

where $\mathsf{v}_j = (v_{1j},v_{2j},v_{3j})^\dagger$ and $\mathsf{t}^{\mathsf{v}_j} = t_1^{v_{1j}}t_2^{v_{2j}}t_3^{v_{3j}}$.
Then

- **i**. $\Phi^\flat(t_0\cdot\mathsf{u}) \;=\; \Phi^\flat(\mathsf{u})$
- **ii**. $\Phi^\flat(\mathsf{t}\star\mathsf{u}) \;=\; \Phi^\flat(\mathsf{u})$

- **iii**. $\mathcal{P}_{\mathcal{V},t_0\cdot\mathsf{u}}(x_1,x_2,x_3) \;=\; t_0\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)$
- **iv**. $\mathcal{P}_{\mathcal{V},\mathsf{t}\star\mathsf{u}}(x_1,x_2,x_3) \;=\; \mathcal{P}_{\mathcal{V},\mathsf{u}}(t_1x_1,t_2x_2,t_3x_3)$

Equation **iii** implies that the K3 surfaces $\mathcal{X}_{\mathcal{V},t_0\cdot\mathsf{u}}$ and $\mathcal{X}_{\mathcal{V},\mathsf{u}}$ are the same.
Equation **iv** implies that the K3 surfaces $\mathcal{X}_{\mathcal{V},\mathsf{t}\star\mathsf{u}}$ and $\mathcal{X}_{\mathcal{V},\mathsf{u}}$ are isomorphic.

Thus, with the action as in $\blacklozenge$,

$$\mathbb{C}\otimes\mathbb{C}^{*N-1}/\mathbb{C}^*\otimes\mathbb{C}^{*3}$$

is a (naive?) **moduli space** for the family of K3 surfaces $\{\mathcal{X}_{\mathcal{V},\mathsf{u}}\}$

Case 1: $\mathcal{V} = \big[\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_4\big] = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$, $\quad \mathsf{Fano}(\mathcal{V}) = \mathbb{P}^3$,

$$\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3) \;=\; u_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_1^{-1} x_2^{-1} x_3^{-1}.$$

$\ell_1 \mathsf{v}_1 + \ldots + \ell_N \mathsf{v}_N = \mathsf{0} \;\Rightarrow\; \ell_1 = \ell_2 = \ell_3 = \ell_4\,, \quad \ell_0 = -4\ell_1$

$\mathsf{v}_1 \delta_1 + \ldots + \mathsf{v}_N \delta_N = \mathsf{0} \;\Rightarrow\; \delta_1 = \delta_2 = \delta_3 = \delta_4\,, \quad \delta_0 = 4\delta_1$

$\overline{\mathcal{R}}^*(\mathcal{V}) \;=\; \mathbb{Z}\mathsf{1} \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_\infty$

with multiplication: $\;\mathsf{1}$ is the multiplicative unit element

$\delta_1^2 = 4\delta_\infty, \quad \delta_1 \cdot \delta_\infty = \delta_\infty \cdot \delta_\infty = 0$

$$\Phi^\flat(u_0, u_1, u_2, u_3, u_4) = \sum_{\ell_1 \in \mathbb{Z}_{\geq 0}} \frac{(1 + 4\delta_1)_{4\ell_1}}{\big((1+\delta_1)_{\ell_1}\big)^4} \left(u_0^{-4} u_1 u_2 u_3 u_4\right)^{\ell_1 + \delta_1}$$

Its degree 0 component is

$$\begin{aligned} \Phi^\flat(u_0, u_1, u_2, u_3, u_4)^{\deg 0} \;&=\; \sum_{\ell_1 \geq 0} \frac{(4\ell_1)!}{\big(\ell_1!\big)^4} \left(u_0^{-4} u_1 u_2 u_3 u_4\right)^{\ell_1} \\ &= \frac{1}{(2\pi i)^3} \oint_{|x_1|=|x_2|=|x_3|=1} \frac{u_0}{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3} \end{aligned}$$

The matrix of the bilinear form w.r.t. the basis $\mathsf{1}, \delta_1, \delta_\infty$ is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Case 1 **continued:**
Upon homogenization and scaling coordinates the K3 surface $\mathcal{X}_{\mathcal{V},\mathsf{u}}$ in $\mathbb{P}^3$ is defined by

$$X^2YZ + XY^2Z + XYZ^2 + W^4 + \lambda WXYZ = 0$$

with $\lambda = u_0(u_1u_2u_3u_4)^{-1/4}$. We see:

$$\Phi^\flat(u_0,u_1,u_2,u_3,u_4) = \lambda^{-4\delta_1} \sum_{\ell_1 \in \mathbb{Z}_{\geq 0}} \frac{(\frac{1}{4}+\delta_1)_{\ell_1}(\frac{1}{2}+\delta_1)_{\ell_1}(\frac{3}{4}+\delta_1)_{\ell_1}}{\left((1+\delta_1)_{\ell_1}\right)^3} \left(\frac{4^4}{\lambda^4}\right)^{\ell_1}$$

$$\Phi^\flat(u_0,u_1,u_2,u_3,u_4)^{\deg 0} \;=\; {}_3F_2\left(\begin{array}{ccc} \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ & 1 & 1 \end{array} ; \frac{4^4}{\lambda^4}\right)$$

where ${}_3F_2$ is a **generalized Gauss hypergeometric function**.

Case 6: $\mathcal{V} = \left[\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_4, \mathsf{v}_5, \mathsf{v}_6\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$

$\mathsf{Fano}(\mathcal{V}) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$$\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3) \;=\; u_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_2^{-1} + u_5 x_3^{-1} + u_6 x_1^{-1}$$

$$\ell_1 \mathsf{v}_1 + \ldots + \ell_N \mathsf{v}_N = \mathsf{0} \;\Rightarrow\; \ell_1 = \ell_6, \quad \ell_2 = \ell_4, \quad \ell_3 = \ell_5, \quad \ell_0 = -2\ell_1 - 2\ell_2 - 2\ell_3$$
$$\mathsf{v}_1 \delta_1 + \ldots + \mathsf{v}_N \delta_N = \mathsf{0} \;\Rightarrow\; \delta_1 = \delta_6, \quad \delta_2 = \delta_4, \quad \delta_3 = \delta_5, \quad \delta_0 = 2\delta_1 + 2\delta_2 + 2\delta_3$$

$$\overline{\mathcal{R}}^*(\mathcal{V}) \;=\; \mathbb{Z}\mathsf{1} \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\delta_3 \oplus \mathbb{Z}\delta_\infty$$

with multiplication: $\mathsf{1}$ is the multiplicative unit element,

$$\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_\infty^2 = \delta_1\delta_\infty = \delta_2\delta_\infty = \delta_3\delta_\infty = 0, \quad \delta_1\delta_2 = \delta_1\delta_3 = \delta_2\delta_3 = 2\delta_\infty.$$

The bilinear form w.r.t. the basis $\mathsf{1}, \delta_1, \delta_2, \delta_3, \delta_\infty$ is $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

Case 6 continued:

$$\Phi^\flat(u_0,u_1,u_2,u_3,u_4,u_5,u_6) \ =$$

$$\sum_{\ell_1,\ell_2,\ell_3\geq 0} \frac{(1+2\delta_1+2\delta_2+2\delta_3)_{2\ell_1+2\ell_2+2\ell_3}}{\big((1+\delta_1)_{\ell_1}(1+\delta_2)_{\ell_2}(1+\delta_3)_{\ell_3}\big)^2}\ (u_0^{-2}u_1u_6)^{\ell_1+\delta_1}(u_0^{-2}u_2u_4)^{\ell_2+\delta_2}(u_0^{-2}u_3u_5)^{\ell_3+\delta_3}$$

$$= \lambda_1^{-2\delta_1}\lambda_2^{-2\delta_2}\lambda_3^{-2\delta_3}\times$$

$$\sum_{\ell_1,\ell_2,\ell_3\geq 0} \frac{(1+\delta_1+\delta_2+\delta_3)_{\ell_1+\ell_2+\ell_3}(\frac{1}{2}+\delta_1+\delta_2+\delta_3)_{\ell_1+\ell_2+\ell_3}}{\big((1+\delta_1)_{\ell_1}(1+\delta_2)_{\ell_2}(1+\delta_3)_{\ell_3}\big)^2}\ \left(\frac{4}{\lambda_1^2}\right)^{\ell_1}\left(\frac{4}{\lambda_2^2}\right)^{\ell_2}\left(\frac{4}{\lambda_3^2}\right)^{\ell_3}$$

where $\lambda_1 = u_0(u_1u_6)^{-1/2}$, $\lambda_2 = u_0(u_2u_4)^{-1/2}$, $\lambda_3 = u_0(u_3u_5)^{-1/2}$.

Its degree 0 component is

$$\Phi^\flat(u_0,u_1,u_2,u_3,u_4,u_5,u_6)^{\deg 0} \ =$$

$$= \sum_{\ell_1,\ell_2,\ell_3\geq 0} \frac{(2\ell_1+2\ell_2+2\ell_3)!}{\big(\ell_1!\,\ell_2!\,\ell_3!\big)^2}\ (u_0^{-2}u_1u_6)^{\ell_1}(u_0^{-2}u_2u_4)^{\ell_2}(u_0^{-2}u_3u_5)^{\ell_3}$$

$$= \sum_{\ell_1,\ell_2,\ell_3\geq 0} \frac{(1)_{\ell_1+\ell_2+\ell_3}(\frac{1}{2})_{\ell_1+\ell_2+\ell_3}}{\big(\ell_1!\,\ell_2!\,\ell_3!\big)^2}\ \left(\frac{4}{\lambda_1^2}\right)^{\ell_1}\left(\frac{4}{\lambda_2^2}\right)^{\ell_2}\left(\frac{4}{\lambda_3^2}\right)^{\ell_3}$$

$$= F_C\left(1,\tfrac{1}{2};1,1,1\ \middle|\ \frac{4}{\lambda_1^2},\frac{4}{\lambda_2^2},\frac{4}{\lambda_3^2}\right)$$

$$= \frac{1}{(2\pi i)^3}\oint_{|x_1|=|x_2|=|x_3|=1} \frac{u_0}{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)}\,\frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}$$

where F_C is a **Lauricella hypergeometric function of type** C.

Case 6 continued:

In ±1987 Chris Peters and I investigated the family of K3 surfaces

$$x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_3 + x_3^{-1} = s$$

This is Case 6 with $u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 1$ and $u_0 = -s$.

We found that for <u>general</u> s the transcendental lattice is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 12 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

$$\Phi^\flat(u_0, 1, 1, 1, 1, 1, 1) =$$

$$\sum_{m \geq 0} \left(\sum_{\ell_1+\ell_2+\ell_3=m} \frac{(1+2\delta)_{2m}}{\left((1+\delta_1)_{\ell_1}(1+\delta_2)_{\ell_2}(1+\delta_3)_{\ell_3}\right)^2} \right) u_0^{-2(m+\delta)}$$

where $\delta = \delta_1 + \delta_2 + \delta_3$.
The coefficients are invariant under permutations of $\delta_1, \delta_2, \delta_3$
and, hence, linear combinations of 1, $\delta_1 + \delta_2 + \delta_3$ and $\delta_1\delta_2 + \delta_1\delta_3 + \delta_2\delta_3$.

Therefore $\Phi^\flat(u_0, 1, 1, 1, 1, 1, 1)$ takes values in $\left(\mathbb{Z}\mathbf{1} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\delta_\infty\right) \otimes \mathbb{C}$.

The bilinear form on $\mathbb{Z}\mathbf{1} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\delta_\infty$ is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 12 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

GKZ hypergeometric systems

The **GKZ system of differential equations** associated with $\mathcal{V}$ and additional parameters $\beta_0 \in \mathbb{C}$ and $\underline{\beta} = [\beta_1, \beta_2, \beta_3]^\dagger \in \mathbb{C}^3$ is the system of differential equations for functions $\Phi(u_0, \ldots, u_N)$ of $N+1$ complex variables:

- For every $(\ell_1, \ldots, \ell_N) \in \mathbb{L}$ there is one differential equation:

$$\prod_{\ell_i<0} \left(\frac{\partial}{\partial u_i}\right)^{-\ell_i} \Phi \;=\; \prod_{\ell_i>0} \left(\frac{\partial}{\partial u_i}\right)^{\ell_i} \Phi \qquad (\star)$$

  where in the products $0 \leq i \leq N$ and $\ell_0 = -(\ell_1 + \ldots + \ell_N)$

- The system of four linear differential equations:

$$\sum_{j=0}^{N} u_j \frac{\partial}{\partial u_j}\Phi = \beta_0\,\Phi\,, \qquad \sum_{j=1}^{N} \mathsf{v}_j\, u_j \frac{\partial}{\partial u_j}\Phi = \underline{\beta}\,\Phi \qquad (\star\star)$$

The system we need for our purpose has $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$.

Differential equations $(\star\star)$ are then equivalent with the invariance property

$$\Phi(t_0 \cdot \mathsf{u}) = \Phi(\mathsf{u})\,, \qquad \Phi(\mathsf{t} \star \mathsf{u}) = \Phi(\mathsf{u})\,, \qquad (\star\star\star)$$

where for $t_0 \in \mathbb{C}^*$, $\mathsf{t} = (t_1, t_2, t_3) \in \mathbb{C}^{*3}$ and $\mathsf{u} = (u_0, u_1, \ldots, u_N) \in \mathbb{C}^N$

$$t_0 \cdot \mathsf{u} = (t_0 u_0, t_0 u_1, \ldots, t_0 u_N), \qquad \mathsf{t} \star \mathsf{u} = (u_0, \mathsf{t}^{\mathsf{v}_1} u_1, \ldots, \mathsf{t}^{\mathsf{v}_N} u_N) \qquad \blacklozenge$$

THEOREM A

Consider the expression

$$\Phi(u_0,\ldots,u_N) \;=\; \sum_{\ell\in\mathbb{L}} \frac{\Gamma(1-\overline{\mathsf{D}}_0)}{\Gamma(\ell_0+1-\overline{\mathsf{D}}_0)}\, u_0^{\ell_0-\overline{\mathsf{D}}_0}\cdot \prod_{j=1}^{N} \frac{\Gamma(1+\overline{\mathsf{D}}_j)}{\Gamma(\ell_j+1+\overline{\mathsf{D}}_j)}\, u_j^{\ell_j+\overline{\mathsf{D}}_j}$$

with $\ell=(\ell_1,\ldots,\ell_N)\in\mathbb{L}$, $\ell_0=-(\ell_1+\ldots+\ell_N)$,
$\overline{\mathsf{D}}_1,\ldots,\overline{\mathsf{D}}_N\in\mathsf{Pic}\big(\mathsf{Fano}(\mathcal{V})\big)$ and $\overline{\mathsf{D}}_0=\overline{\mathsf{D}}_1+\ldots+\overline{\mathsf{D}}_N$.

i. It defines a (multivalued) function on a non-empty domain in $\mathbb{C}^{N+1}$ with values in $\mathsf{CH}^*\big(\mathsf{Fano}(\mathcal{V})\big)\otimes\mathbb{C}$.

ii. It satisfies the GKZ system $(\star)-(\star\star\star)$ associated with $\mathcal{V}$, $\beta_0=0$, $\underline{\beta}=\mathsf{0}$.

iii. Define for $\Xi\in\mathsf{CH}^*\big(\mathsf{Fano}(\mathcal{V})\big)$ the function $\langle\!\langle\,\Xi\,,\,\Phi(u_0,\ldots,u_N)\rangle\!\rangle$ by

$$\big(\Xi\cdot\Phi(u_0,\ldots,u_N)\big)^{\deg 3} = \langle\!\langle\,\Xi\,,\,\Phi(u_0,\ldots,u_N)\rangle\!\rangle\,\overline{\mathsf{D}}_1\cdot\overline{\mathsf{D}}_2\cdot\overline{\mathsf{D}}_3$$

Then the assignment

$$\Xi\mapsto\langle\!\langle\,\Xi\,,\,\Phi(u_0,\ldots,u_N)\rangle\!\rangle$$

establishes an isomorphism

$$\mathsf{CH}^*\big(\mathsf{Fano}(\mathcal{V})\big)\otimes\mathbb{C}\;\stackrel{\simeq}{\longrightarrow}\;\big\{\text{solutions of GKZ system }(\star)-(\star\star\star)\big\}$$

THEOREM B

The defining expression for the function $\Phi(u_0,\ldots,u_N)$ can be rewritten as

$$\Phi(u_0,\ldots,u_N) = u_1^{\overline{\mathsf{D}}_1}\cdot\ldots\cdot u_N^{\overline{\mathsf{D}}_N} - \overline{\mathsf{D}}_0\Phi^{\Diamond}(u_0,\ldots,u_N)$$

with

$$\begin{aligned}\Phi^{\Diamond}(u_0,\ldots,u_N) &= \left(\log(u_0) - \tfrac{1}{2}\log^2(u_0)\delta_0 + \tfrac{1}{6}\log^3(u_0)\delta_0^2\right)\cdot u_1^{\delta_1}\cdot\ldots\cdot u_N^{\delta_N} \\ &\quad - \sum_{\ell\in\mathbb{L},\ell\neq 0}(-1)^{\ell_0}\prod_{k=1}^{-\ell_0-1}(\delta_0+k)\cdot\prod_{j=1}^{N}\frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j+1+\delta_j)}\,u_j^{\ell_j+\delta_j}\cdot u_0^{\ell_0-\delta_0}\end{aligned}$$

Here $\delta_0,\delta_1,\ldots,\delta_N \in \mathcal{R}^*(\mathcal{V})/\mathsf{Ann}(\overline{\mathsf{D}}_0) \subset \overline{\mathcal{R}}^*(\mathcal{V})$.

$\Phi^{\Diamond}(u_0,\ldots,u_N)$ is a function of $u_0,\ldots,u_N$ with values in $\overline{\mathcal{R}}^*(\mathcal{V})\otimes\mathbb{C}$.

COROLLARY A:

$$\begin{aligned}\frac{\partial}{\partial u_0}\Phi(u_0,\ldots,u_N) &= -\overline{\mathsf{D}}_0\cdot\frac{\partial}{\partial u_0}\Phi^{\Diamond}(u_0,\ldots,u_N) \\ u_0\frac{\partial}{\partial u_0}\Phi^{\Diamond}(u_0,\ldots,u_N) &= \Phi^{\flat}(u_0,\ldots,u_N)\end{aligned}$$

Recall

$$\Phi^{\flat}(u_0,\ldots,u_N) = \sum_{\ell\in\mathbb{L}}(-1)^{\ell_0}\prod_{k=1}^{-\ell_0}(\delta_0+k)\cdot\prod_{j=1}^{N}\frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j+1+\delta_j)}\,u_j^{\ell_j+\delta_j}\cdot u_0^{\ell_0-\delta_0}$$

COROLLARY B:

i. $\left\langle\!\!\left\langle\, \overline{\mathsf{D}}_1 \cdot \overline{\mathsf{D}}_2 \cdot \overline{\mathsf{D}}_3 \,,\; \Phi(u_0, \ldots, u_N) \right\rangle\!\!\right\rangle \;=\; \Phi(u_0, \ldots, u_N)^{\deg 0} \;=\; 1 \qquad \text{(constant)}$

ii. For an edge $\mathsf{conv}(\mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V})$

$$\left\langle\!\!\left\langle\, \overline{\mathsf{D}}_i \cdot \overline{\mathsf{D}}_j \,,\; \Phi(u_0, \ldots, u_N) \right\rangle\!\!\right\rangle \;=\; \log(u_i^a u_j^b u_k u_m) - \left\langle \delta_i, \delta_j \right\rangle \Phi^{\diamondsuit}(u_0, \ldots, u_N)^{\deg 0}$$

where $\mathsf{conv}(\mathsf{v}_i, \mathsf{v}_j, \mathsf{v}_k)$ and $\mathsf{conv}(\mathsf{v}_i, \mathsf{v}_j, \mathsf{v}_m)$ are the two faces of $\mathsf{conv}(\mathcal{V})$ adjacent to edge $\mathsf{conv}(\mathsf{v}_i, \mathsf{v}_j)$ and $a, b \in \mathbb{Z}_{>0}$ are determined by the unique linear relation $\quad a\mathsf{v}_i + b\mathsf{v}_j + \mathsf{v}_k + \mathsf{v}_m = 0.$

iii. The constant function 1 and the functions $\log\big(u_1^{a_1} \cdot \ldots \cdot u_N^{a_N}\big)$ with $(a_1, \ldots, a_N) \in \mathbb{L}$ such that $a_1 + \ldots + a_N = 0$ generate an $(N-3)$-dimensional $\mathbb{C}$-subspace of the solution space of the GKZ system $(\star) - (\star\star\star)$.

iv. The isomorphism

$$\mathsf{CH}^*\big(\mathsf{Fano}(\mathcal{V})\big) \otimes \mathbb{C} \stackrel{\simeq}{\longrightarrow} \big\{\text{solutions of GKZ system } (\star) - (\star\star\star)\big\}$$

defined by the assignment $\qquad \Xi \mapsto \left\langle\!\!\left\langle\, \Xi \,,\; \Phi(u_0, \ldots, u_N) \right\rangle\!\!\right\rangle$
restricts to an isomorphism

$$\mathsf{Ann}(\overline{\mathsf{D}}_0) \otimes \mathbb{C} \stackrel{\simeq}{\longrightarrow} \ker\left(\frac{\partial}{\partial u_0}\right)$$

THEOREM C

i. The relative cohomology group

$$\mathsf{H}^4\left(\mathbb{C}^{*4}\ \mathsf{rel}\ \mathcal{Z}\big(x_0 - \mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3)\big)\right)$$

is a module over the ring of differential operators $\mathcal{D} = \mathbb{C}[u_0, \ldots, u_N, \frac{\partial}{\partial u_0}, \ldots, \frac{\partial}{\partial u_N}]$. It is generated as a $\mathcal{D}$-module by the cohomology class $\left[\frac{dx_0 dx_1 dx_2 dx_3}{x_0 x_1 x_2 x_3}\right]$ of the differential form $\dfrac{dx_0 dx_1 dx_2 dx_3}{x_0 x_1 x_2 x_3}$.

ii. There is an isomorphism of $\mathcal{D}$-modules

$$\begin{aligned} \mathsf{H}^4\left(\mathbb{C}^{*4}\ \mathsf{rel}\ \mathcal{Z}\big(x_0 - \mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3)\big)\right) &\overset{\simeq}{\longrightarrow} \mathsf{CH}^*\big(\mathsf{Fano}(\mathcal{V})\big) \otimes \mathcal{O} \\ \left[\frac{dx_0 dx_1 dx_2 dx_3}{x_0 x_1 x_2 x_3}\right] &\mapsto \Phi(u_0, \ldots, u_N) \end{aligned}$$

where $\mathcal{O}$ is an appropriate ring of functions in the variables $u_0, \ldots, u_N$.

iii. The isomorphism in **ii** restricts to an isomorphism

$$\ker\left(\frac{\partial}{\partial u_0}\right) \overset{\simeq}{\longrightarrow} \mathsf{Ann}(\overline{\mathsf{D}}_0) \otimes \mathcal{O}$$

Mahler measure

$$\Phi^{\Diamond}(u_0,\ldots,u_N) = \left(\log(u_0) - \tfrac{1}{2}\log^2(u_0)\delta_0 + \tfrac{1}{6}\log^3(u_0)\delta_0^2\right)\cdot u_1^{\delta_1}\cdot\ldots\cdot u_N^{\delta_N}$$
$$- \sum_{\ell\in\mathbb{L},\ell\neq 0}(-1)^{\ell_0}\prod_{k=1}^{-\ell_0-1}(\delta_0+k)\cdot\prod_{j=1}^{N}\frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j+1+\delta_j)}\,u_j^{\ell_j+\delta_j}\cdot u_0^{\ell_0-\delta_0}$$

Whence

$$\Phi^{\Diamond}(u_0,\ldots,u_N)^{\deg 0} = \log(u_0) - \sum_{\ell\in\mathbb{L}^+}(-1)^{\ell_0}(-\ell_0-1)!\,u_0^{\ell_0}\prod_{j=1}^{N}\frac{u_j^{\ell_j}}{\ell_j!}$$

where $\mathbb{L}^+ = \mathbb{L}\cap\left(\mathbb{Z}_{\geq 0}^N\setminus\{\mathsf{0}\}\right)$, $\ell_0 = -(\ell_1+\ldots+\ell_N)$.

The series converges if $|u_0| > |u_1|+\ldots+|u_N|$ and is then equal to the integral

$$(2\pi i)^{-3}\oint_{|x_1|=|x_2|=|x_3|=1}\log\left(\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)\right)\frac{dx_1}{x_1}\frac{dx_2}{x_2}\frac{dx_3}{x_3}.$$

If $u_0,\ldots,u_N$ are real numbers, then the integral is equal to

$$\int_0^1\int_0^1\int_0^1\log\left(\left|\mathcal{P}_{\mathcal{V},\mathsf{u}}(e^{2\pi i\xi_1},e^{2\pi i\xi_2},e^{2\pi i\xi_3})\right|\right)d\xi_1 d\xi_2 d\xi_3.$$

The latter integral is called the **logarithmic Mahler measure** of the Laurent polynomial $\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)$.

$\mathbb{C}^{*4}$	γ	$\int$	$\frac{dx_0 dx_1 dx_2 dx_3}{x_0 x_1 x_2 x_3}$
$\supset$	$\downarrow \partial$		$\uparrow d$
$\{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3) = x_0 \neq 0\}$	$\partial\gamma$	$\oint$	$\log(x_0)\frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3}$
$=$	$=$		$=$
$\mathbb{C}^{*3} \setminus \{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3) = 0\}$	$\partial\gamma$	$\oint$	$\log\big(\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)\big)\frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3}$
$\subset$			$\downarrow u_0 \frac{\partial}{\partial u_0}$
$\mathbb{C}^{*3}$	$\partial\gamma$	$\oint$	$\frac{u_0}{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)}\frac{dx_1 dx_2 dx_3}{x_1 x_2 x_3}$
$\supset$			$\downarrow$ residue
$\{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3) = 0\}$	residue $\partial\gamma$	$\oint$	$\frac{u_0}{d\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}$