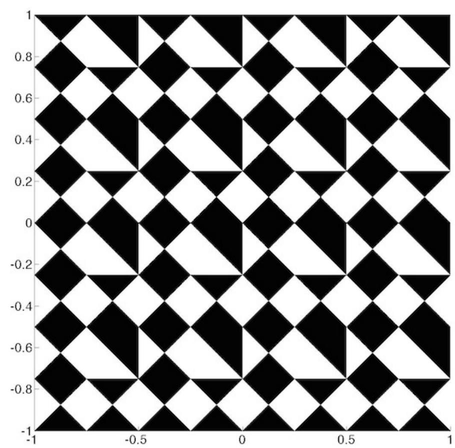
ZHEGALKIN ZEBRA MOTIVES

Algebra and Geometry
in Black and White

Jan Stienstra



Nijmegen Geometry Seminar,
November 1, 2018

Motives in *Algebraic Geometry*
need (technically very complicated)

intersection theory of algebraic cycles

....

Zhegalkin zebra motives
just need the usual operations

intersection/union/complement
for *subsets of the plane* !!

subsets of $\mathbb{R}^2$ $\quad =$
$\mathbb{F}_2$-valued functions on $\mathbb{R}^2$

Zhegalkin (1927):
Boolean formalism for subsets
$\Leftrightarrow$
usual $\times$ and $+$
for $\mathbb{F}_2$-valued functions

Zebra with frequency $\mathfrak{v} \in \mathbb{R}^2 \setminus \{\mathsf{0}\}$

$$\mathsf{Z}^{\mathfrak{v}} \ : \qquad \mathbb{R}^2 \longrightarrow \mathbb{F}_2$$

$$\mathsf{Z}^{\mathfrak{v}}(\mathbf{x}) \ = \ \lfloor 2\mathbf{x} \cdot \mathfrak{v} \rfloor \bmod 2$$

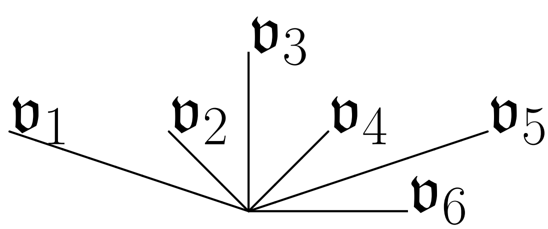
bands $\perp$ $\mathfrak{v}$, width $\frac{1}{2|\mathfrak{v}|}$

Zebras $\mathsf{Z}^{\mathfrak{v}}$ with frequencies $\mathfrak{v} \in \mathbb{Z}^2$ generate a subring of the ring of $\mathbb{F}_2$-valued functions on $\mathbb{R}^2$.

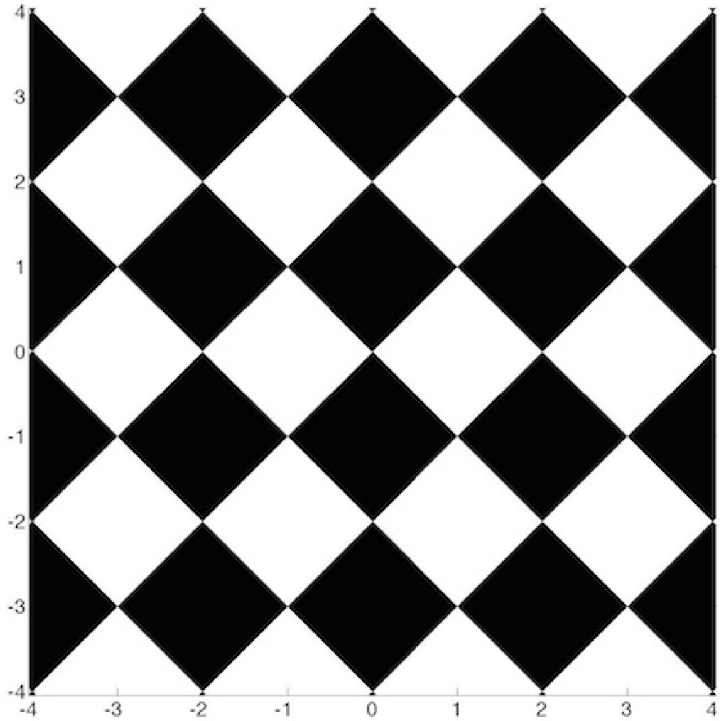
I call the elements of this subring *Zhegalkin zebra motives.*

Only use zebras with frequencies positive integer multiples of the six basic frequencies

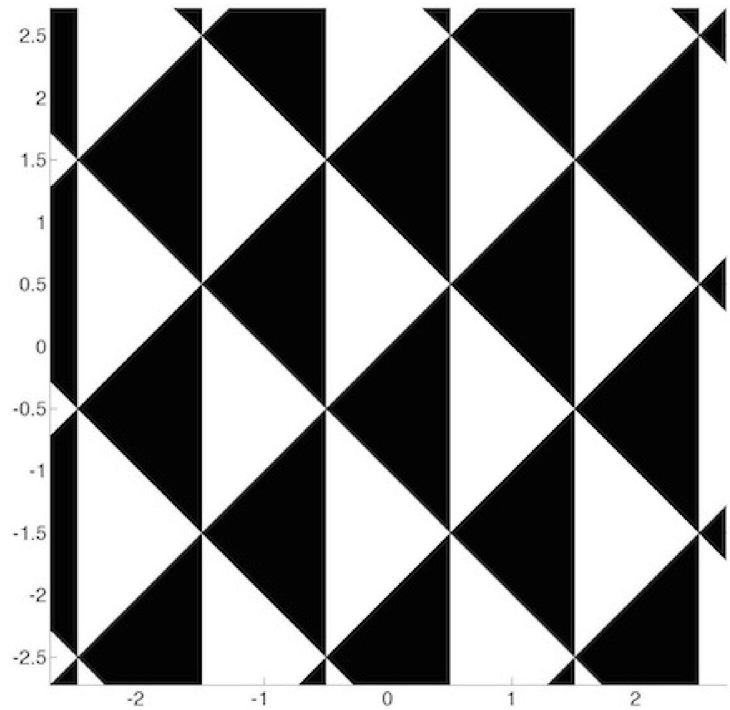
$$\mathfrak{v}_1 = \begin{pmatrix} -3 \\ 1 \end{pmatrix}, \qquad \mathfrak{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$
$$\mathfrak{v}_3 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \qquad \mathfrak{v}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$\mathfrak{v}_5 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \qquad \mathfrak{v}_6 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$



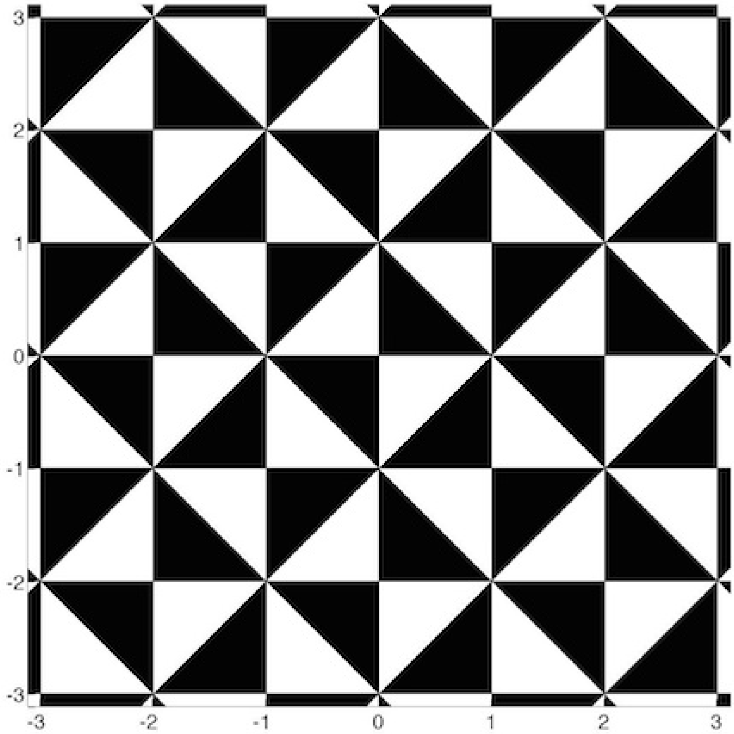
Short hand notation: $\mathsf{Z}^{jk} = \mathsf{Z}^{k\mathfrak{v}_j}$



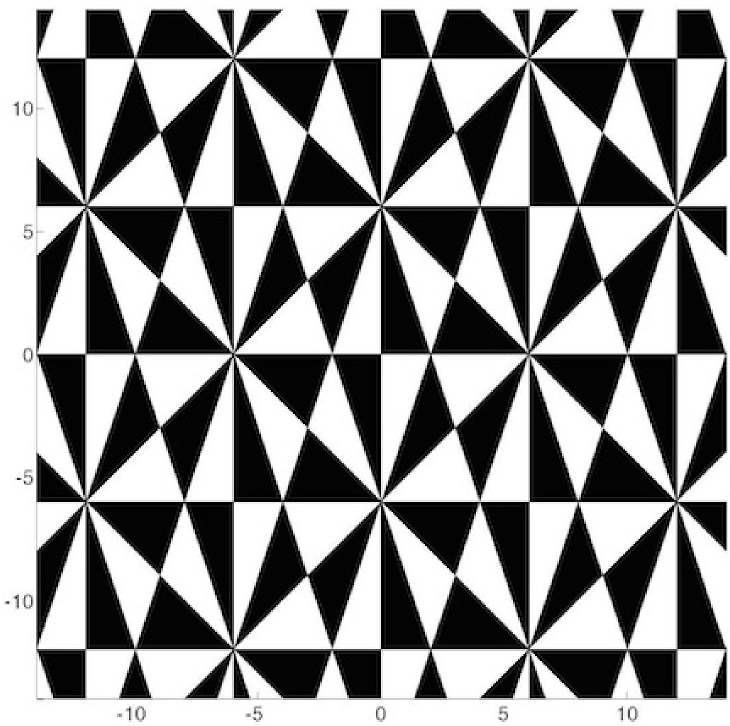
$$\mathcal{F}_2 = \mathsf{Z}^{21} + \mathsf{Z}^{41}$$



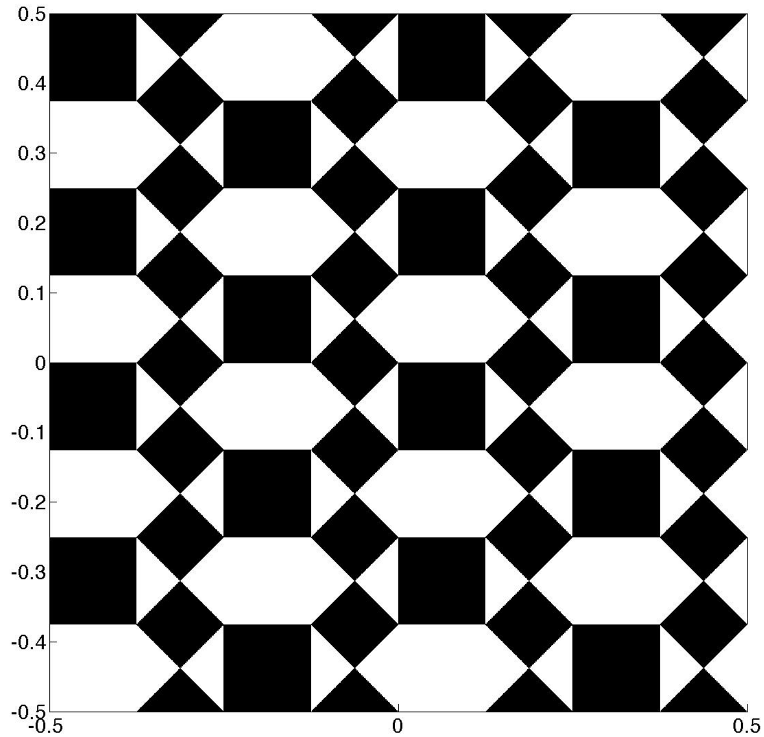
$$\mathcal{F}_3 = \mathsf{Z}^{21} + \mathsf{Z}^{41} + \mathsf{Z}^{61}$$



$$\mathcal{F}_4 = \mathsf{Z}^{21} + \mathsf{Z}^{31} + \mathsf{Z}^{41} + \mathsf{Z}^{61}$$



$$\mathcal{F}_6 = \mathsf{Z}^{11} + \mathsf{Z}^{21} + \mathsf{Z}^{31} + \mathsf{Z}^{41} + \mathsf{Z}^{51} + \mathsf{Z}^{61}$$

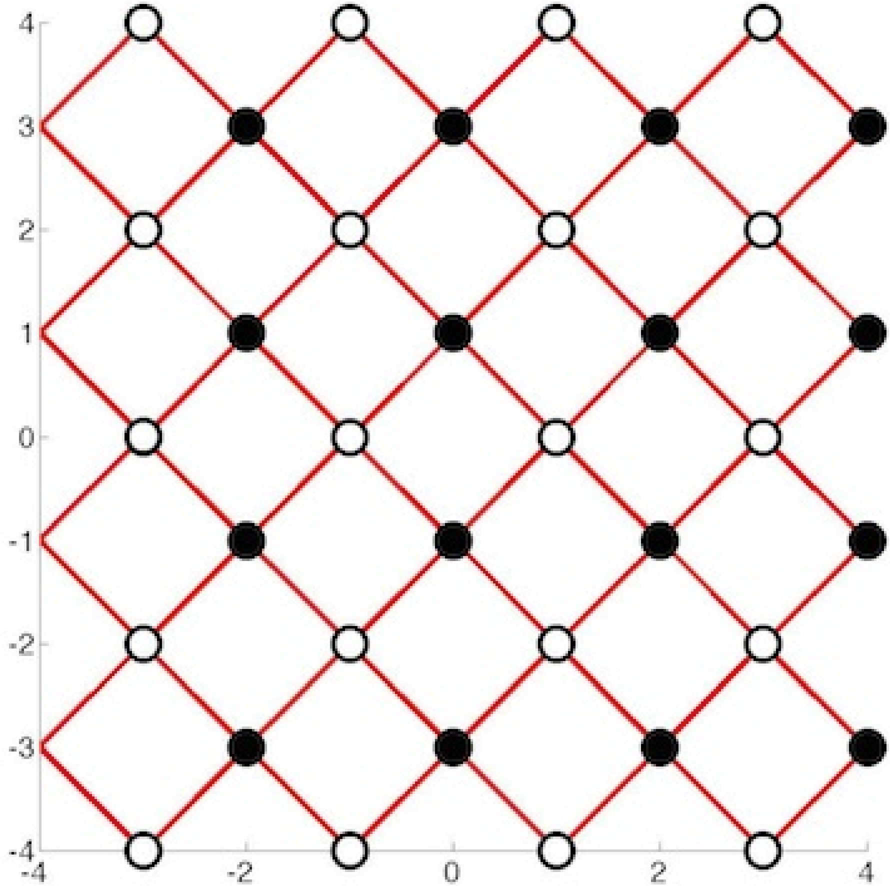


$$(1 + \mathsf{Z}^{62})(\mathsf{Z}^{32} + \mathsf{Z}^{61}) + \mathsf{Z}^{62}(\mathsf{Z}^{24} + \mathsf{Z}^{44})$$

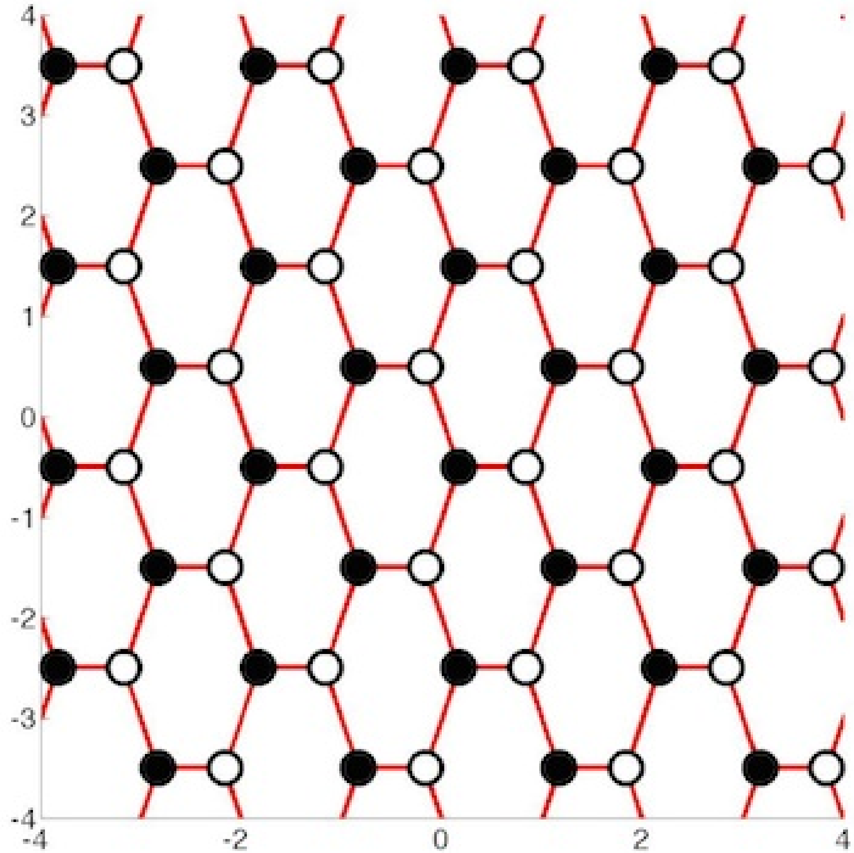
$$\left.\begin{array}{l}\text{vertices}\\ \text{(oriented) edges}\end{array}\right\}\quad \text{quiver } \Gamma$$

$$\left.\begin{array}{l}\text{black/white polygons}\\ \text{edges}\end{array}\right\}\quad \text{bipartite graph } \Gamma^{\vee}$$

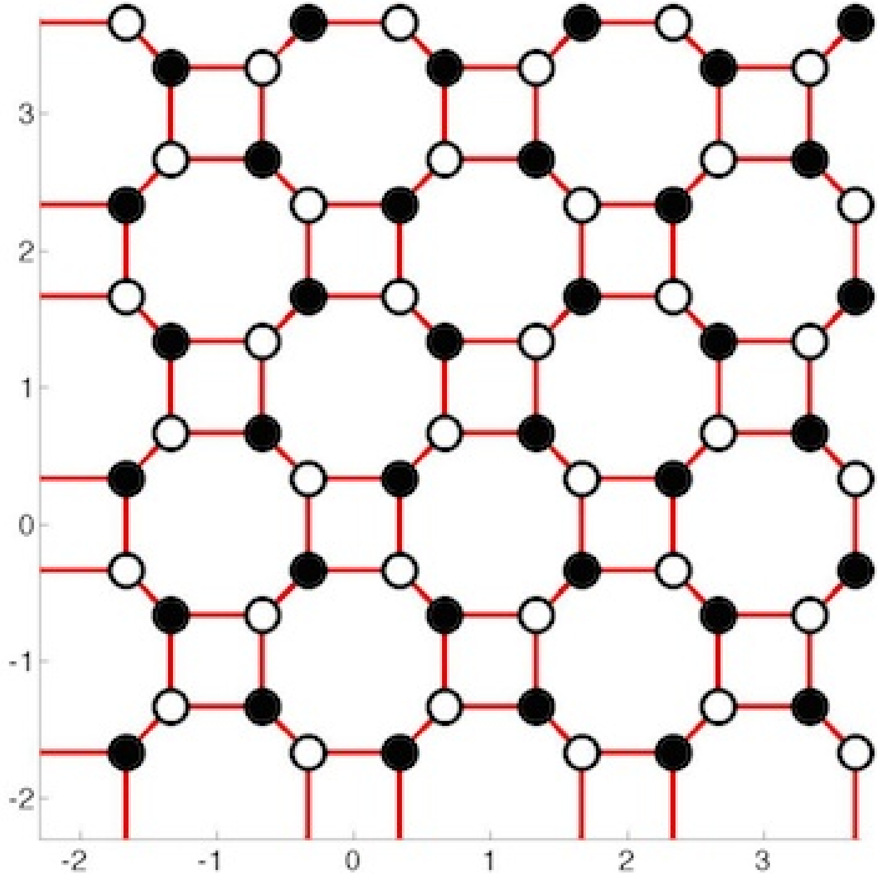
Γ and $\Gamma^{\vee}$ are dual graphs



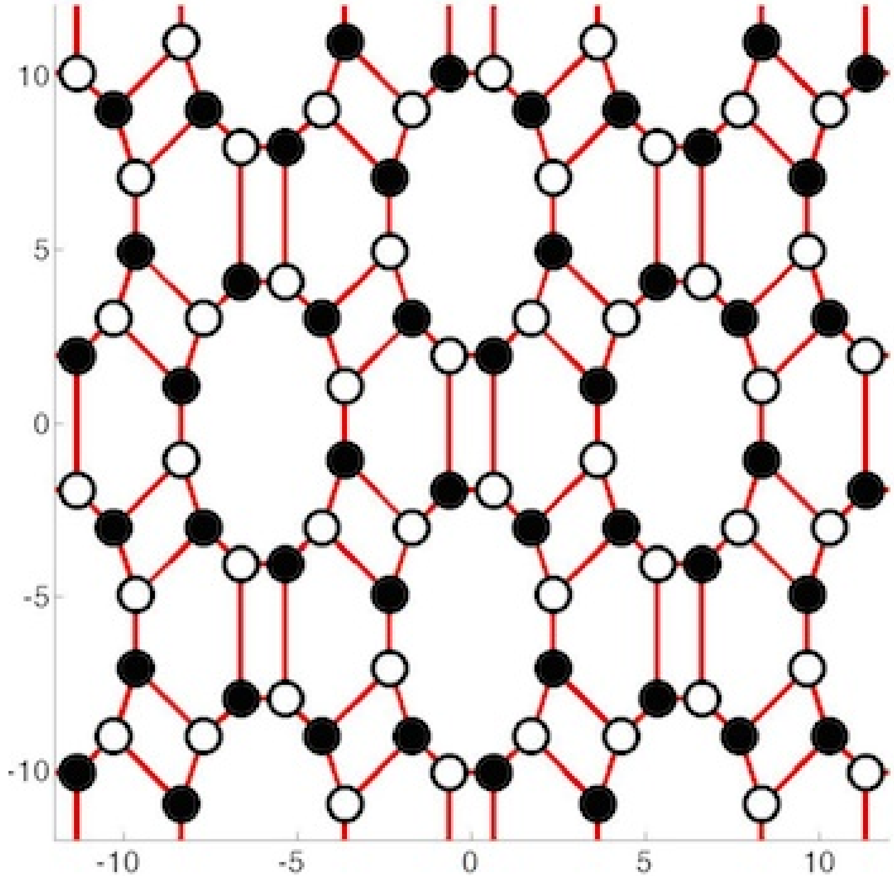
$$\mathcal{F}_2 = \mathsf{Z}^{21} + \mathsf{Z}^{41}$$



$$\mathcal{F}_3 = \mathsf{Z}^{21} + \mathsf{Z}^{41} + \mathsf{Z}^{61}$$



$$\mathcal{F}_4 = \mathsf{Z}^{21} + \mathsf{Z}^{31} + \mathsf{Z}^{41} + \mathsf{Z}^{61}$$



$$\mathcal{F}_6 = \mathsf{Z}^{11} + \mathsf{Z}^{21} + \mathsf{Z}^{31} + \mathsf{Z}^{41} + \mathsf{Z}^{51} + \mathsf{Z}^{61}$$

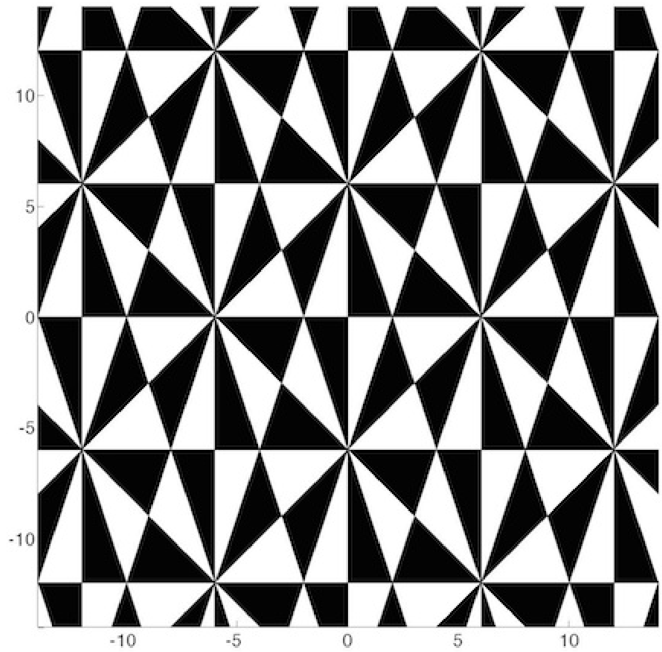
Positive integer weight function

$$\nu : \{\text{edges of } \Gamma\} \longrightarrow \mathbb{Z}_{>0}$$

such that for every polygon P

$$\sum_{e \text{ edge } P} \nu(e) = \deg(\nu)$$

with $\deg(\nu)$ independent of P.



10
5
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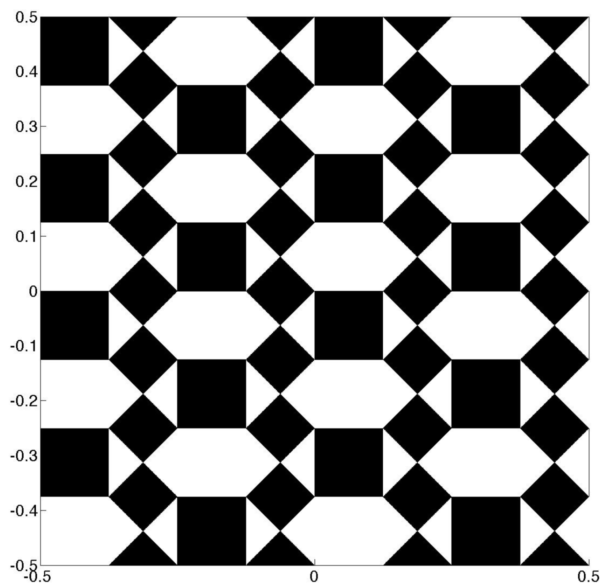
-10
-10
-5
0
5
10

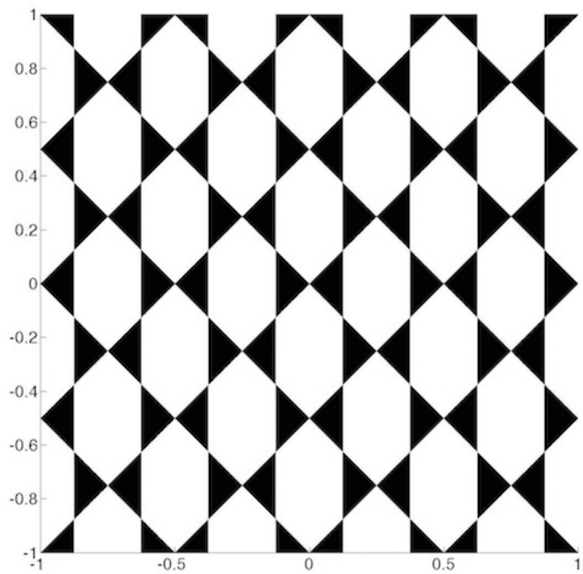


0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
-0.5
0
0.5

Necessary for existence of a positive integer weight function:

$\sharp$ black $=$ $\sharp$ white polygons

This fails for



$$\mathsf{Z}^{21} + \mathsf{Z}^{41} + \mathsf{Z}^{61} + \mathsf{Z}^{62}$$

Positive integer weight function ν marks a point in every polygon P by:

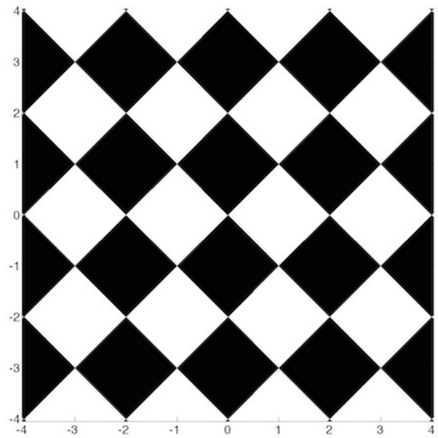
$$\sum_{e \text{ edge } P} \frac{\nu(e)}{\deg(\nu)} (\text{midpoint of } e)$$

Automorphism group of Zhegalkin zebra motive $\mathcal{F}$

$$\mathsf{Aut}(\mathcal{F}) =$$
$$\{\, \tau \in \mathbb{R}^2 \,|\, \mathcal{F}(\mathbf{x} + \tau) = \mathcal{F}(\mathbf{x})\,,\ \forall \mathbf{x} \in \mathbb{R}^2 \,\}\,.$$

This is a lattice in $\mathbb{R}^2$ if all polygons in the tiling for $\mathcal{F}$ are convex.

For sublattice $\Lambda \subset \mathsf{Aut}(\mathcal{F})$ the tiling of $\mathbb{R}^2$ by black and white polygons descends to a black-white polygonal tiling on the torus $\mathbb{R}^2/\Lambda$.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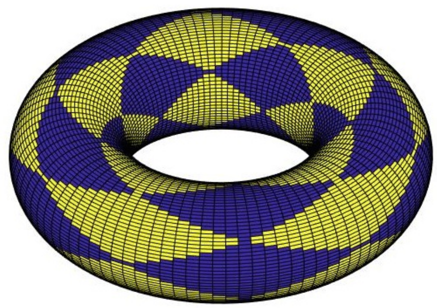
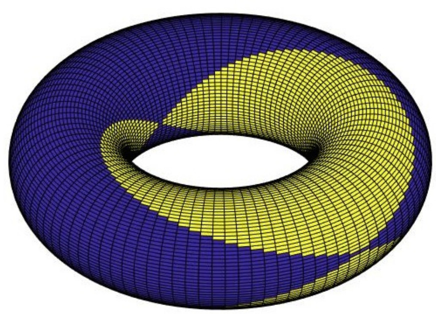
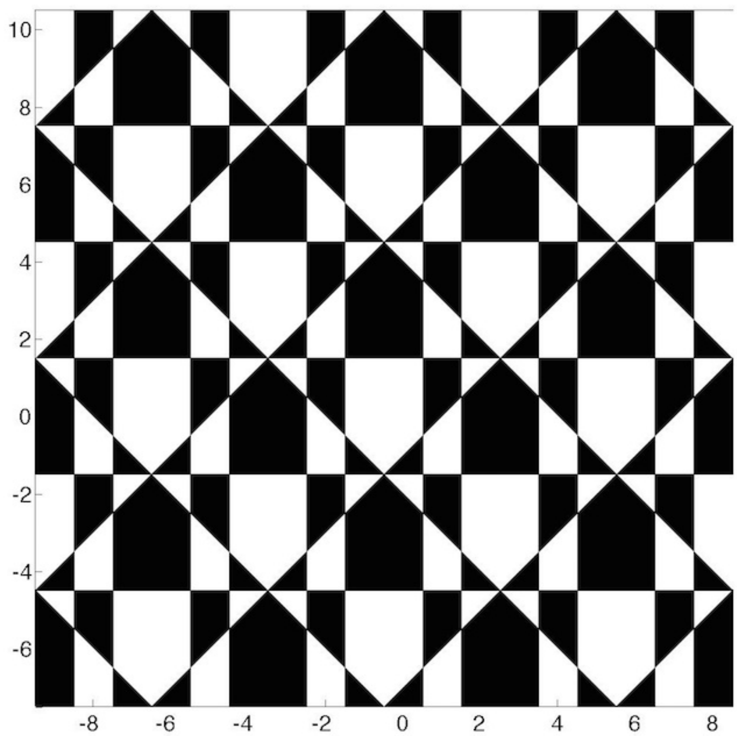
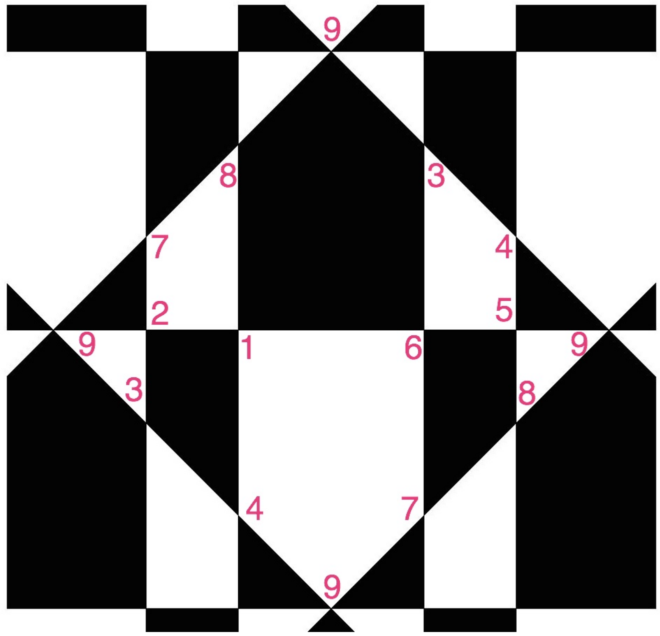
4







$$\mathsf{Z}^{21} + \mathsf{Z}^{31} + \mathsf{Z}^{41} + \mathsf{Z}^{61} + \mathsf{Z}^{63}$$



e	$s(e)$	$t(e)$	$\omega_1(e)$	$\omega_2(e)$	ν
1	2	9	-1	0	2
2	7	2	0	-1	2
3	2	1	1	0	1
4	1	8	0	2	1
5	6	1	-2	0	1
6	9	3	1	-1	1
7	3	6	0	-2	1
8	3	2	0	1	2
9	6	5	1	0	1
10	4	3	-1	1	1
11	5	4	0	1	2
12	1	4	0	-2	1
13	9	5	-1	0	2
14	4	9	1	-1	1
15	9	7	1	1	1
16	7	6	0	2	1
17	8	7	-1	-1	1
18	5	8	0	-1	2
19	8	9	1	1	1

Weight functions with values in $\mathbb{Z}_{\geq 0}$:

$$\begin{aligned} \nu_1(e) &= \omega_1(e) + 2\nu(e) \\ \nu_2(e) &= \omega_2(e) + 2\nu(e) \\ \nu_3(e) &= 2\nu(e) \end{aligned}$$

$\deg \nu_1 = \deg \nu_2 = \deg \nu_3 = 10.$

Convert table to 9×9-matrix:

edge e contributes monomial

$$u_1^{\nu_1(e)} u_2^{\nu_2(e)} u_3^{\nu_3(e)}$$

to matrix entry in position $(s(e), t(e))$

$$
\begin{bmatrix}
0 & 0 & 0 & u_1^2u_3^2 & 0 & 0 & 0 & u_1^2u_2^4u_3^2 & 0 \\
u_1^3u_2^2u_3^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & u_1^3u_2^4u_3^4 \\
0 & u_1^4u_2^5u_3^4 & 0 & 0 & 0 & u_1^2u_3^2 & 0 & 0 & 0 \\
0 & 0 & u_1u_2^3u_3^2 & 0 & 0 & 0 & 0 & 0 & u_1^3u_2u_3^2 \\
0 & 0 & 0 & u_1^4u_2^5u_3^4 & 0 & 0 & 0 & u_1^4u_2^3u_3^4 & 0 \\
u_2^2u_3^2 & 0 & 0 & 0 & u_1^3u_2^2u_3^2 & 0 & 0 & 0 & 0 \\
0 & u_1^4u_2^3u_3^4 & 0 & 0 & 0 & u_1^2u_2^4u_3^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & u_1u_2u_3^2 & 0 & u_1^3u_2^3u_3^2 \\
0 & 0 & u_1^3u_2u_3^2 & 0 & u_1^3u_2^4u_3^4 & 0 & u_1^3u_2^3u_3^2 & 0 & 0
\end{bmatrix}
$$

Contribution of edge e:

matrix $\Phi_{\nu_1,\nu_2,\nu_3}(e)$
with $(\mathbf{s},\mathbf{t})$-entry

$$
\begin{array}{ll}
u_1^{\nu_1(e)}u_2^{\nu_2(e)}u_3^{\nu_3(e)} & \text{if } \ \mathbf{s}=s(e),\ \mathbf{t}=t(e)\,, \\
0 & \text{otherwise.}
\end{array}
$$

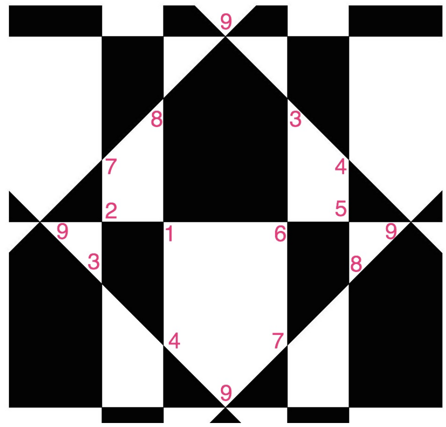
$e \mapsto \Phi_{\nu_1,\nu_2,\nu_3}(e)$
defines algebra homomorphism

$$\begin{aligned} \Phi_{\nu_1,\nu_2,\nu_3} : &\quad \text{Path algebra of } \Gamma_\Lambda \\ \longrightarrow &\quad \mathsf{Mat}_{\mathsf{P}^\star_\Lambda}(\mathbb{Z}[u_1, u_2, u_3]) \end{aligned}$$

Image of Φ_{ν_1,ν_2,ν_3}

$$\simeq \quad \textbf{Jacobi algebra} \text{ of } (\mathcal{F}, \Lambda)$$

$$= \mathbb{Z}[\mathsf{path}(\Gamma_\Lambda)] / \langle \mathsf{D}^\circ(e) \mid e \in \mathcal{E}_\Lambda \rangle$$



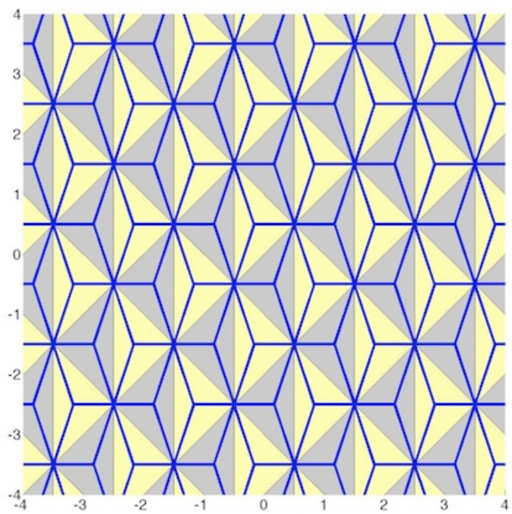
2-sided ideal generated by

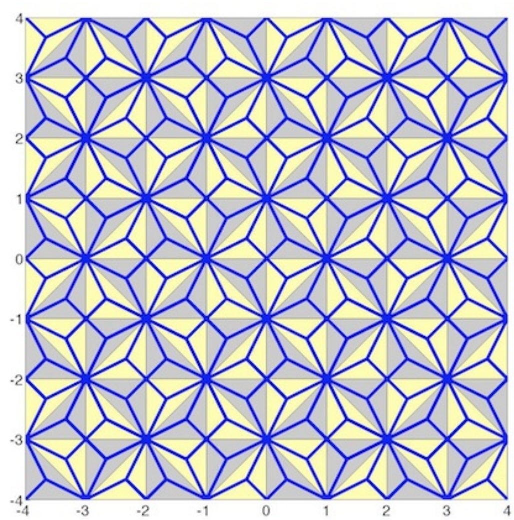
$$\mathsf{D}^{\circ}(e) = \prod_{e''\neq e:w(e'')=w(e)}^{\circlearrowleft} e'' - \prod_{e'\neq e:b(e')=b(e)}^{\circlearrowright} e' \,.$$

$\rightsquigarrow$

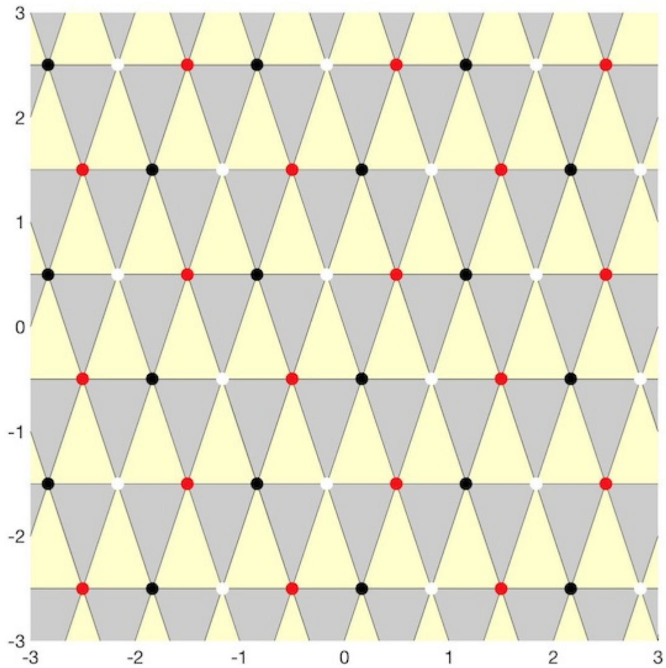
Non-commutative algebraic geometry

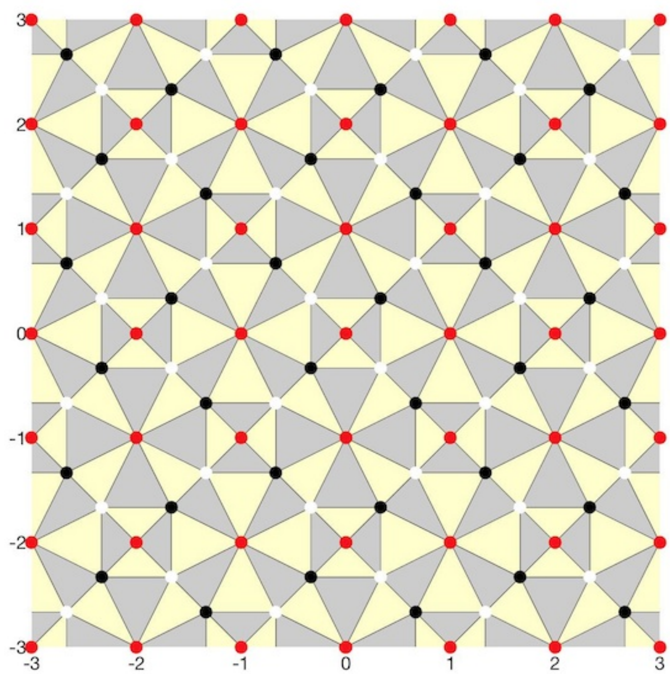
By connecting the marked points in the polygons to the vertices one obtains a tiling by quadrangles





or





The latter is a triangulation of the plane with vertices in 3 colors such that every triangle has one vertex of each color.

Modulo period lattice Λ get similar triangulation of the torus $\mathbb{R}^2/\Lambda$.

This gives a ramified covering

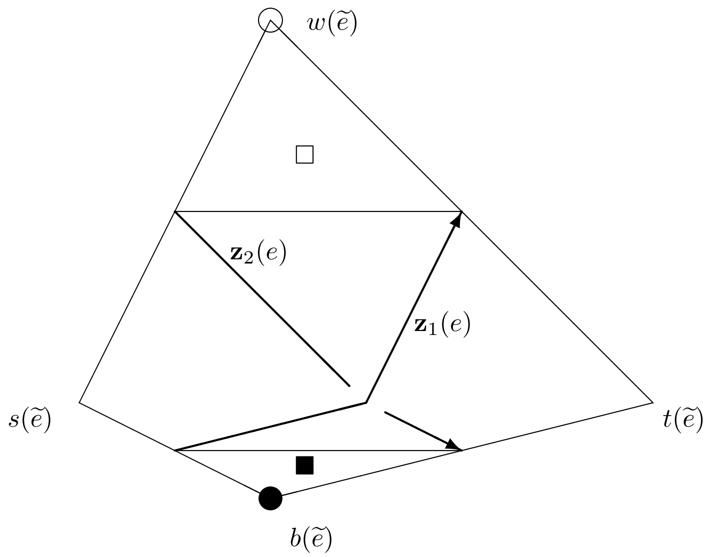
$$\mathbb{R}^2/\Lambda \longrightarrow \mathbb{P}^1 = \mathbb{S}^2$$

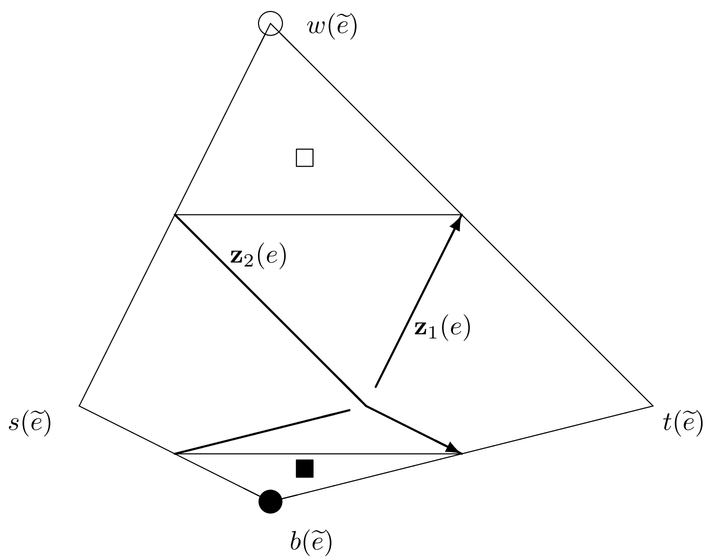
ramified only over 0, 1, ∞.

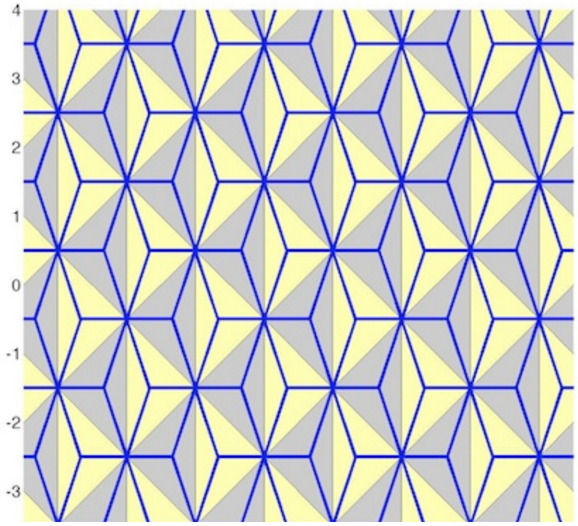
$$\rightsquigarrow$$

Dessins d'enfants, Belyi maps

Decorations on quadrangles







4
3
2
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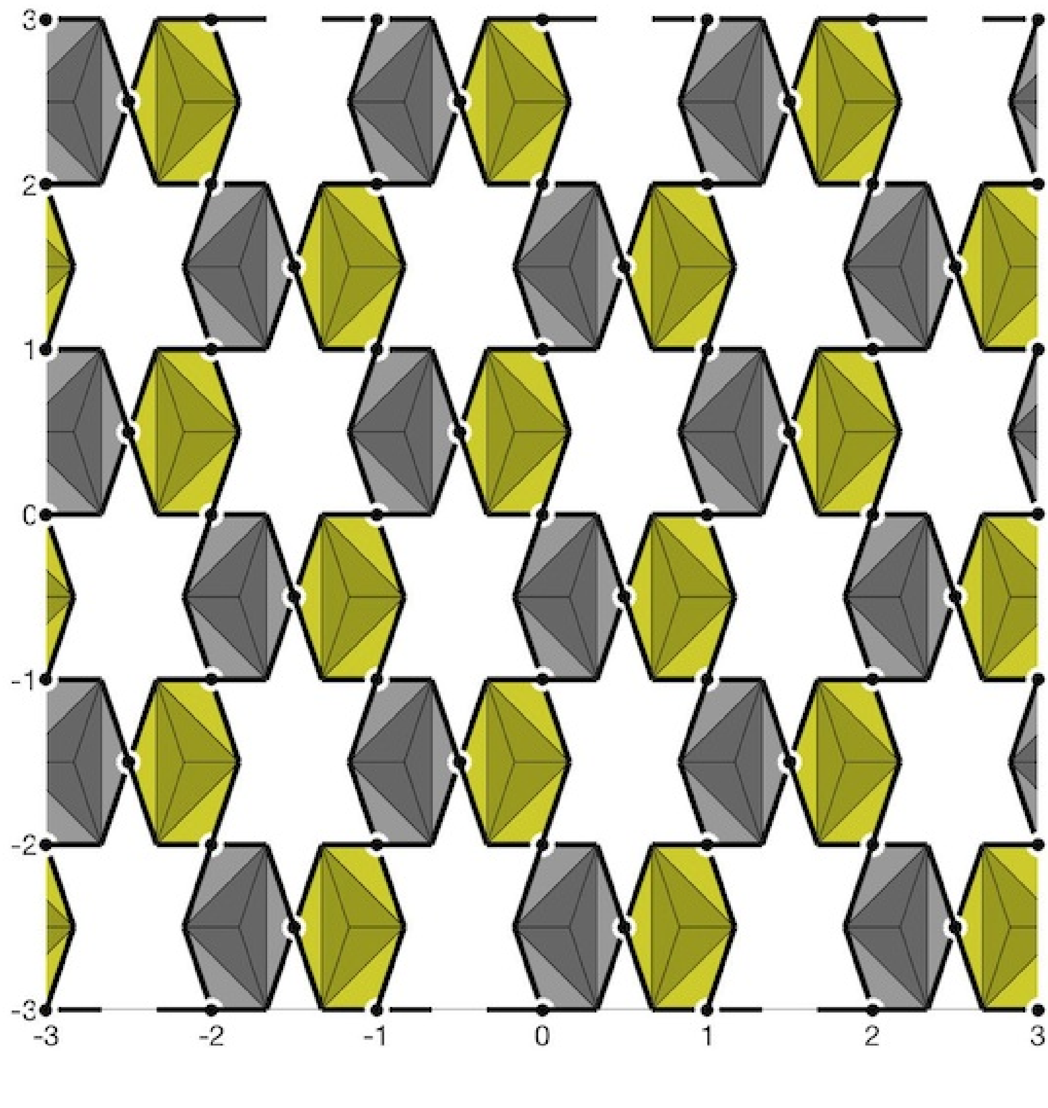
-1
-2
-3



3
2
1
0
-1
-2
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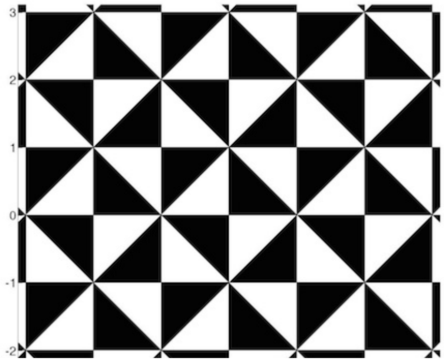
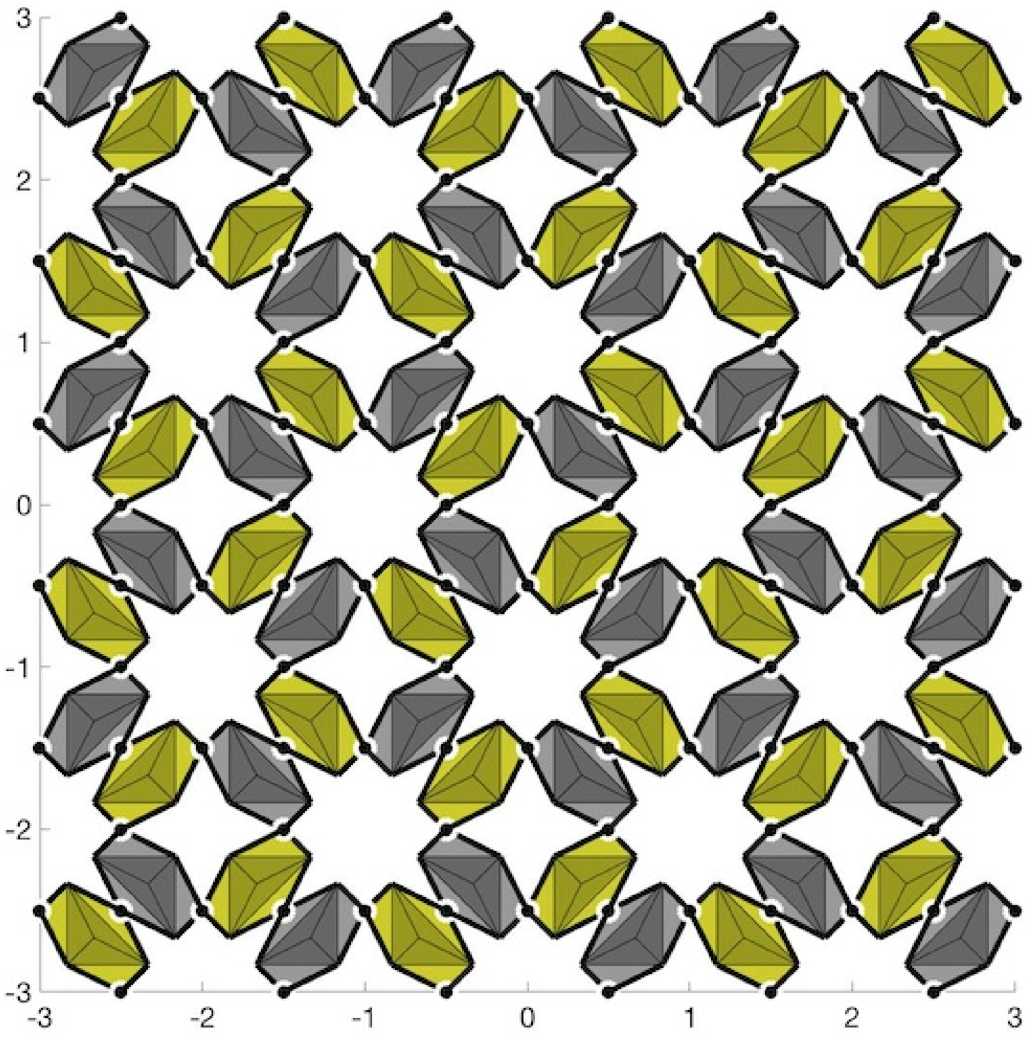
-3
-2
-1
0
1
2
3





3
2
1
0
-1
-2
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-3
-2
-1
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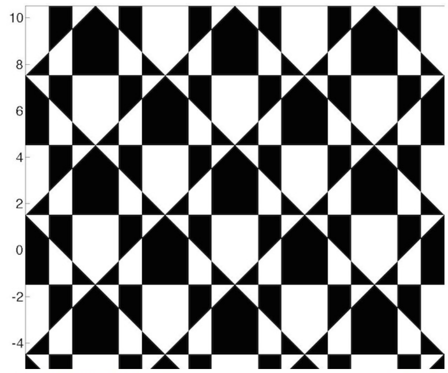
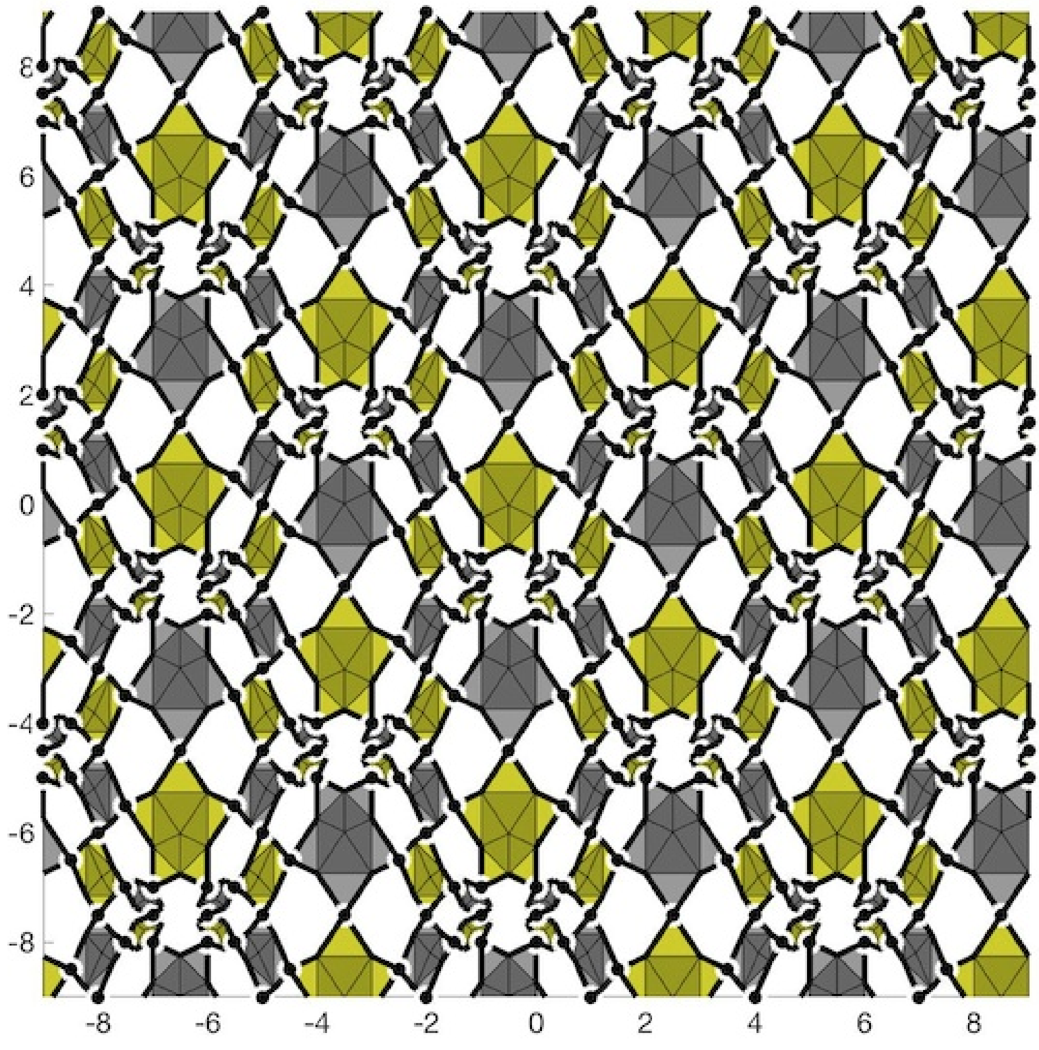
1
2
3


10
8
6
4
2
0
-2
-4

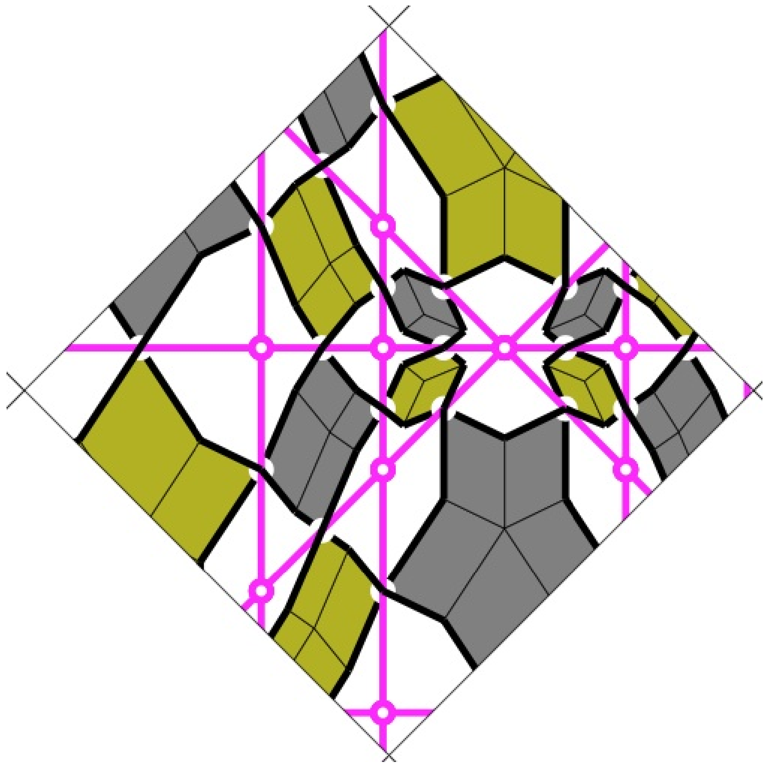

8
6
4
2
0
-2
-4
-6
-8
-8 -6 -4 -2 0 2 4 6 8

These are typically the pictures of a *Seifert surface for a link* constructed from a link projection.

Taken modulo a period lattice Λ this is a link and Seifert surface in $\mathbb{S}^3$.

$\rightsquigarrow$

Knot theory



What is the relation between the Jacobi algebra of $(\mathcal{F}, \Lambda)$ *and the fundamental group of the complement of the link in* $\mathbb{S}^3$*?*

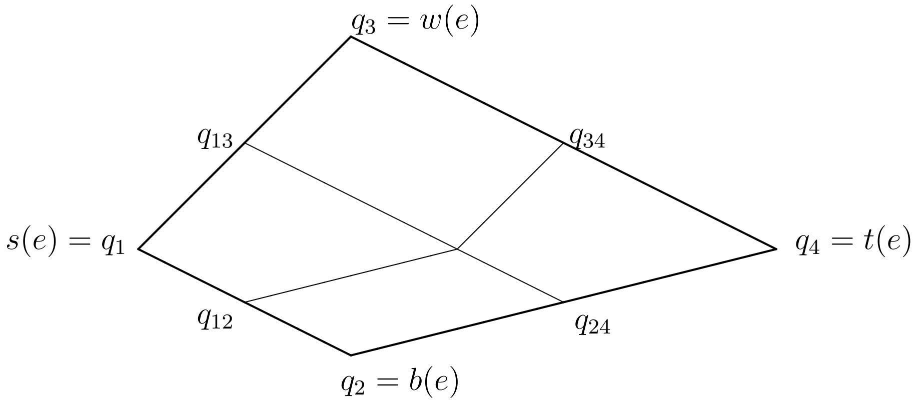
Bipartite graph $\Gamma_\Lambda^\vee$ is deformation retract of Seifert surface.

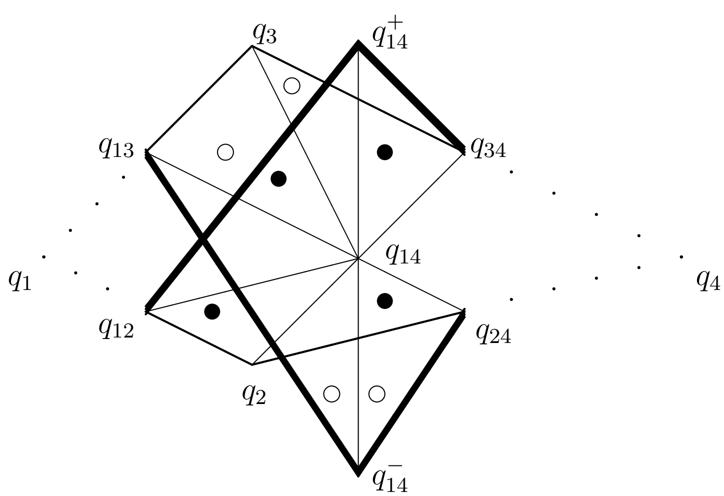
$\Rightarrow$

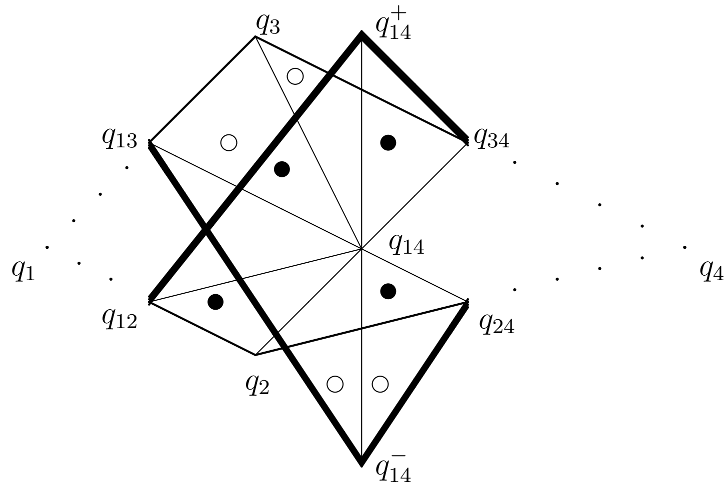
- Seifert form on $H_1(\Gamma_\Lambda^\vee, \mathbb{Z})$
- Poisson structure on torus $H^1(\Gamma_\Lambda^\vee, \mathbb{C}^*)$
- non-commutative (quantum) torus

$\rightsquigarrow$

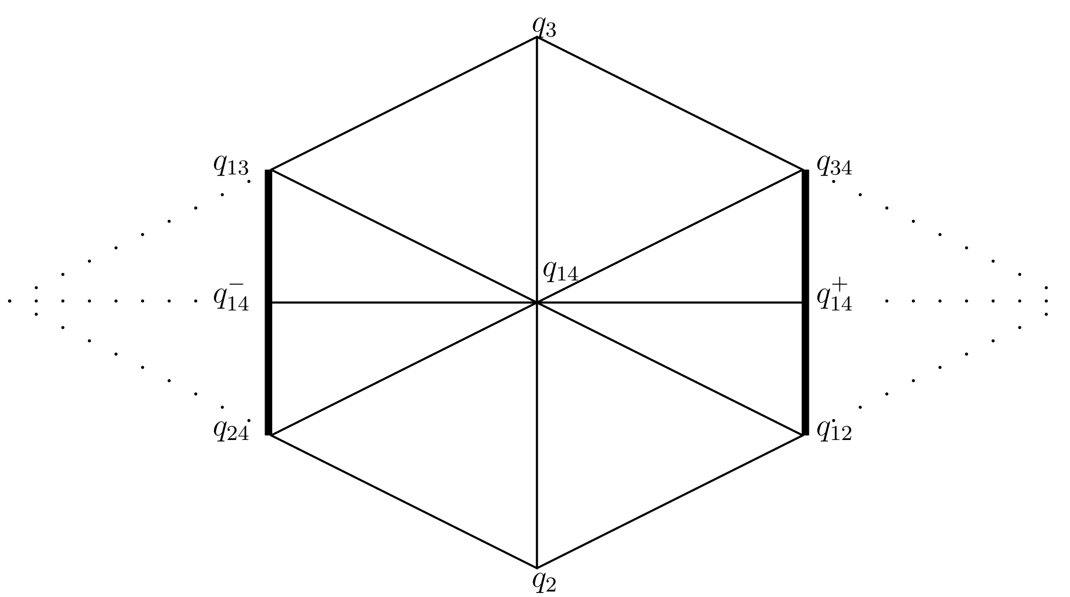
Cluster integrable systems
(work of Goncharov and Kenyon)


$q_3 = w(e)$
q_{13}
q_{34}
$s(e) = q_1$
$q_4 = t(e)$
q_{12}
q_{24}
$q_2 = b(e)$


q_3
q_{14}^+
q_{13}
q_{34}
q_{14}
q_1
q_4
q_{12}
q_{24}
q_2
q_{14}^-

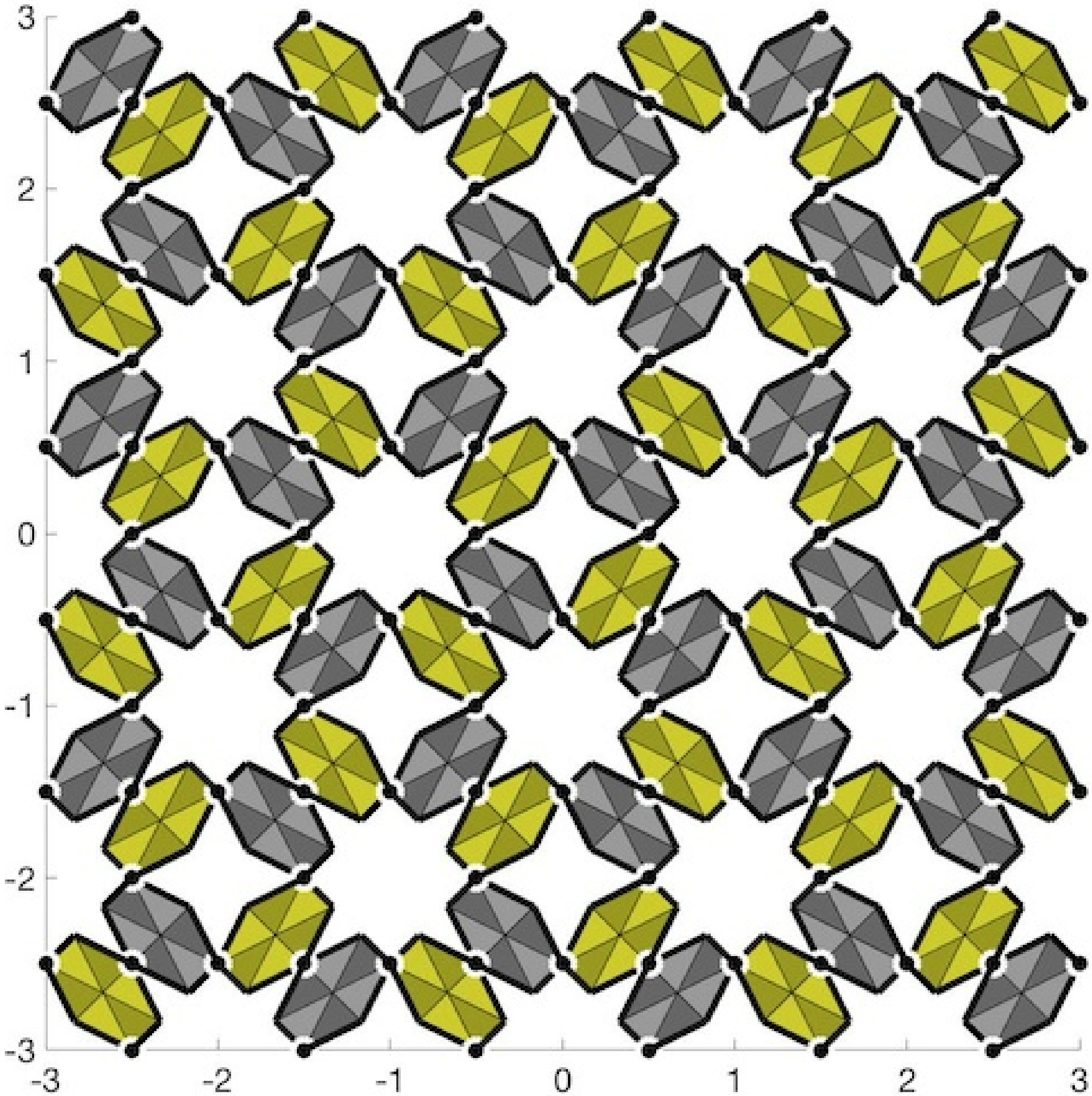

q_3
q_{14}^+
q_{13}
q_{34}
q_{14}
q_1
q_4
q_{12}
q_{24}
q_2
q_{14}^-

untwist

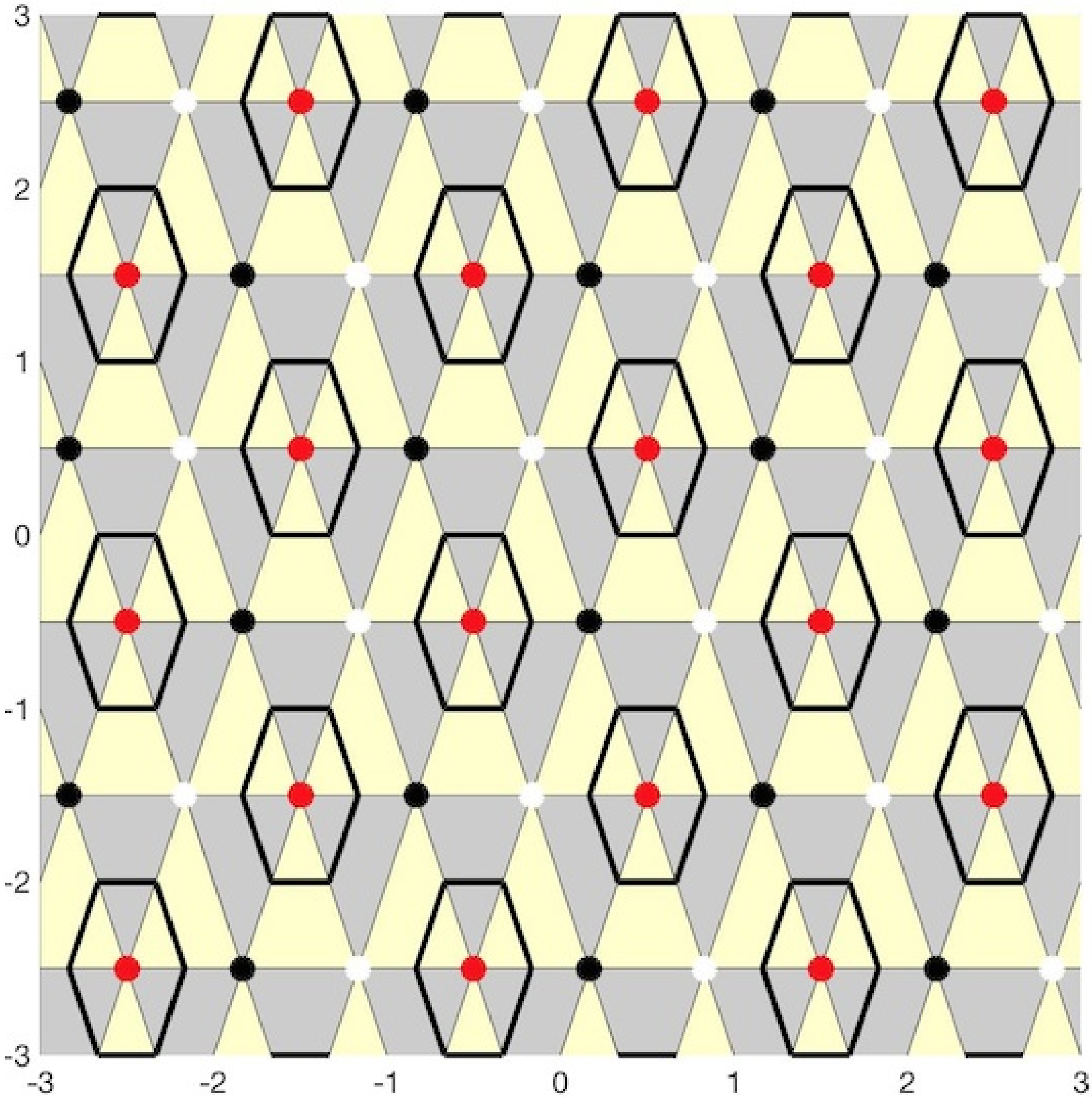

q_3
q_{13}
q_{34}
q_{14}
q_{14}^-
q_{14}^+
q_{24}
q_{12}
q_2

Seifert surface for

$$\mathcal{F}_4 = \mathsf{Z}^{21} + \mathsf{Z}^{31} + \mathsf{Z}^{41} + \mathsf{Z}^{61}$$



untwist "is"



dessin for $\mathcal{F}_3 = \mathsf{Z}^{21} + \mathsf{Z}^{41} + \mathsf{Z}^{61}$

From this we see that there is a ramified covering map from the Seifert surface to the disc

$$\{z \in \mathbb{C} \mid |z - \tfrac{1}{2}| \leq 1\}$$

which ramifies only over 0 and 1.

$$\begin{array}{ccc} \text{Seifert surface} & \hookrightarrow & \mathbb{S}^3 \\ \downarrow & & \\ \mathbb{S}^2 & & \end{array}$$

The inverse image of the boundary circle of the disc is the link.

$\rightsquigarrow$???!!!!???