Hypergeometric systems, Laurent polynomials, Chow quotients and dimers

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$$\mathbb{L} \quad \text{subgroup of} \quad \mathbb{Z}^N$$

$$\mathbb{L} \perp (1, 1, \ldots, 1)$$

$$\mathbb{Z}^N \;/\; \mathbb{L} \quad \text{torsion free}$$

$$\operatorname{rank} \mathbb{L} \;=\; k$$

$$\mathbb{T}^N \;=\; \mathrm{Hom}(\mathbb{Z}^N, \mathbb{C}^*) \;=\; \mathbb{C}^{*N}$$

$$\mathbb{T}_A \;=\; \mathrm{Hom}(\mathbb{Z}^N \;/\; \mathbb{L}\,,\, \mathbb{C}^*)$$

$\mathbb{T}_A \;\subset\; \mathbb{T}^N \;=\; \mathbb{C}^{*N}$ act on $\mathbb{C}^N$
by coordinatewise multiplication

$\mathbb{L} \perp (1, 1, \ldots, 1)$ implies

$$\mathbb{C}^* \hookrightarrow \mathbb{T}_A \hookrightarrow \mathbb{T}^N = \mathbb{C}^{*N}$$
$$t \in \mathbb{C}^* \quad \mapsto \quad (t, t, \ldots, t) \in \mathbb{C}^{*N}$$

$$\overline{\mathbb{T}}_A \ = \ \mathbb{T}_A \ / \ \mathbb{C}^*$$
$$\overline{\mathbb{T}}^N \ = \ \mathbb{T}^N \ / \ \mathbb{C}^*$$

act on projective space $\mathbb{P}^{N-1}$

orbit space for $\overline{\mathbb{T}}_A$ acting on $\mathbb{P}^{N-1}$

$=$

orbit space for $\mathbb{T}_A$ acting on $\mathbb{C}^N \setminus \{0\}$

? **Geometry of this orbit space** ?

contains at least the torus

$$\mathbb{T}^N \;/\; \mathbb{T}_A \;=\; \mathrm{Hom}(\mathbb{L}, \mathbb{C}^*)$$

What more does it sensibly contain?

e.g.

the so-called secondary fan in $\mathrm{Hom}(\mathbb{L}, \mathbb{R})$ provides a toric compactification.

Investigate geometry by intersecting orbits with linear subspaces of $\mathbb{P}^{N-1}$

e.g.
there is a kind of discriminantal set consisting of orbits of which the closure intersects the linear subspace

$$\mathbb{P}(\mathbb{L} \otimes \mathbb{C}) \;=\; ((\mathbb{L} \otimes \mathbb{C}) \setminus \{0\}) \;/\; \mathbb{C}^*$$

i.e.
image of $\mathbb{P}(\mathbb{L} \otimes \mathbb{C})$ in the orbit space.

Gelfand-Kapranov-Zelevinsky's
Principal A-determinant
is equation for discriminantal set.

Fix

$$\mathbb{Z}^N \;/\; \mathbb{L} \;\simeq\; \mathbb{Z}^{N-k}$$
$$\mathsf{e}_j \bmod \mathbb{L} \leftrightarrow \mathsf{a}_j = (a_{ij})$$

Then

$$\mathbb{C}^{*N-k} = \mathbb{T}_A \hookrightarrow \mathbb{T}^N = \mathbb{C}^{*N}$$
$$\mathsf{t} \;\mapsto\; (\mathsf{t}^{\mathsf{a}_1}, \ldots, \mathsf{t}^{\mathsf{a}_N})$$

where

$$\mathsf{t} = (t_1, \ldots, t_{N-k}), \quad \mathsf{t}^{\mathsf{a}_j} = \prod_{i=1}^{N-k} t_i^{a_{ij}}$$

Orbit through $\mathsf{u} = (u_1, \ldots, u_N) \in \mathbb{C}^N$ is

$$\mathbb{T}_A \mathsf{u} \;=\; \{(u_1 \mathsf{t}^{\mathsf{a}_1}, \ldots, u_N \mathsf{t}^{\mathsf{a}_N}) \mid \mathsf{t} \in \mathbb{T}_A\}$$

Intersections with hyperplane $x_1 + \ldots + x_N = 0$

$$\mathbb{T}_A \mathsf{u} \quad \bigcap \quad \{x_1 + \ldots + x_N = 0\}$$

boils down to

Laurent polynomial equation

$$u_1 \mathsf{t}^{\mathsf{a}_1} + \ldots + u_N \mathsf{t}^{\mathsf{a}_N} = 0$$

Geometry of orbit space from **variation of Hodge structure** of these intersections

e.g. for **Calabi-Yau threefolds** this leads to **special Kähler geometry** on the orbit space.

Variation of Hodge structure is given by **Gelfand-Kapranov-Zelevinsky** system of **hypergeometric** differential equations associated with $\mathbb{L}$ and a vector $\mathsf{c} \in \mathbb{C}^{N-k}$

This is system of PDE's for functions $\Phi(u_1, \ldots, u_N)$

- for $(\ell_1, \ldots, \ell_N) \in \mathbb{L} \subset \mathbb{Z}^N$

$$\prod_{\ell_j > 0} \left(\frac{\partial}{\partial u_j} \right)^{\ell_j} \Phi \;=\; \prod_{\ell_j < 0} \left(\frac{\partial}{\partial u_j} \right)^{-\ell_j} \Phi$$

-
$$\sum_{j=1}^{N} \mathsf{a}_j u_j \frac{\partial}{\partial u_j} \Phi \;=\; \mathsf{c} \Phi$$

'Generically'
dimension solution space =
volume of convex hull of $\{\mathsf{a}_1, \ldots, \mathsf{a}_N\}$

For solutions Φ_1, Φ_2 of GKZ system:

$$\sum_{j=1}^{N} \mathsf{a}_j u_j \frac{\partial}{\partial u_j} \left(\frac{\Phi_1}{\Phi_2} \right) \; = \; 0$$

This means that $\frac{\Phi_1}{\Phi_2}$ is $\mathbb{T}_A$-invariant

Consequence:
Locally evaluation of functions maps orbit space into projectivization of dual of solution space of GKZ system
Globally must mod out by monodromy action on solution space

Intersections with $k-1$-dim. linear subspaces

For $X \subset \mathbb{P}^{N-1}$ irreducible subvariety of dimension $N-k-1$ and degree d consider the set $Z(X)$ of all $k-1$-dimensional linear subspaces of $\mathbb{P}^{N-1}$ which intersect X.

$Z(X)$ is an irreducible hypersurface of degree d in the Grassmannian $G(k, N)$ of all $k-1$-dimensional linear subspaces of $\mathbb{P}^{N-1}$, called the **associated hypersurface** of X.

$Z(X)$ is given by the vanishing of a (up to scalar multiple) unique form of degree d in the Plücker coordinates coordinates on $G(k, N)$:
the **Chow form** of X.

Apply to closures of maximal $\overline{\mathbb{T}}_A$-orbits in $\mathbb{P}^{N-1}$ and assign to each maximal $\overline{\mathbb{T}}_A$-orbit its Chow form.

Get map from dense open part of orbit space into the projectivation of the space of degree d forms in the (Plücker) coordinate ring of $G(k, N)$.

Closure of image is **Chow quotient** for the action of $\overline{\mathbb{T}}_A$ on $\mathbb{P}^{N-1}$.

Other models for a quotient of $\mathbb{P}^{N-1}$ by the action of $\overline{\mathbb{T}}_A$ are provided by **Geometric Invariant Theory**.

Kapranov-Sturmfels-Zelevinsky:

The Chow quotient is the toric variety constructed from the *secondary fan*. The secondary fan is the normal fan to the *secondary polytope*.

The Chow quotient is the smallest toric variety which maps onto all GIT quotients (associated to particular linearizations) orbit of $[\mathsf{u}] = [u_1 : \ldots : u_N] \in \mathbb{P}^{N-1}$

$$\overline{\mathbb{T}}_A[\mathsf{u}] = \{[u_1\mathsf{t}^{\mathsf{a}_1} : \ldots : u_N\mathsf{t}^{\mathsf{a}_N}] \mid \mathsf{t} \in \mathbb{T}_A\}$$

Linear embedding $\mathbb{P}^{k-1} \hookrightarrow \mathbb{P}^{N-1}$ given by rank k $k \times N$ complex matrix:

$$Y = (y_{ij})$$

This linear subspace intersects orbit $\overline{\mathbb{T}}_A[\mathsf{u}]$ if and only if there is $\mathsf{t} \in \mathbb{T}_A$ such that matrix ♠ has rank k;

$$\begin{bmatrix} u_1\mathsf{t}^{\mathsf{a}_1} & \ldots & \ldots & u_N\mathsf{t}^{\mathsf{a}_N} \\ y_{11} & \ldots & \ldots & y_{1N} \\ \vdots & \vdots & \vdots & \vdots \\ y_{k1} & \ldots & \ldots & y_{kN} \end{bmatrix} \qquad \spadesuit$$

Linear subspace intersects orbit $\overline{\mathbb{T}}_A[\mathsf{u}]$ if and only if there is $\mathsf{t} \in \mathbb{T}_A$ such that all $(k+1) \times (k+1)$-subdeterminants of matrix $\spadesuit$ vanish.

The Plücker coordinates of the point of $G(k, N)$ corresponding to the linear subspace are the $k \times k$-subdeterminants of the matrix Y.

$(k+1) \times (k+1)$-subdeterminants of $\spadesuit$ are homogeneous linear in each of the sets $\{u_1, \ldots, u_N\}$, $\{\mathsf{t}^{\mathsf{a}_1}, \ldots, \mathsf{t}^{\mathsf{a}_N}\}$ and $\{$Plücker coordinates of $Y\}$.

In order to find Chow form we must eliminate t.

From now on $\operatorname{rank}\mathbb{L} = k = 2$**.**

The 3×3-subdeterminants of ♠ are then

$$u_h \mathsf{t}^{\mathsf{a}_h} Y_{ij} + u_i \mathsf{t}^{\mathsf{a}_i} Y_{jh} + u_j \mathsf{t}^{\mathsf{a}_j} Y_{hi}$$

with $1 \leq h < i < j \leq N$ en

$$Y_{ij} = y_{1i} y_{2j} - y_{2i} y_{1j}$$

If all $u_i \neq 0$ the linear subspace intersects the orbit $\overline{\mathbb{T}}_A[\mathsf{u}]$ if and only if there is $\mathsf{t} \in \mathbb{T}_A$ such that

$$\begin{aligned} \clubsuit_{hij} &= Y_{hi} u_h^{-1} u_i^{-1} \mathsf{t}^{-\mathsf{a}_h - \mathsf{a}_i} \\ &\quad + Y_{ij} u_i^{-1} u_j^{-1} \mathsf{t}^{-\mathsf{a}_i - \mathsf{a}_j} \\ &\quad + Y_{jh} u_j^{-1} u_h^{-1} \mathsf{t}^{-\mathsf{a}_j - \mathsf{a}_h} \\ &= 0 \end{aligned}$$

for all $1 \leq h < i < j \leq N$.

Represent $\mathbb{L} \subset \mathbb{Z}^N$ as the image of a $2 \times N$-matrix

$$B \;=\; \begin{bmatrix} b_{11} & \dots & \dots & b_{1N} \\ b_{21} & \dots & \dots & b_{2N} \end{bmatrix}$$

Interpret the skew-symmetric matrix

$$Q \;=\; B^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} B$$

as the (oriented) adjacency matrix of a quiver $\mathcal{Q}$.

The so-called **inverse algorithm** constructs a **superpotential for the quiver** $\mathcal{Q}$ (Hannany et al., J.S., Gulotta)

In geometric terms the superpotential is an embedding of $\mathcal{Q}$ into an oriented surface without boundary $\mathcal{S}$.

This induces on $\mathcal{S}$ cell structure with

$$\begin{aligned} &\text{0-cells} = \mathcal{Q}^0 = \text{vertices of } \mathcal{Q} \\ &\text{1-cells} = \mathcal{Q}^1 = \text{arrows of } \mathcal{Q} \\ &\text{2-cells} = \text{conn. components of } \mathcal{S} \setminus \mathcal{Q} \end{aligned}$$

The 2-cells come in two kinds depending on whether the orientation on the boundary induced from $\mathcal{S}$ is the same as (+) or opposite to (-) the orientation induced from $\mathcal{Q}$

$$\begin{aligned} \mathcal{Q}^{2\bullet} &= \text{2-cells with} + \text{boundary} \\ \mathcal{Q}^{2\circ} &= \text{2-cells with} - \text{boundary} \end{aligned}$$

In present situation

$$\sharp\mathcal{Q}^{2\bullet} = \sharp\mathcal{Q}^{2\circ} = \text{vol. conv.}\{\mathsf{a}_1, \ldots, \mathsf{a}_N\}$$

maps

$$s, t : \mathcal{Q}^1 \to \mathcal{Q}^0$$

assign to arrow its source resp. target

maps

$$\begin{aligned} \mathsf{b} &: \mathcal{Q}^1 \to \mathcal{Q}^{2\bullet} \\ \mathsf{w} &: \mathcal{Q}^1 \to \mathcal{Q}^{2\circ} \end{aligned}$$

assign to arrow e the $+$ resp. $-$ 2-cell with e in boundary

Superpotential faithfully described by **bi-adjacency matrix** $\mathbb{K}(Z_{ij})$ with rows (resp. columns) $1:1$ corresponding with $\mathcal{Q}^{2\circ}$ (resp. $\mathcal{Q}^{2\bullet}$)

$$(\mathsf{w},\mathsf{b})\text{-entry of } \mathbb{K}(Z_{ij}) = \sum_{e\in\mathcal{Q}^1,\, \mathsf{w}(e)=\mathsf{w},\, \mathsf{b}(e)=\mathsf{b}} Z_{s(e),t(e)}$$

where $\{Z_{ij} \mid 1 \leq i,j \leq N\}$ is a set of variables and $\mathcal{Q}^0$ has been identified with $\{1,\dots,N\}$.

Theorem

1. Chow form of closure of $\overline{\mathbb{T}}_A$-orbit of $[u_1 : \ldots : u_N]$ with all $u_j \neq 0$ is
$$\det(u_1 \cdot \ldots \cdot u_N \cdot \mathbb{K}(Y_{ij} u_i^{-1} u_j^{-1}))$$
with
Y_{ij} Plücker coordinates on $G(k, N)$
2. Principal A-determinant is
$$\det(u_1 \cdot \ldots \cdot u_N \cdot \mathbb{K}(Y_{ij} u_i^{-1} u_j^{-1}))$$
with
$$\begin{aligned} Y_{ij} &= \sharp\{\text{arrows } i \to j\} \\ &= \text{Plücker coords } \mathbb{L} \subset \mathbb{Z}^N \end{aligned}$$
3. The *Newton polygon* of
$$\det(u_1 \cdot \ldots \cdot u_N \cdot \mathbb{K}(Y_{ij} u_i^{-1} u_j^{-1}))$$
w.r.t. variables $u_1, \ldots, u_N$ is the *secondary polytope*.

Ideas of proof for 1.

- for every 2-cell C sums of $\clubsuit_{hij}$ yield

$$\sum_{e\in\partial C} Y_{s(e)t(e)} u_{s(e)}^{-1} u_{t(e)}^{-1} \mathsf{t}^{-\mathsf{a}_{s(e)}-\mathsf{a}_{t(e)}}$$

  where the sum runs over all 1-cells in the boundary of C.

- Consequently, if the linear subspace of $\mathbb{P}^{N-1}$ given by matrix Y intersects the orbit $\overline{\mathbb{T}}_A[\mathsf{u}]$ then the matrix

$$\mathbb{K}(Y_{ij} u_i^{-1} u_j^{-1} \mathsf{t}^{-\mathsf{a}_i-\mathsf{a}_j})$$

  has non-trivial kernel and, hence,

$$\det \mathbb{K}(Y_{ij} u_i^{-1} u_j^{-1} \mathsf{t}^{-\mathsf{a}_i-\mathsf{a}_j}) = 0$$

- For every $\mathsf{b} \in \mathcal{Q}^{2\bullet}$ and $\mathsf{w} \in \mathcal{Q}^{2\circ}$ there are ε_{b}, $\varepsilon_{\mathsf{w}} \in \mathbb{Z}^{N-k}$ such that for every $e \in \mathcal{Q}^1$

$$\mathsf{a}_{s(e)} + \mathsf{a}_{t(e)} = \varepsilon_{\mathsf{b}(e)} - \varepsilon_{\mathsf{w}(e)}$$

- Consequently

$$\det \mathbb{K}(Y_{ij} u_i^{-1} u_j^{-1} \mathsf{t}^{-\mathsf{a}_i - \mathsf{a}_j}) =$$

$$\mathsf{t}^{\epsilon} \det \mathbb{K}(Y_{ij} u_i^{-1} u_j^{-1})$$

with

$$\epsilon = \sum_{\mathsf{w} \in \mathcal{Q}^{2\circ}} \varepsilon_{\mathsf{w}} - \sum_{\mathsf{b} \in \mathcal{Q}^{2\bullet}} \varepsilon_{\mathsf{b}}$$

- $\det \mathbb{K}(Y_{ij} u_i^{-1} u_j^{-1})$ has the correct degree in the Plücker coordinates

■

Example related to Somos 4

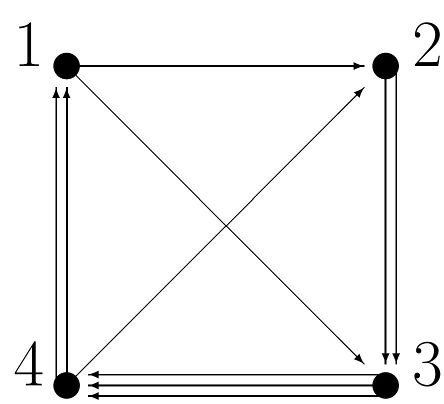
$$\mathbb{L} = \mathbb{Z}^2 \begin{bmatrix} 0 & 1 & 1 & -2 \\ -1 & 0 & 2 & -1 \end{bmatrix} \subset \mathbb{Z}^4$$

$$[\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3, \mathsf{a}_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 0 & 2 & 1 \end{bmatrix}$$

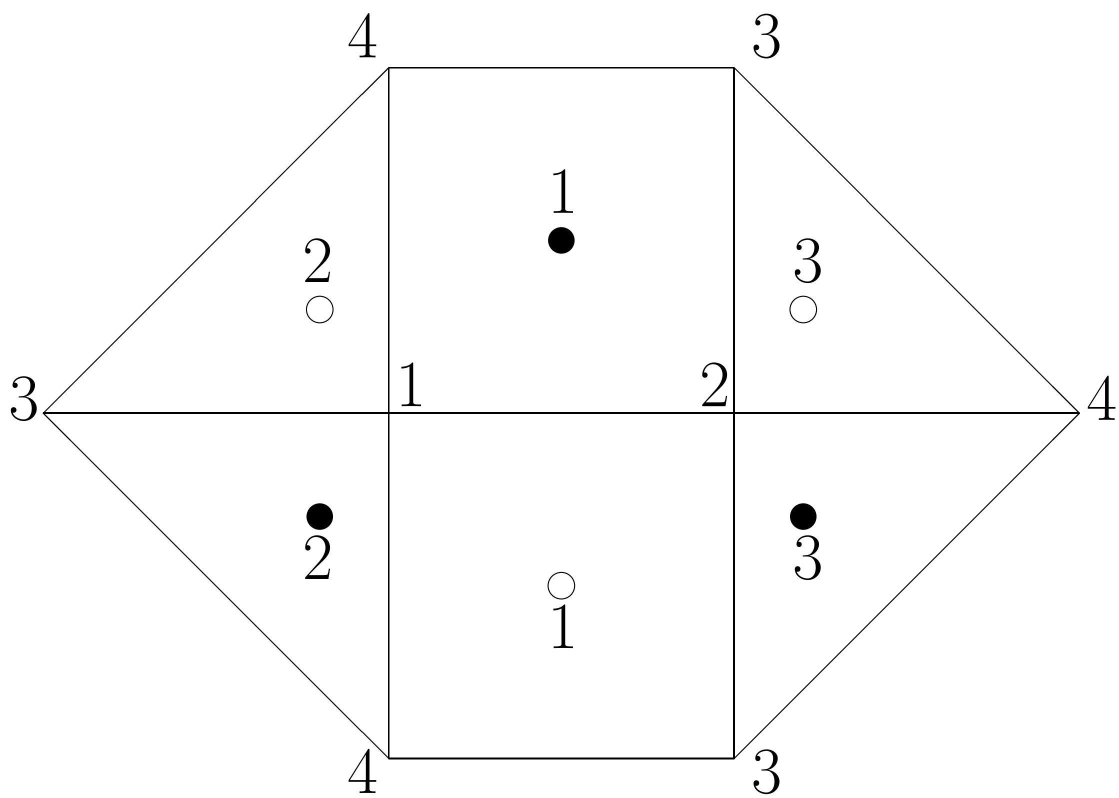
Laurent polynomial equation

$$(u_1 t_2^3 + u_2 + u_3 t_2^2 + u_4 t_2) t_1 = 0$$

Quiver



Superpotential=
Quiver in torus (identify opposite sides)



$$\mathbb{K}(Z_{ij}) = \begin{bmatrix} Z_{12}+Z_{34} & Z_{41} & Z_{23} \\ Z_{41} & Z_{13} & Z_{34} \\ Z_{23} & Z_{34} & Z_{42} \end{bmatrix}$$

$$\det \mathbb{K}(Z_{ij}) =$$

$$Z_{12}Z_{13}Z_{42} + Z_{34}Z_{13}Z_{42} + 2\,Z_{41}Z_{34}Z_{23}$$
$$- Z_{12}Z_{34}^2 - Z_{34}^3 - Z_{13}Z_{23}^2 - Z_{42}Z_{41}^2$$

Chow form of closure of $\overline{\mathbb{T}}_A$-orbit of $[u_1 : u_2 : u_3 : u_4]$ with all $u_j \neq 0$ is

$$\det(u_1u_2u_3u_4 \cdot \mathbb{K}(Y_{ij}u_i^{-1}u_j^{-1})) =$$
$$Y_{12}Y_{13}Y_{42}u_1u_2u_3^2u_4^2 - Y_{34}^3u_1^3u_2^3$$
$$- Y_{13}Y_{23}^2u_1^2u_2u_4^3 - Y_{42}Y_{41}^2u_1u_2^2u_3^3$$
$$+(Y_{34}Y_{13}Y_{42} + 2\,Y_{41}Y_{34}Y_{23} - Y_{12}Y_{34}^2)$$
$$\times u_1^2u_2^2u_3u_4$$

with Y_{ij} Plücker coordinates on $G(2,4)$

Principal A-determinant is

$$u_1 u_2 u_3^2 u_4^2 + 18 u_1^2 u_2^2 u_3 u_4 +$$
$$- 27 u_1^3 u_2^3 - 4 u_1^2 u_2 u_4^3 - 4 u_1 u_2^2 u_3^3$$

This is $u_1 u_2$ times the discriminant of the polynomial

$$u_1 t^3 + u_3 t^2 + u_4 t + u_2$$

Secondary polygon