The transcendental part of K3 surfaces associated with 3D Fano Polytopes

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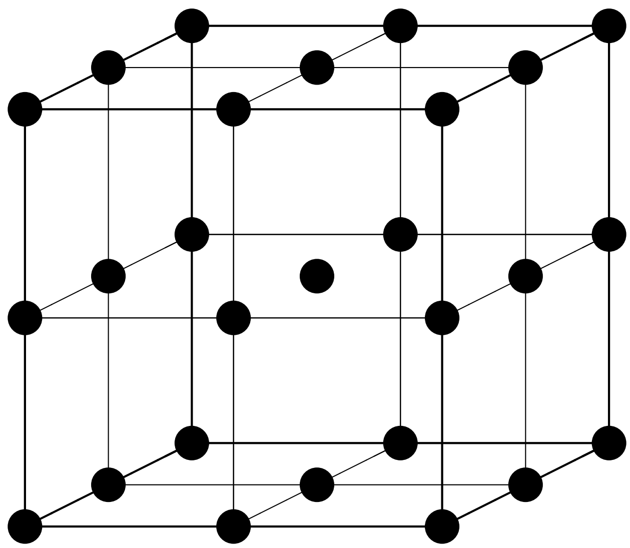

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	$\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3) = u_0 + \ \mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1,x_2,x_3)$
1	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_1^{-1}x_2^{-1}x_3^{-1}$
2	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1}x_3^{-1} + u_5x_1^{-1}$
3	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_1^{-1}x_2^{-1}x_3^{-1} + u_5x_1^{-1}$
4	$u_0 + \ u_1x_1 + u_2x_1^{-1}x_2 + u_3x_1^{-1}x_3 + u_4x_2^{-1}x_3^{-1} + u_5x_1^{-1}$
5	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1}x_3^{-1} + u_5x_1^{-1}x_2^{-1}x_3^{-1}$
6	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1} + u_5x_3^{-1} + u_6x_1^{-1}$
7	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1}x_3^{-1} + u_5x_3^{-1} + u_6x_1^{-1}x_3^{-1}$
8	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1}x_3^{-1} + u_5x_3^{-1} + u_6x_1^{-1}x_3$
9	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1}x_3^{-1} + u_5x_3^{-1} + u_6x_1^{-1}$
10	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1}x_3^{-1} + u_5x_3^{-1} + u_6x_1^{-1}x_2^{-1}x_3^{-1}$
11	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_3^{-1} + u_5x_2x_3^{-1} + u_6x_1^{-1}x_2^{-1}x_3^{-1}$
12	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_3^{-1} + u_5x_2x_3^{-1} + u_6x_1^{-1}x_2^{-1}$
13	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1} + u_5x_2^{-1}x_3^{-1} + u_6x_3^{-1} + u_7x_1^{-1}$
14	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1} + u_5x_2^{-1}x_3^{-1} + u_6x_3^{-1} + u_7x_1^{-1}x_2^{-1}$
15	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1} + u_5x_2^{-1}x_3^{-1} + u_6x_3^{-1} + u_7x_1^{-1}x_2$
16	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1} + u_5x_2^{-1}x_3^{-1} + u_6x_3^{-1} + u_7x_1^{-1}x_2^{-1}x_3^{-1}$
17	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1}x_3 + u_5x_2^{-1} + u_6x_3^{-1} + u_7x_2x_3^{-1} + u_8x_1^{-1}$
18	$u_0 + \ u_1x_1 + u_2x_2 + u_3x_3 + u_4x_2^{-1}x_3 + u_5x_2^{-1} + u_6x_3^{-1} + u_7x_2x_3^{-1} + u_8x_1^{-1}x_2x_3^{-1}$

ArXiv:2208.01465v1 **T. Matsumura, A. Nagano:**
Elliptic fibrations on toric K3 hypersurfaces and mirror symmetry derived from Fano polytopes

The Laurent polynomials $\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3)$ are characterized by their **Newton polytope** is a **3D Fano Polytope**;

i.e. a convex polytope P in $\mathbb{R}^3$ such that

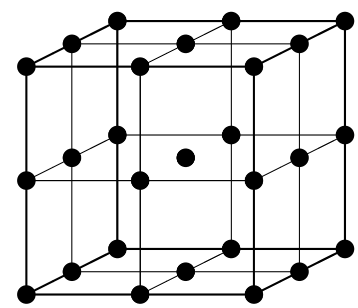
- all faces of P are triangles
- $P \cap \mathbb{Z}^3$ consists of the vertices of P and one point in its interior

Exponents of $\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3)$: $\mathsf{0}$, $\mathsf{v}_1, \ldots, \mathsf{v}_N \quad (N = 4, 5, 6, 7, 8)$
Set of vertices $\mathcal{V} = \left[\mathsf{v}_1, \ldots, \mathsf{v}_N\right]$
Interior point $\mathsf{0}$

$$\sharp\{\text{vertices}\} = N\,, \qquad \sharp\{\text{edges}\} = 3N - 6\,, \qquad \sharp\{\text{faces}\} = 2N - 4$$

Up to affine transformations in $\mathbb{Z}^3$ there are 18 3D Fano Polytopes.
These correspond to the 18 deformation classes of
smooth toric Fano Threefolds

case	polytope	Fano threefold
1	$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$	$\mathbb{P}^3$
2	$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathbb{P}^2$
6	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$
9	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathsf{dP}_1$
13	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathsf{dP}_2$
17	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 \end{bmatrix}$	$\mathbb{P}^1 \times \mathsf{dP}_3$



All Laurent polynomials in the list can in an obvious way be put in homogeneous form which completes their zero locus to

- a surface of degree 4 in $\mathbb{P}^3$
- a surface of degree $(2,3)$ in $\mathbb{P}^1 \times \mathbb{P}^2$
- a surface of degree $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Smooth surfaces with these degrees are K3 surfaces

Less obviously the zero locus of the Laurent polynomial can also be completed to a so-called **double sextic K3 surface** i.e. a double cover of $\mathbb{P}^2$ branched along a curve of degree 6

General facts about K3 surface $\mathcal{X}$:

$$\mathsf{H}^2(\mathcal{X},\mathbb{C}) = \mathsf{H}^2(\mathcal{X},\mathbb{Z})\otimes\mathbb{C}, \qquad \mathsf{H}^2(\mathcal{X},\mathbb{Z}) \simeq \mathsf{U}^{\oplus 3}\bigoplus(-\mathsf{E}_8)^{\oplus 2}$$

Hodge Filtration:

$$0 \subset \mathsf{Fil}^2_{\mathsf{Hodge}} \subset \mathsf{Fil}^1_{\mathsf{Hodge}} \subset \mathsf{Fil}^0_{\mathsf{Hodge}} = \mathsf{H}^2(\mathcal{X},\mathbb{C}),$$

$$\mathsf{Fil}^2_{\mathsf{Hodge}} = \mathsf{H}^0\big(\mathcal{X},\Omega^2_{\mathcal{X}/\mathbb{C}}\big), \qquad \mathsf{Fil}^1_{\mathsf{Hodge}} = {\mathsf{Fil}^2_{\mathsf{Hodge}}}^{\perp},$$

$$\mathsf{Fil}^1_{\mathsf{Hodge}}/\mathsf{Fil}^2_{\mathsf{Hodge}} = \mathsf{H}^1\big(\mathcal{X},\Omega^1_{\mathcal{X}/\mathbb{C}}\big), \qquad \mathsf{Fil}^0_{\mathsf{Hodge}}/\mathsf{Fil}^1_{\mathsf{Hodge}} = \mathsf{H}^2\big(\mathcal{X},\mathcal{O}_{\mathcal{X}}\big)$$

Hodge diamond: $\begin{matrix} & 1 & \\ 1 & 20 & 1 \\ & 1 & \end{matrix}$

Chow ring and the **cohomology ring** of K3 surface $\mathcal{X}$

$$\begin{aligned} \mathsf{CH}^*(\mathcal{X}) &= \mathsf{CH}^0(\mathcal{X}) \oplus \mathsf{CH}^1(\mathcal{X}) \oplus \mathsf{CH}^2(\mathcal{X}), \qquad \mathsf{CH}^1(\mathcal{X}) = \mathsf{Pic}(\mathcal{X}) \\ \mathsf{H}^*(\mathcal{X}, \mathbb{Z}) &= \mathsf{H}^0(\mathcal{X}, \mathbb{Z}) \oplus \mathsf{H}^2(\mathcal{X}, \mathbb{Z}) \oplus \mathsf{H}^4(\mathcal{X}, \mathbb{Z}) \end{aligned}$$

Cycle class map $\mathsf{CH}^*(\mathcal{X}) \longrightarrow \mathsf{H}^*(\mathcal{X}, \mathbb{Z})$ is ring homomorphism,

$$\mathsf{CH}^0(\mathcal{X}) = \mathsf{H}^0(\mathcal{X}, \mathbb{Z}), \qquad \mathsf{CH}^1(\mathcal{X}) \hookrightarrow \mathsf{H}^2(\mathcal{X}, \mathbb{Z}), \qquad \mathsf{CH}^2(\mathcal{X}) \twoheadrightarrow \mathsf{H}^4(\mathcal{X}, \mathbb{Z})$$

Theorem of Beauville and Voisin:

Points on $\mathcal{X}$ which lie on some (possibly singular) rational curve all have the same class in $\mathsf{CH}^2(\mathcal{X})$.

Corollary:

The cycle class map restricts to a ring isomorphism between

- the subring $\mathsf{H}^0(\mathcal{X}, \mathbb{Z}) \oplus \mathsf{Pic}(\mathcal{X}) \oplus \mathsf{H}^4(\mathcal{X}, \mathbb{Z})$ of $\mathsf{H}^*(\mathcal{X}, \mathbb{Z})$
- a subring $\mathsf{BV}(\mathcal{X})$ **(Beauville-Voisin ring)** of $\mathsf{CH}^*(\mathcal{X})$

$$\mathsf{H}^*(\mathcal{X},\mathbb{Z}) = \mathsf{BV}(\mathcal{X}) \oplus \mathsf{Tr}(\mathcal{X}), \qquad \mathsf{CH}^*\big(\mathcal{X}\big) = \mathsf{BV}(\mathcal{X}) \oplus \mathsf{CH}^2\big(\mathcal{X}\big)_{\deg 0}$$

where

- $\mathsf{Tr}(\mathcal{X}) = \mathsf{Pic}(\mathcal{X})^\perp$ is the **transcendental lattice**
- $\mathsf{CH}^2\big(\mathcal{X}\big)_{\deg 0}$ is the **Albanese kernel**; i.e. the group of degree 0 zero cycles on $\mathcal{X}$ modulo rational equivalence

"Manifestations" of the transcendental part of $\mathcal{X}$:

- transcendental lattice $\mathsf{Tr}(\mathcal{X})$ + Hodge structure on $\mathsf{Tr}(\mathcal{X}) \otimes \mathbb{C}$
- Albanese kernel $\mathsf{CH}^2\big(\mathcal{X}\big)_{\deg 0}$
- Brauer group $\mathsf{H}^2_{\text{ét}}\big(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}\big)$
- complement of the union of all rational curves in $\mathcal{X}$

Hodge Structure is about **period integrals**.

<u>BASIC EXAMPLE</u>

$$\left(\tfrac{1}{2\pi i}\right)^3 \int_{\gamma_0} \frac{u_0}{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}$$

with $\gamma_0 = \left\{ (x_1,x_2,x_3) \in \mathbb{C}^3 \middle| \quad |x_1| = |x_2| = |x_3| = 1 \right\}$

$|u_0| > |u_1| + \ldots + |u_N| \quad \rightsquigarrow$

$\rightsquigarrow \quad \gamma_0$ lies in $\mathbb{C}^{*3} \setminus \mathcal{Z}(\mathcal{P}_{\mathcal{V},\mathsf{u}})$

$\rightsquigarrow \quad \left| u_0^{-1} \mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1,x_2,x_3) \right| < 1 \quad$ on γ_0.

$\rightsquigarrow \quad$ convergent series expansions with $T = u_0^{-1}$:

$$\left(\tfrac{1}{2\pi i}\right)^3 \int_{|x_1|=|x_2|=|x_3|=1} \frac{u_0}{u_0 + \mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1,x_2,x_3)} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}$$

$$= \sum_{m \geq 0} (-T)^m \cdot \left(\tfrac{1}{2\pi i}\right)^3 \int_{|x_1|=|x_2|=|x_3|=1} \big(\mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1,x_2,x_3)\big)^m \frac{dx_1}{x_1} \frac{dx_2}{x_2} \frac{dx_3}{x_3}$$

$$= \sum_{m \geq 0} (-T)^m \cdot \textbf{constant term w.r.t. } x_1,x_2,x_3 \textbf{ in } \big(\mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1,x_2,x_3)\big)^m$$

$$= \sum_{m \geq 0} (-1)^m T^m \cdot \sum_{\substack{\ell_1,\ldots,\ell_N \geq 0 \\ \ell_1 + \ldots + \ell_N = m \\ \ell_1 \mathsf{v}_1 + \ldots + \ell_N \mathsf{v}_N = \mathsf{0}}} \frac{m!}{\ell_1! \cdot \ldots \cdot \ell_N!} u_1^{\ell_1} \cdot \ldots \cdot u_N^{\ell_N}$$

Now first look at

$$\tfrac{1}{2\pi i}\int_{|x_1|=1}\frac{u_0}{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)}\,\frac{dx_1}{x_1}$$

for fixed values of x_2 and x_3 such that $|x_2|=|x_3|=1$.

The integrand has only poles at the roots ξ_a and ξ_b of the quadratic polynomial $x_1\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)$.

$|u_0|>|u_1|+\ldots+|u_N|$ and $|x_2|=|x_3|=1$ imply $|\xi_a|+|\xi_b|>1+|\xi_a||\xi_b|$ and, hence, say, $|\xi_a|<1<|\xi_b|$

The residue theorem now gives

$$\tfrac{1}{2\pi i}\int_{|x_1|=1}\frac{u_0}{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)}\,\frac{dx_1}{x_1}=\frac{u_0/u_1}{\xi_a-\xi_b}.$$

So:

$$\left(\tfrac{1}{2\pi i}\right)^3\int_{|x_1|=|x_2|=|x_3|=1}\frac{u_0}{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)}\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\,\frac{dx_3}{x_3}\;=\;\left(\tfrac{1}{2\pi i}\right)^2\int_{\alpha_0}\omega_{\mathcal{V},\mathsf{u}}$$

with

$$\begin{aligned}\omega_{\mathcal{V},\mathsf{u}}\;&=\;\frac{u_0}{d\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)}\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\,\frac{dx_3}{x_3}\\ \alpha_0\;&=\;\left\{(x_1,x_2,x_3)\in\mathbb{C}^3\;\middle|\;\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)=0,\;|x_1|<|x_2|=|x_3|=1\right\}.\end{aligned}$$

In all cases (with the exception of cases 3 and 4)

$$x_1\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3) = u_1x_1^2 + \big(u_0 + \sum_{j=2}^{N-1} u_j x_2^{v_{2,j}} x_3^{v_{3,j}}\big)x_1 + u_N x_2^{v_{2,N}} x_3^{v_{3,N}}$$

and, hence,

$$\xi_a - \xi_b = \frac{1}{u_1}\sqrt{\big(u_0 + \sum_{j=2}^{N-1} u_j x_2^{v_{2,j}} x_3^{v_{3,j}}\big)^2 - 4u_1u_N x_2^{v_{2,N}} x_3^{v_{3,N}}}.$$

With $T = u_0^{-1}$, as before, the period integral now becomes

$$\left(\tfrac{1}{2\pi i}\right)^2 \int_{\alpha_0} \frac{dx_2\, dx_3}{\sqrt{\big(x_2x_3 + T\sum_{j=2}^{N-1} u_j x_2^{1+v_{2,j}} x_3^{1+v_{3,j}}\big)^2 - 4u_1u_N x_2^{2+v_{2,N}} x_3^{2+v_{3,N}} T^2}}.$$

For fixed $T, u_1, \ldots, u_N$ the expression under the square root is a polynomial in x_2 and x_3 of degree 6.

So, it may be useful to complete the surface $\mathcal{Z}(\mathcal{P}_{\mathcal{V},\mathsf{u}})$ in $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ defined by $\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3) = 0$ to a **double sextic K3 surface**; i.e. double cover of $\mathbb{P}^2$ branched along a curve of degree 6 .

The condition $|u_0| > |u_1| + \ldots + |u_N|$ implies that the topological 2-cycle $|x_2| = |x_3| = 1$ does not intersect the branch locus and has two connected components. The the domain of integration α_0 for the period integral is one of these components.

$$
\left(\tfrac{1}{2\pi i}\right)^2 \int_{\alpha_0} \frac{dx_2\, dx_3}{\sqrt{\left(x_2x_3 + T\sum_{j=2}^{N-1} u_j x_2^{1+v_{2,j}} x_3^{1+v_{3,j}}\right)^2 - 4u_1u_N x_2^{2+v_{2,N}} x_3^{2+v_{3,N}} T^2}}
$$

$$
= \sum_{m\geq 0} (-1)^m C_m T^m
$$

where

$$
\begin{aligned}
C_m &= \textbf{constant term w.r.t. } x_1, x_2, x_3 \textbf{ in } \left(\mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1,x_2,x_3)\right)^m \\
&= \sum_{\substack{\ell_1,\ldots,\ell_N \geq 0 \\ \ell_1+\ldots+\ell_N = m \\ \ell_1\mathsf{v}_1+\ldots+\ell_N\mathsf{v}_N = \mathsf{0}}} \frac{m!}{\ell_1!\cdot\ldots\cdot\ell_N!}\, u_1^{\ell_1}\cdot\ldots\cdot u_N^{\ell_N}
\end{aligned}
$$

Integrating w.r.t. T gives for $\tau \in \mathbb{C}$ such that $|\tau|^{-1} > |u_1| + \ldots + |u_N|$:

$$
\left(\tfrac{1}{2\pi i}\right)^2 \int_0^\tau \int_{\alpha_0} \frac{dx_2\, dx_3\, dT}{\sqrt{\left(x_2x_3 + T\sum_{j=2}^{N-1} u_j x_2^{1+v_{2,j}} x_3^{1+v_{3,j}}\right)^2 - 4u_1u_N x_2^{2+v_{2,N}} x_3^{2+v_{3,N}} T^2}}
$$

$$
= \sum_{m\geq 1} (-1)^{m-1} C_{m-1} \frac{\tau^m}{m}
$$

The series

$$\mathfrak{L}(\tau) = \sum_{m\geq 1} (-1)^{m-1} C_{m-1} \frac{\tau^m}{m}$$

is the **logarithm for a 1-dimensional formal group law** $\mathfrak{G}(\tau_1, \tau_2)$ over the ring $\mathbb{Z}[u_1, \ldots, u_N]$. This means

$$\mathfrak{G}(\tau_1, \tau_2) \;=\; \mathfrak{L}^{-1}\big(\mathfrak{L}(\tau_1) + \mathfrak{L}(\tau_2)\big) \;\in\; \mathbb{Z}[u_1, \ldots, u_N][[\tau_1, \tau_2]]$$

After specializing $u_1, \ldots, u_N$ to values in some field K
$\mathfrak{G}(\tau_1, \tau_2)$ becomes the **formal Brauer group** of the K3 surface over K defined by

$$\mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1, x_2, x_3) = 0.$$

Jan Stienstra, Formal Group Laws arising from Algebraic Varieties, Amer.J.Math. 109 (1987) 907-925

Fano threefold

$$\mathsf{Fano}(\mathcal{V}) \xrightarrow{\simeq} \mathbb{C}^N_{\mathcal{F}}/\mathbb{L}\otimes\mathbb{C}^*$$

$\mathcal{F}$ denotes the set of faces of $\mathsf{conv}(\mathcal{V})$

$\mathbb{C}^N_{\mathcal{F}} = \bigcup_{[h,i,j]\in\mathcal{F}} \mathbb{C}^N_{[h,i,j]}$

$\mathbb{C}^N_{[h,i,j]} = \left\{(z_1,\ldots,z_N)\in\mathbb{C}^N \;\middle|\; z_m \neq 0 \text{ if } m\neq h,i,j \right\}$

$\mathbb{L} = \left\{(\ell_1,\ldots,\ell_N)\in\mathbb{Z}^N \;\middle|\; \ell_1\mathsf{v}_1+\ldots+\ell_N\mathsf{v}_N = \mathsf{0}\right\}, \qquad \mathsf{rank}\,\mathbb{L} = N-3$

$\mathbb{L}\otimes\mathbb{C}^* \quad \hookrightarrow \quad {\mathbb{C}^*}^N$

${\mathbb{C}^*}^N$ acts on $\mathbb{C}^N$ and $\mathbb{C}^N_{\mathcal{F}}$ by componentwise multiplication

$\mathbb{C}^3 \xrightarrow{\simeq} \mathbb{C}^N_{[h,i,j]}/\mathbb{L}\otimes\mathbb{C}^* \qquad (y_1,y_2,y_3)\mapsto(z_1,\ldots,z_N)$

with $z_h = y_1,\ z_i = y_2,\ z_j = y_3$ and $z_m = 1$ for $m\neq h,i,j$

The coordinates $(z_1, \ldots, z_N)$ on $\mathbb{C}^N$ are **homogeneous coordinates** on $\mathsf{Fano}(\mathcal{V})$.

- For a vertex v_i of the Fano polytope $\mathsf{conv}(\mathcal{V})$

  $z_i = 0$ defines a divisor D_i on $\mathsf{Fano}(\mathcal{V})$.

- For an edge $\mathsf{conv}(\mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V})$

  $z_i = z_j = 0$ defines the intersection $\mathsf{D}_i \cap \mathsf{D}_j$

- For a face $\mathsf{conv}(\mathsf{v}_h, \mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V})$

  $z_h = z_i = z_j = 0$ defines the intersection $\mathsf{D}_h \cap \mathsf{D}_i \cap \mathsf{D}_j$

The Chow ring $\mathsf{CH}^*\big(\mathsf{Fano}(\mathcal{V})\big) \;=\; \bigoplus_{m=0}^{3} \mathsf{CH}^m\big(\mathsf{Fano}(\mathcal{V})\big)$

and cohomology ring $\mathsf{H}^*\big(\mathsf{Fano}(\mathcal{V}), \mathbb{Z}\big) \;=\; \bigoplus_{m=0}^{3} \mathsf{H}^{2m}\big(\mathsf{Fano}(\mathcal{V}), \mathbb{Z}\big)$

are isomorphic as graded rings with

$$\mathcal{R}^*(\mathcal{V}) \;=\; \mathbb{Z}\big[\mathsf{D}_1, \ldots, \mathsf{D}_N\big] / \big(\mathcal{I} + \mathcal{J}\big)$$

- ideal $\mathcal{I}$ generated by products $\mathsf{D}_a \cdot \mathsf{D}_b$ and $\mathsf{D}_c \cdot \mathsf{D}_d \cdot \mathsf{D}_e$ with $\{a, b\}$ and $\{c, d, e\}$ not contained in any $[h, i, j] \in \mathcal{F}$
- ideal $\mathcal{J}$ generated by the three linear forms in the system

  $\mathsf{v}_1\mathsf{D}_1 + \ldots + \mathsf{v}_N\mathsf{D}_N$

$\mathsf{rank}\mathcal{R}^0(\mathcal{V}) = \mathsf{rank}\mathcal{R}^3(\mathcal{V}) = 1, \qquad \mathsf{rank}\mathcal{R}^1(\mathcal{V}) = \mathsf{rank}\mathcal{R}^2(\mathcal{V}) = N - 3,$

$\mathsf{rank}\mathcal{R}^*(\mathcal{V}) \;=\; 2N - 4$

Let $\overline{\mathsf{D}}_1, \ldots, \overline{\mathsf{D}}_N \in \mathcal{R}^1(\mathcal{V})$ denote the residue classes $\mathsf{D}_1, \ldots, \mathsf{D}_N \bmod \mathcal{I} + \mathcal{J}$. Then

- $1 \in \mathcal{R}^0(\mathcal{V}) \quad \Leftrightarrow$ equivalence class of $\mathsf{Fano}(\mathcal{V})$ in $\mathsf{CH}^0\big(\mathsf{Fano}(\mathcal{V})\big)$.
- $\overline{\mathsf{D}}_i \in \mathcal{R}^1(\mathcal{V}) \quad \Leftrightarrow$
  equivalence class of the divisor D_i in $\mathsf{CH}^1\big(\mathsf{Fano}(\mathcal{V})\big) = \mathsf{Pic}\big(\mathsf{Fano}(\mathcal{V})\big)$
- $\overline{\mathsf{D}}_i \cdot \overline{\mathsf{D}}_j \in \mathcal{R}^2(\mathcal{V})$ for an edge $\mathsf{conv}(\mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V}) \quad \Leftrightarrow$
  equivalence class of the curve $\mathsf{D}_i \cap \mathsf{D}_j$ in $\mathsf{CH}^2\big(\mathsf{Fano}(\mathcal{V})\big)$
- $\overline{\mathsf{D}}_h \cdot \overline{\mathsf{D}}_i \cdot \overline{\mathsf{D}}_j \in \mathcal{R}^3(\mathcal{V})$ for a face $\mathsf{conv}(\mathsf{v}_h, \mathsf{v}_i, \mathsf{v}_j)$ of $\mathsf{conv}(\mathcal{V}) \quad \Leftrightarrow$
  equivalence class of the point $\mathsf{D}_h \cap \mathsf{D}_i \cap \mathsf{D}_j$ in $\mathsf{CH}^3\big(\mathsf{Fano}(\mathcal{V})\big)$
- All points $\mathsf{D}_h \cap \mathsf{D}_i \cap \mathsf{D}_j$ for $[h, i, j] \in \mathcal{F}$ are rationally equivalent,

i.e. $\overline{\mathsf{D}}_h \cdot \overline{\mathsf{D}}_i \cdot \overline{\mathsf{D}}_j = \overline{\mathsf{D}}_1 \cdot \overline{\mathsf{D}}_2 \cdot \overline{\mathsf{D}}_3$ in $\mathsf{CH}^3\big(\mathsf{Fano}(\mathcal{V})\big)$

The intersection form $\langle , \rangle$ on $\mathsf{Pic}\big(\mathsf{Fano}(\mathcal{V})\big) = \mathcal{R}^1(\mathcal{V})$ is given by:

for c, $\mathsf{c}' \in \mathcal{R}^1(\mathcal{V})$

$$\mathsf{c} \cdot \mathsf{c}' \cdot \overline{\mathsf{D}}_0 \ = \ \big\langle \mathsf{c}, \mathsf{c}' \big\rangle \cdot \overline{\mathsf{D}}_1 \cdot \overline{\mathsf{D}}_2 \cdot \overline{\mathsf{D}}_3.$$

where $\overline{\mathsf{D}}_0 = \overline{\mathsf{D}}_1 + \ldots + \overline{\mathsf{D}}_N$ is the *anti-canonical class.*

case	$\left(\langle \overline{\mathsf{D}}_i, \overline{\mathsf{D}}_j \rangle\right)_{1\le i,j\le N}$
1 $\mathbb{P}^3$	$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \end{bmatrix}$
2 $\mathbb{P}^1 \times \mathbb{P}^2$	$\begin{bmatrix} 0 & 3 & 3 & 3 & 0 \\ 3 & 2 & 2 & 2 & 3 \\ 3 & 2 & 2 & 2 & 3 \\ 3 & 2 & 2 & 2 & 3 \\ 0 & 3 & 3 & 3 & 0 \end{bmatrix}$
6 $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$\begin{bmatrix} 0 & 2 & 2 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 2 & 2 & 0 \end{bmatrix}$

case	$\left(\langle \overline{\mathsf{D}}_i, \overline{\mathsf{D}}_j \rangle\right)_{1\le i,j\le N}$
9 $\mathbb{P}^1 \times \mathsf{dP}_1$	$\begin{bmatrix} 0 & 2 & 3 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 & 2 & 2 \\ 3 & 2 & 2 & 2 & 0 & 3 \\ 2 & 0 & 2 & 0 & 2 & 2 \\ 1 & 2 & 0 & 2 & -2 & 1 \\ 0 & 2 & 3 & 2 & 1 & 0 \end{bmatrix}$
13 $\mathbb{P}^1 \times \mathsf{dP}_2$	$\begin{bmatrix} 0 & 2 & 2 & 1 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 & 0 & 2 & 2 \\ 2 & 2 & 0 & 2 & 0 & 0 & 2 \\ 1 & 0 & 2 & -2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 2 & -2 & 2 & 1 \\ 1 & 2 & 0 & 0 & 2 & -2 & 1 \\ 0 & 2 & 2 & 1 & 1 & 1 & 0 \end{bmatrix}$
17 $\mathbb{P}^1 \times \mathsf{dP}_3$	$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 0 & 2 & 1 \\ 1 & 2 & -2 & 2 & 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & -2 & 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 2 & -2 & 2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 2 & -2 & 2 & 1 \\ 1 & 2 & 0 & 0 & 0 & 2 & -2 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$

An *anti-canonical* **K3 surface** in $\mathsf{Fano}(\mathcal{V})$ is the zero set of a global section of the anti-canonical bundle on $\mathsf{Fano}(\mathcal{V})$.

examples

anti-canonical K3 in $\mathbb{P}^3 \Leftrightarrow$ surface of degree 4

anti-canonical K3 in $\mathbb{P}^1 \times \mathbb{P}^2 \Leftrightarrow$ surface of degree $(2, 3)$

anti-canonical K3 in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \Leftrightarrow$ surface of degree $(2, 2, 2)$

Every anti-canonical K3 surface $\mathcal{Y}$ carries a collection of curves cut out by the divisors $\mathsf{D}_1, \ldots, \mathsf{D}_N$ on $\mathsf{Fano}(\mathcal{V})$.

The classes $\delta_1, \ldots, \delta_N$ of these curves generate a sublattice in $\mathsf{Pic}(\mathcal{Y})$ with intersection form given by $\left\langle \delta_i, \delta_j \right\rangle = \left\langle \overline{\mathsf{D}}_i, \overline{\mathsf{D}}_j \right\rangle$

For a general anti-canonical K3 surface $\mathcal{Y}$ this sublattice is all of $\mathsf{Pic}(\mathcal{Y})$ and the **Beauville-Voisin ring** $\mathsf{BV}(\mathcal{Y})$ is isomorphic with the graded ring

$$\overline{\mathcal{R}}^*(\mathcal{V}) \;=\; \overline{\mathcal{R}}^0(\mathcal{V}) \oplus \overline{\mathcal{R}}^1(\mathcal{V}) \oplus \overline{\mathcal{R}}^2(\mathcal{V})$$

with

$\overline{\mathcal{R}}^0(\mathcal{V}) = \mathbb{Z}\mathsf{1}, \qquad \overline{\mathcal{R}}^2(\mathcal{V}) = \mathbb{Z}\delta_\infty,$

$\overline{\mathcal{R}}^1(\mathcal{V}) = \mathbb{Z}\delta_1 \oplus \ldots \oplus \mathbb{Z}\delta_N$ modulo the relations $\mathsf{v}_1\delta_1 + \ldots + \mathsf{v}_N\delta_N = \mathsf{0}$.

The multiplication is given by:

$$\mathsf{1}\cdot\mathsf{1} = \mathsf{1}\,, \quad \mathsf{1}\cdot\delta_j = \delta_j\,, \quad \mathsf{1}\cdot\delta_\infty = \delta_\infty\,, \quad \delta_\infty\cdot\delta_j = \delta_\infty\cdot\delta_\infty = 0,$$
$$\delta_i\cdot\delta_j = \left\langle \delta_i, \delta_j \right\rangle \delta_\infty$$

for $i, j = 1, \ldots, N$.

The main result of **Matsumura-Nagano** (arXiv:2208.01465) can be stated as follows:

Let $\mathcal{X}_{\mathcal{V},\mathsf{u}}$ be a K3 surface which contains the surface $\mathcal{Z}(\mathcal{P}_{\mathcal{V},\mathsf{u}})$ in $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ as a dense open subset.

Then for general coefficients $\mathsf{u} = (u_0, u_1, \dots, u_N)$ there is an isomorphism of lattices

$$\mathsf{Tr}(\mathcal{X}_{\mathcal{V},\mathsf{u}}) \;\simeq\; \mathsf{BV}(\mathcal{Y})$$

between the **transcendental lattice of** $\mathcal{X}_{\mathcal{V},\mathsf{u}}$ and the **Beauville-Voisin ring** $\mathsf{BV}(\mathcal{Y})$ of a general anti-canonical K3 surface $\mathcal{Y}$ in $\mathsf{Fano}(\mathcal{V})$.

The bilinear form $\langle , \rangle$ on $\mathsf{BV}(\mathcal{Y}) = \overline{\mathcal{R}}^*(\mathcal{V})$ is defined by

$$(\xi \cdot \eta)^{\deg 2} \;=\; \langle \xi, \eta \rangle \, \delta_\infty$$

The two families

anti-canonical K3 *surfaces* $\mathcal{Y}$ *in* $\mathsf{Fano}(\mathcal{V})$

K3 *surfaces* $\mathcal{X}_{\mathcal{V},\mathsf{u}}$ *of degree* 4 *in* $\mathbb{P}^3$ *defined by polynomials with Newton polytope* $\mathsf{conv}(\mathcal{V})$

form a

- **Mirror Pair** in the sense of **Batyrev**

- **Mirror Pair of lattice polarized K3 surfaces** in the sense of **Dolgachev**

- The map $\mathcal{P}_{\mathcal{V},\mathsf{u}^\star} : \mathbb{C}^{*3} \to \mathbb{C}$ defined by the polynomial $\mathcal{P}_{\mathcal{V},\mathsf{u}^\star}(x_1, x_2, x_3)$ is a **Landau-Ginzburg Mirror of the Fano threefold** $\mathsf{Fano}(\mathcal{V})$ (possibly under restrictive conditions on the coefficients $u_1, \ldots, u_N$)

The variation of Hodge structure on $\mathsf{H}^2(\mathcal{X}_{\mathcal{V},\mathsf{u}},\mathbb{C})$ describes the position of the cohomology class $[\omega_{\mathcal{V},\mathsf{u}}]$ of the differential 2-form

$$\omega_{\mathcal{V},\mathsf{u}} = \frac{u_0}{d\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)}\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\,\frac{dx_3}{x_3}$$

w.r.t. the lattice $\mathsf{H}^2(\mathcal{X}_{\mathcal{V},\mathsf{u}},\mathbb{Z})$ in $\mathsf{H}^2(\mathcal{X}_{\mathcal{V},\mathsf{u}},\mathbb{C})$.

Since $[\omega_{\mathcal{V},\mathsf{u}}] \perp \mathsf{Pic}(\mathcal{X}_{\mathcal{V},\mathsf{u}})$ we actually have $[\omega_{\mathcal{V},\mathsf{u}}] \in \mathsf{Tr}(\mathcal{X}_{\mathcal{V},\mathsf{u}})\otimes\mathbb{C}$

For <u>general</u> $\mathsf{u}=(u_0,u_1,\dots,u_N)$:

$$\mathsf{Tr}(\mathcal{X}_{\mathcal{V},\mathsf{u}})\otimes\mathbb{C} \simeq \overline{\mathcal{R}}^*(\mathcal{V})\otimes\mathbb{C}.$$

<u>THEOREM</u>:

The expression

$$\Phi^\flat(u_0,\dots,u_N) = \sum_{\ell\in\mathbb{L}}(-1)^{\ell_0}\prod_{k=1}^{-\ell_0}(\delta_0+k)\cdot\prod_{j=1}^{N}\frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j+1+\delta_j)}\,u_j^{\ell_j+\delta_j}\cdot u_0^{\ell_0-\delta_0}$$

gives the position of $[\omega_{\mathcal{V},\mathsf{u}}]$ in $\overline{\mathcal{R}}^*(\mathcal{V})\otimes\mathbb{C}$

as a function of the <u>variables</u> $u_0,u_1,\dots,u_N$

$$\Phi^\flat(u_0,\ldots,u_N) = \sum_{\ell\in\mathbb{L}} (-1)^{\ell_0} \prod_{k=1}^{-\ell_0} (\delta_0+k) \cdot \prod_{j=1}^{N} \frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j+1+\delta_j)}\, u_j^{\ell_j+\delta_j} \cdot u_0^{\ell_0-\delta_0}$$

LEGEND:

- for $\ell \in \mathbb{L}$: $\ell = (\ell_1,\ldots,\ell_N)\,, \qquad \ell_0 = -(\ell_1+\ldots+\ell_N)$

- $\delta_1,\ldots,\delta_N,\delta_\infty \in \overline{\mathcal{R}}^*(\mathcal{V})\,, \qquad \delta_0 = \delta_1+\ldots+\delta_N$

- $u_j^{\ell_j+\delta_j} = u_j^{\ell_j}\,\big(1+\delta_j\,\log(u_j) + \frac{1}{2}\big\langle\delta_j,\delta_j\big\rangle\,\delta_\infty\,(\log(u_j))^2\,\big)$

- Pochammer symbol:

  $(s)_0 = 1, \qquad (s)_n = s\cdot(s+1)\cdot\ldots\cdot(s+n-1) \quad \text{if } n>0$

  $$\frac{\Gamma(1+s)}{\Gamma(n+1+s)} = \begin{cases} \dfrac{1}{(1+s)_n} & \text{if } \; n\in\mathbb{Z}_{\geq 0}, \\ (-1)^n(-s)_{-n} & \text{if } \; n\in\mathbb{Z}_{\leq 0}. \end{cases}$$

  RHS also makes sense if s is a nilpotent element in a $\mathbb{Q}$ algebra.

- If one works with complex variables $u_0,\ldots,u_N$ the logarithms lead to multi-valuedness and monodromy.

  Alternatively, one may restrict to positive real values of $u_0,\ldots,u_N$

- $\Phi^\flat(u_0,\ldots,u_N)$ equals $u_0^{-\delta_0}\prod_{j=1}^N u_j^{\delta_j}$ times a power series with non-zero terms only for ℓ in some pointed convex cone in $\mathbb{L}$. This series converges for $|u_0|$ sufficiently much larger than $|u_1|,\ldots,|u_N|$

$$\Phi^\flat(u_0,\ldots,u_N) = \sum_{\ell\in\mathbb{L}}(-1)^{\ell_0}\prod_{k=1}^{-\ell_0}(\delta_0+k)\cdot\prod_{j=1}^{N}\frac{\Gamma(1+\delta_j)}{\Gamma(\ell_j+1+\delta_j)}\,u_j^{\ell_j+\delta_j}\cdot u_0^{\ell_0-\delta_0}$$

Hence:

$$\begin{aligned}\Big\langle\delta_\infty,\Phi^\flat(u_0,\ldots,u_N)\Big\rangle &= \Phi^\flat(u_0,\ldots,u_N)^{\deg 0}\\ &= \sum_{\ell\in\mathbb{L}}(-1)^{\ell_0}\frac{(-\ell_0)!}{\ell_1!\cdot\ldots\cdot\ell_N!}u_1^{\ell_1}\cdot\ldots\cdot u_N^{\ell_N}\cdot u_0^{\ell_0}\\ &= \sum_{m\geq 0}(-1)^m T^m\cdot\sum_{\substack{\ell_1,\ldots,\ell_N\geq 0\\ \ell_1+\ldots+\ell_N=m\\ \ell_1\mathsf{v}_1+\ldots+\ell_N\mathsf{v}_N=0}}\frac{m!}{\ell_1!\cdot\ldots\cdot\ell_N!}\,u_1^{\ell_1}\cdot\ldots\cdot u_N^{\ell_N}\end{aligned}$$

So, in a clear sense, the series expansion of $\Phi^\flat(u_0,\ldots,u_N)$ is a deformation of the series expansion for the integrals

$$\left(\tfrac{1}{2\pi i}\right)^3\int_{\gamma_0}\frac{u_0}{\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1,x_2,x_3)}\,\frac{dx_1}{x_1}\,\frac{dx_2}{x_2}\,\frac{dx_3}{x_3}$$

and

$$\left(\tfrac{1}{2\pi i}\right)^2\int_{\alpha_0}\frac{dx_2\,dx_3}{\sqrt{\left(x_2x_3+T\sum_{j=2}^{N-1}u_jx_2^{1+v_{2,j}}x_3^{1+v_{3,j}}\right)^2-4u_1u_Nx_2^{2+v_{2,N}}x_3^{2+v_{3,N}}T^2}}.$$

Case 1: $\mathcal{V} = \big[\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_4\big] = \left[\begin{smallmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{smallmatrix}\right]$, $\quad \mathsf{Fano}(\mathcal{V}) = \mathbb{P}^3$,

$\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3) \;=\; u_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_1^{-1} x_2^{-1} x_3^{-1}.$

$X^2YZ + XY^2Z + XYZ^2 + W^4 + \lambda WXYZ = 0$
with $\lambda = u_0(u_1u_2u_3u_4)^{-1/4}$.

$\ell_1 \mathsf{v}_1 + \ldots + \ell_N \mathsf{v}_N = \mathsf{0} \;\Rightarrow\; \ell_1 = \ell_2 = \ell_3 = \ell_4\,, \quad \ell_0 = -4\ell_1$

$\mathsf{v}_1 \delta_1 + \ldots + \mathsf{v}_N \delta_N = \mathsf{0} \;\Rightarrow\; \delta_1 = \delta_2 = \delta_3 = \delta_4\,, \quad \delta_0 = 4\delta_1$

$\overline{\mathcal{R}}^*(\mathcal{V}) \;=\; \mathbb{Z}\mathsf{1} \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_\infty$

with multiplication: $\mathsf{1}$ is the multiplicative unit element

$\delta_1^2 = 4\delta_\infty, \quad \delta_1 \cdot \delta_\infty = \delta_\infty \cdot \delta_\infty = 0$

$$\Phi^\flat(u_0, u_1, u_2, u_3, u_4) = \sum_{\ell_1 \in \mathbb{Z}_{\geq 0}} \frac{(1+4\delta_1)_{4\ell_1}}{\big((1+\delta_1)_{\ell_1}\big)^4} \left(u_0^{-4} u_1 u_2 u_3 u_4\right)^{\ell_1 + \delta_1}$$

Its degree 0 component is

$$\Phi^\flat(u_0, u_1, u_2, u_3, u_4)^{\deg 0} = \sum_{\ell_1 \geq 0} \frac{(4\ell_1)!}{\big(\ell_1!\big)^4} \left(u_0^{-4} u_1 u_2 u_3 u_4\right)^{\ell_1}$$

Case 6: $\mathcal{V} = \left[\mathsf{v}_1, \mathsf{v}_2, \mathsf{v}_3, \mathsf{v}_4, \mathsf{v}_5, \mathsf{v}_6\right] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}$

$\mathsf{Fano}(\mathcal{V}) = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

$$\mathcal{P}_{\mathcal{V},\mathsf{u}}(x_1, x_2, x_3) \;=\; u_0 + u_1 x_1 + u_2 x_2 + u_3 x_3 + u_4 x_2^{-1} + u_5 x_3^{-1} + u_6 x_1^{-1}$$

$$\ell_1 \mathsf{v}_1 + \ldots + \ell_N \mathsf{v}_N = \mathsf{0} \;\Rightarrow\; \ell_1 = \ell_6, \quad \ell_2 = \ell_4, \quad \ell_3 = \ell_5, \quad \ell_0 = -2\ell_1 - 2\ell_2 - 2\ell_3$$
$$\mathsf{v}_1 \delta_1 + \ldots + \mathsf{v}_N \delta_N = \mathsf{0} \;\Rightarrow\; \delta_1 = \delta_6, \quad \delta_2 = \delta_4, \quad \delta_3 = \delta_5, \quad \delta_0 = 2\delta_1 + 2\delta_2 + 2\delta_3$$

$$\overline{\mathcal{R}}^*(\mathcal{V}) \;=\; \mathbb{Z}\mathsf{1} \oplus \mathbb{Z}\delta_1 \oplus \mathbb{Z}\delta_2 \oplus \mathbb{Z}\delta_3 \oplus \mathbb{Z}\delta_\infty$$

with multiplication: $\mathsf{1}$ is the multiplicative unit element,

$$\delta_1^2 = \delta_2^2 = \delta_3^2 = \delta_\infty^2 = \delta_1\delta_\infty = \delta_2\delta_\infty = \delta_3\delta_\infty = 0, \quad \delta_1\delta_2 = \delta_1\delta_3 = \delta_2\delta_3 = 2\delta_\infty.$$

The bilinear form w.r.t. the basis $\mathsf{1}, \delta_1, \delta_2, \delta_3, \delta_\infty$ is $\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

Case 6 **continued:**

$$\Phi^\flat(u_0,u_1,u_2,u_3,u_4,u_5,u_6) \;=$$

$$\sum_{\ell_1,\ell_2,\ell_3\geq 0} \frac{(1+2\delta_1+2\delta_2+2\delta_3)_{2\ell_1+2\ell_2+2\ell_3}}{\big((1+\delta_1)_{\ell_1}(1+\delta_2)_{\ell_2}(1+\delta_3)_{\ell_3}\big)^2}\;(u_0^{-2}u_1u_6)^{\ell_1+\delta_1}(u_0^{-2}u_2u_4)^{\ell_2+\delta_2}(u_0^{-2}u_3u_5)^{\ell_3+\delta_3}$$

Its degree 0 component is

$$\Phi^\flat(u_0,u_1,u_2,u_3,u_4,u_5,u_6)^{\deg 0} \;=$$

$$= \sum_{\ell_1,\ell_2,\ell_3\geq 0} \frac{(2\ell_1+2\ell_2+2\ell_3)!}{\big(\ell_1!\,\ell_2!\,\ell_3!\big)^2}\;(u_0^{-2}u_1u_6)^{\ell_1}(u_0^{-2}u_2u_4)^{\ell_2}(u_0^{-2}u_3u_5)^{\ell_3}$$

Case 6 continued:

Peters, C., J. Stienstra,
A pencil of K3 surfaces related to Apéry's recurrence for $\zeta(3)$ and Fermi surfaces for potential zero,
in: Arithmetic of Complex Manifolds, Lecture Notes in Mathematics 1399, Springer Verlag (1989).

Chris Peters and I investigated the family of K3 surfaces

$$x_1 + x_1^{-1} + x_2 + x_2^{-1} + x_3 + x_3^{-1} = s$$

This is Case 6 with $u_1 = u_2 = u_3 = u_4 = u_5 = u_6 = 1$ and $u_0 = -s$.

We found that for general s the transcendental lattice is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 12 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

$$\Phi^\flat(u_0, 1, 1, 1, 1, 1, 1) =$$

$$\sum_{m \geq 0} \left(\sum_{\ell_1+\ell_2+\ell_3=m} \frac{(1+2\delta)_{2m}}{\left((1+\delta_1)_{\ell_1}(1+\delta_2)_{\ell_2}(1+\delta_3)_{\ell_3}\right)^2} \right) u_0^{-2(m+\delta)}$$

where $\delta = \delta_1 + \delta_2 + \delta_3$.
The coefficients are invariant under permutations of $\delta_1, \delta_2, \delta_3$
and, hence, linear combinations of 1, $\delta_1 + \delta_2 + \delta_3$ and $\delta_1\delta_2 + \delta_1\delta_3 + \delta_2\delta_3$.

Therefore $\Phi^\flat(u_0, 1, 1, 1, 1, 1, 1)$ takes values in $\left(\mathbb{Z}\mathbf{1} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\delta_\infty\right) \otimes \mathbb{C}$.

The bilinear form on $\mathbb{Z}\mathbf{1} \oplus \mathbb{Z}\delta \oplus \mathbb{Z}\delta_\infty$ is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 12 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.