ZHEGALKIN ZEBRA MOTIVES

digital recordings of
Mirror Symmetry

Jan Stienstra

HIM workshop, March 26-30, 2018
"Picard-Fuchs equations and
Hypergeometric Motives"

Chow motives ....?!?!

Wouldn't it be nice if we could "do" motives without the burden of the technically complicated intersection theory of algebraic cycles

....

and instead use the usual operations intersection/union/complement on sets?

subsets of $\mathbb{R}^2$

$=$

$\mathbb{F}_2$-valued functions on $\mathbb{R}^2$

Zhegalkin (1927):
Boolean formalism for subsets

$\Leftrightarrow$

usual multiplication and addition
of $\mathbb{F}_2$-valued functions

Zebra with frequency $\mathfrak{v} \in \mathbb{R}^2 \setminus \{\mathsf{0}\} = \mathbb{C}^*$

$$\mathsf{Z}^{\mathfrak{v}} \; : \quad \mathbb{R}^2 \longrightarrow \mathbb{F}_2$$

$$\mathsf{Z}^{\mathfrak{v}}(\mathbf{x}) \; = \; \lfloor 2\mathbf{x} \cdot \mathfrak{v} \rfloor \bmod 2$$

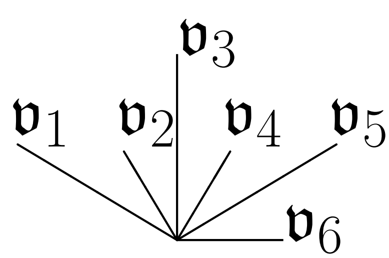
$\cdot$ = dot product on $\mathbb{R}^2$; $\lfloor r \rfloor$ = floor of r

bands $\perp$ $\mathfrak{v}$, width $\frac{1}{2|\mathfrak{v}|}$

Use only zebras with frequencies positive integer multiples of the basic frequencies

$$\mathfrak{v}_1 = \sqrt{3}\,\varepsilon^5, \quad \mathfrak{v}_2 = \varepsilon^4, \quad \mathfrak{v}_3 = \sqrt{3}\,\mathbf{i}\,,$$
$$\mathfrak{v}_4 = \varepsilon^2, \quad \mathfrak{v}_5 = \sqrt{3}\,\varepsilon, \quad \mathfrak{v}_6 = 1\,,$$

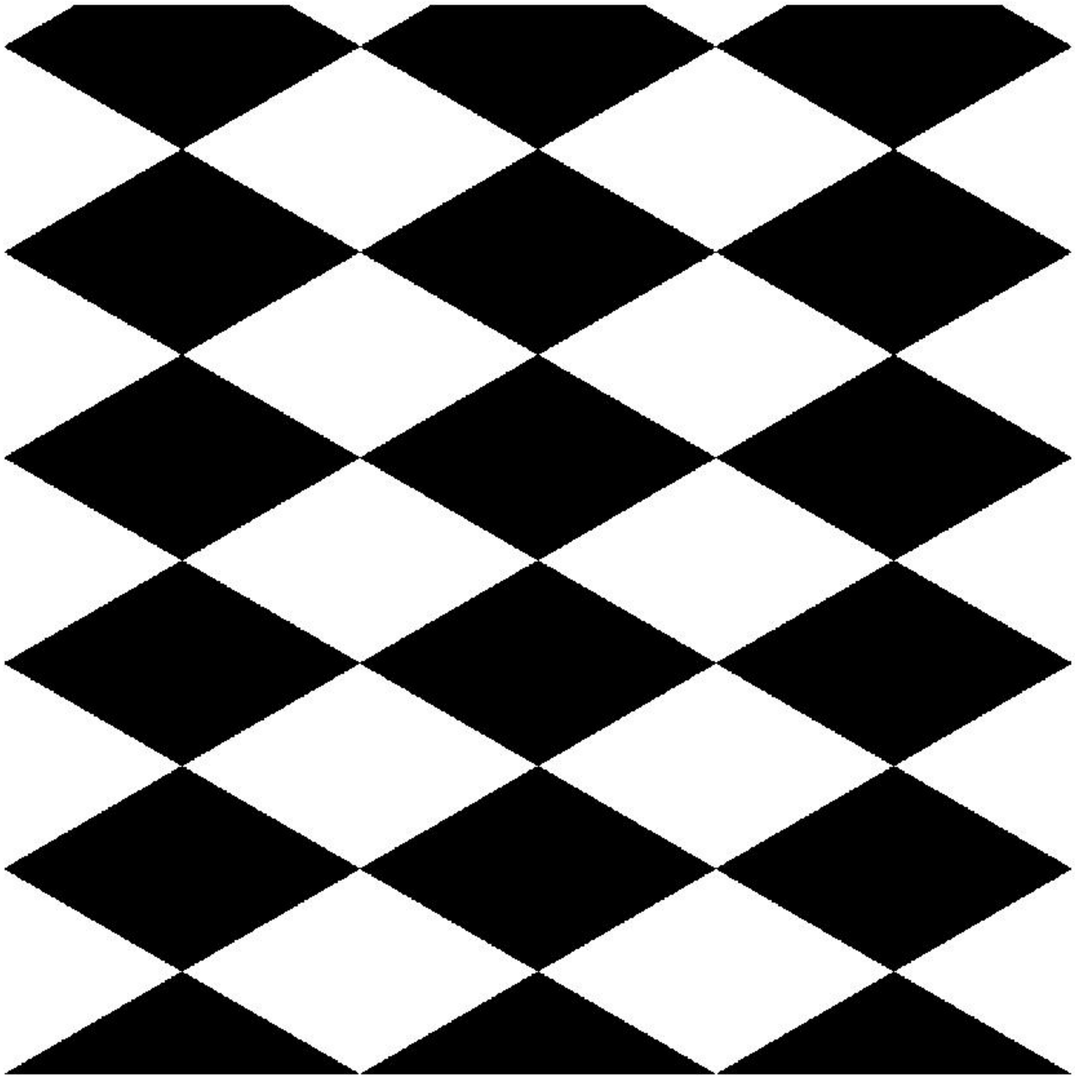
with $\varepsilon = e^{\pi \mathbf{i}/6}$;



Z^{jk} denotes the zebra $\mathsf{Z}^{k\mathfrak{v}_j}$

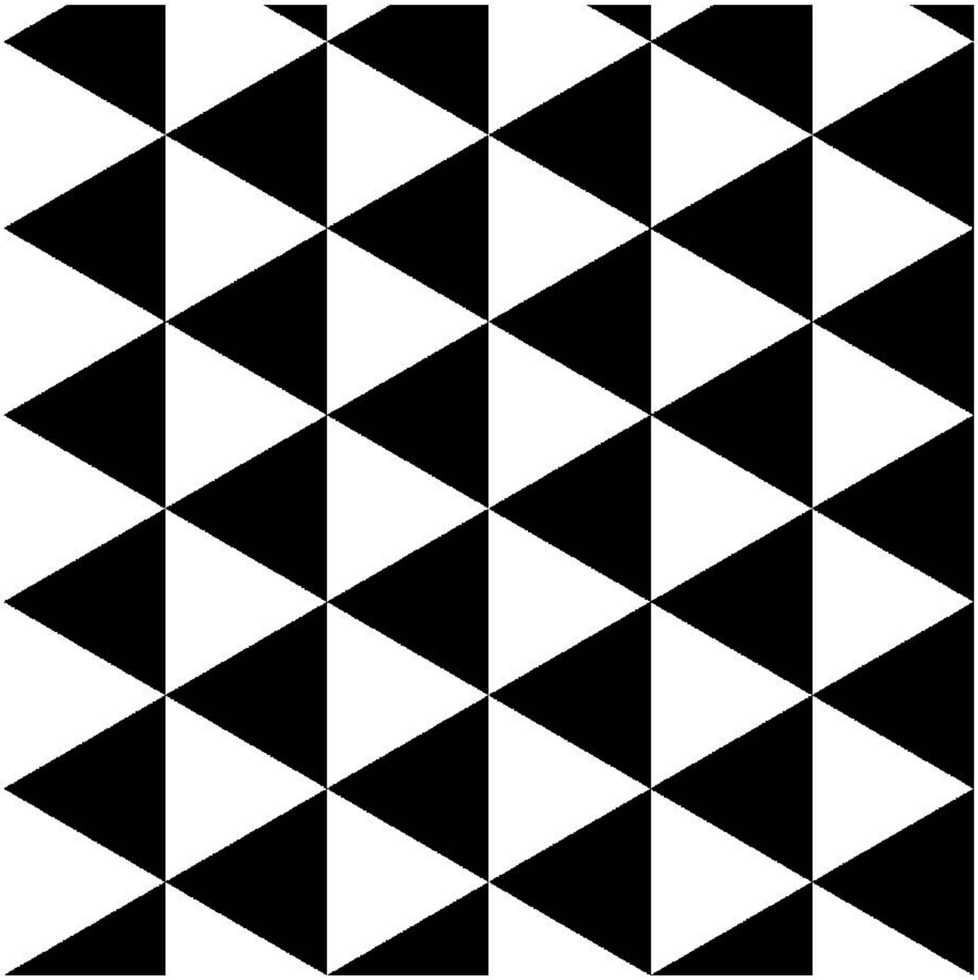
Definition:

A *Zhegalkin zebra motive* is an element of the ring generated by the zebras Z^{jk} with $j = 1, \ldots, 6$, $k \in \mathbb{Z}_{>0}$



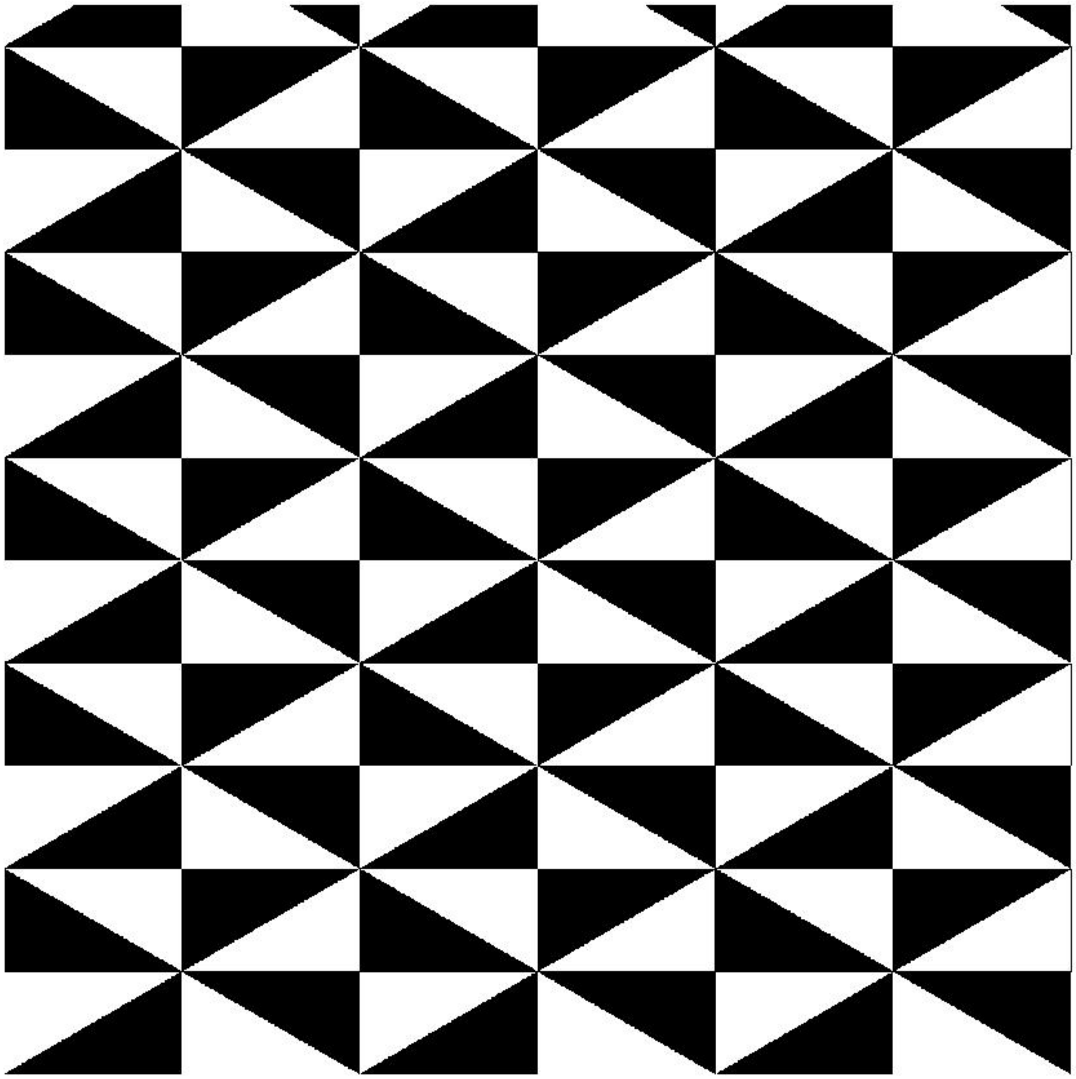
$$\mathcal{F}_2 = \mathsf{Z}^{21} + \mathsf{Z}^{41}$$

$$\mathsf{Aut}^\circ \mathcal{F}_2 = \mathbb{Z}\mathfrak{v}_6 \oplus \tfrac{1}{3}\mathbb{Z}\mathfrak{v}_3$$



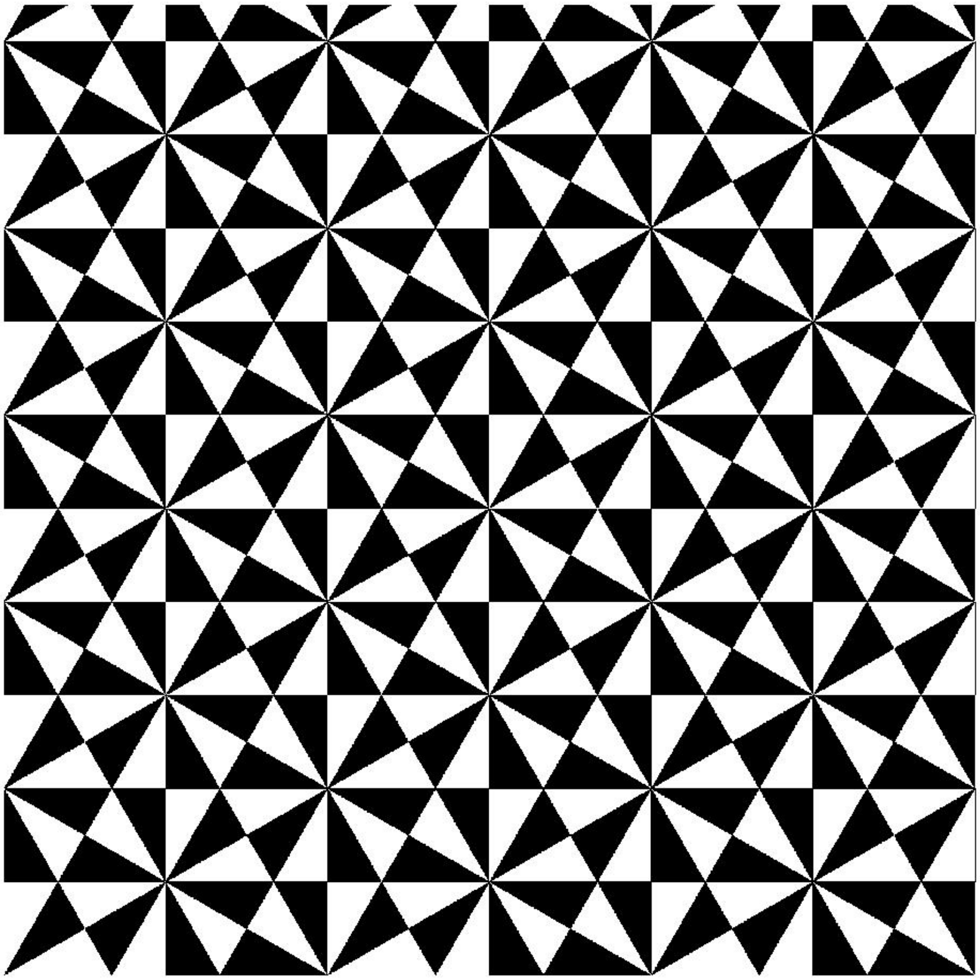
$$\mathcal{F}_3 = \mathsf{Z}^{21} + \mathsf{Z}^{41} + \mathsf{Z}^{61}$$

$$\mathsf{Aut}^\circ\mathcal{F}_3 = \mathbb{Z}\mathfrak{v}_6 \oplus \frac{1}{3}\mathbb{Z}\mathfrak{v}_5$$



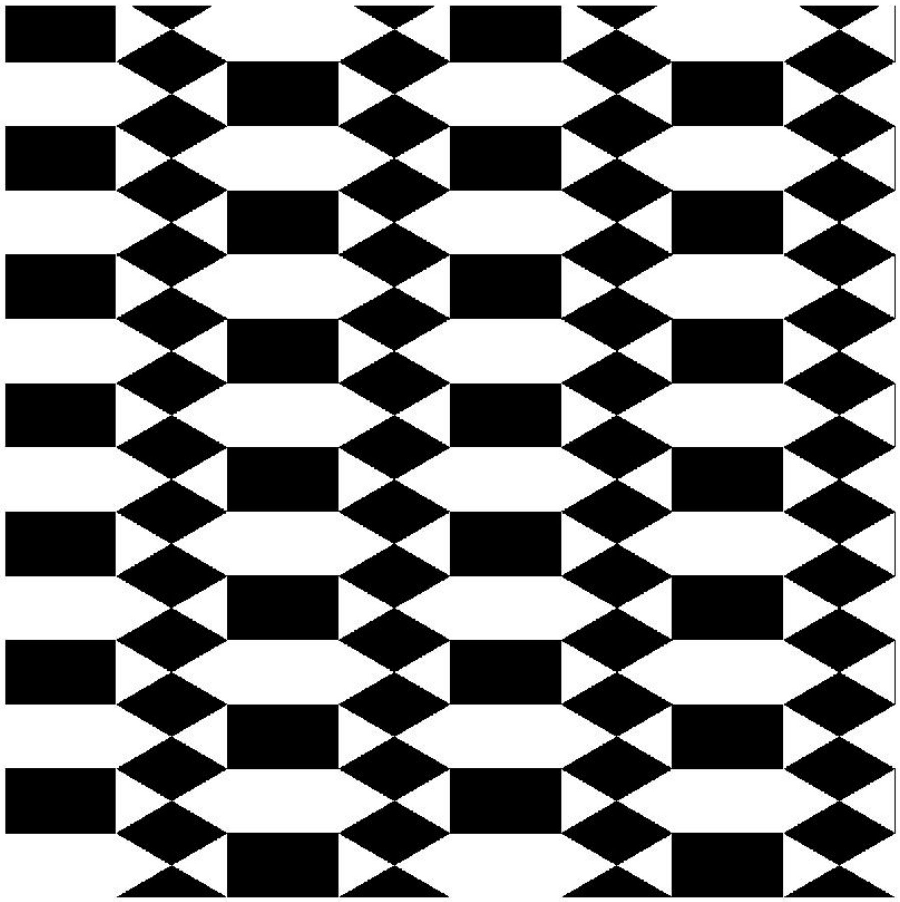
$$\mathcal{F}_4 = \mathsf{Z}^{21} + \mathsf{Z}^{31} + \mathsf{Z}^{41} + \mathsf{Z}^{61}$$

$$\mathsf{Aut}^{\circ}\mathcal{F}_4 = \mathbb{Z}\mathfrak{v}_6 \oplus \frac{1}{3}\mathbb{Z}\mathfrak{v}_3$$



$$\mathcal{F}_6 = \mathsf{Z}^{11} + \mathsf{Z}^{21} + \mathsf{Z}^{31} + \mathsf{Z}^{41} + \mathsf{Z}^{51} + \mathsf{Z}^{61}$$

$$\mathsf{Aut}^{\circ}\mathcal{F}_6 = \frac{1}{3}\mathbb{Z}\mathfrak{v}_1 \oplus \frac{1}{3}\mathbb{Z}\mathfrak{v}_3$$

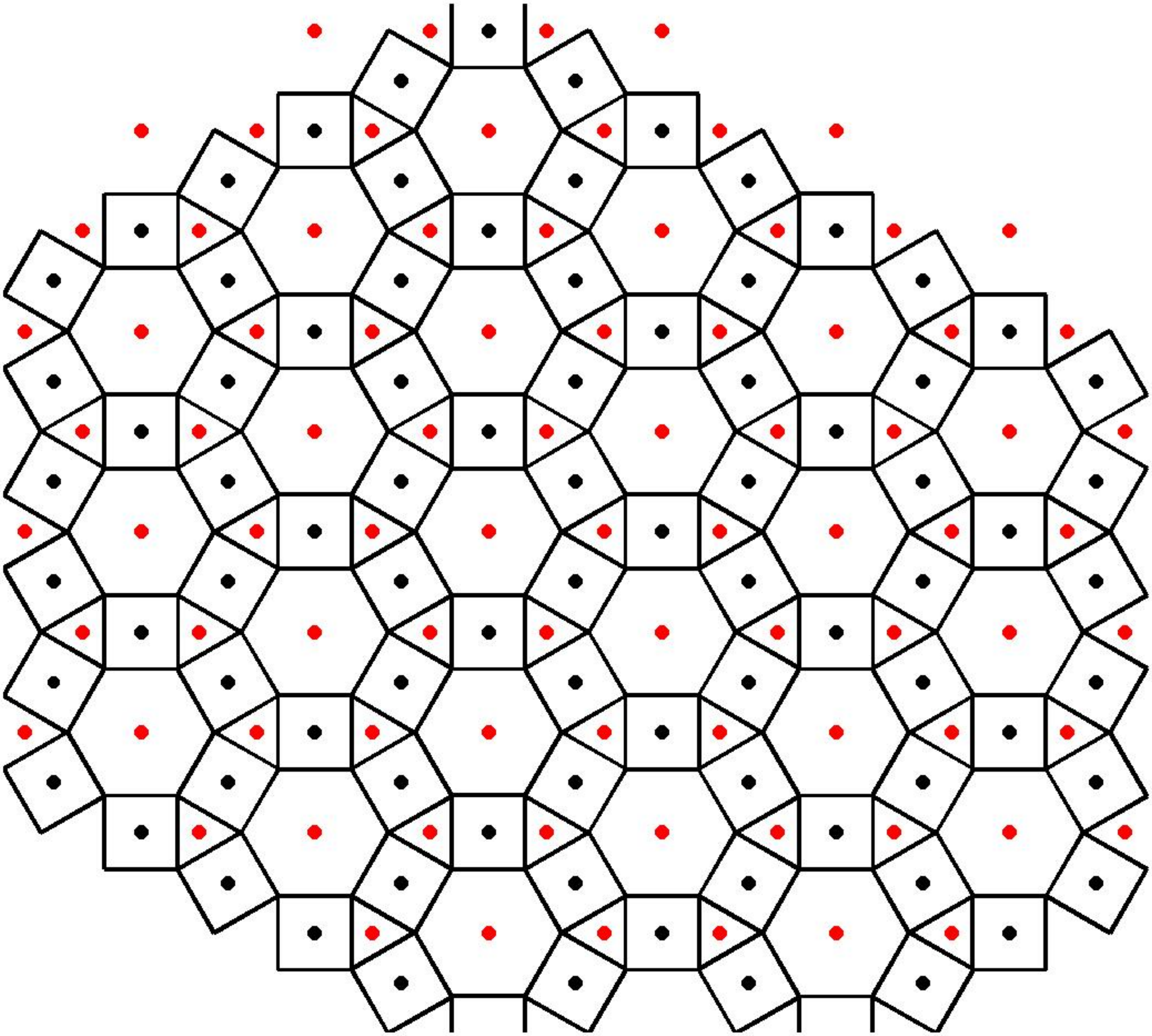


$$(1 + \mathsf{Z}^{62})(\mathsf{Z}^{32} + \mathsf{Z}^{61}) + \mathsf{Z}^{62}(\mathsf{Z}^{24} + \mathsf{Z}^{44})$$

$$\left.\begin{array}{l}\text{vertices}\\ \text{(oriented) edges}\end{array}\right\}\quad \text{quiver } \Gamma$$

$$\left.\begin{array}{l}\text{black/white polygons}\\ \text{edges}\end{array}\right\}\quad \text{bipartite graph } \Gamma^{\vee}$$

$$\left.\begin{array}{l}\text{black/white polygons}\\ \text{edges}\end{array}\right\}\quad \text{superpotential}$$



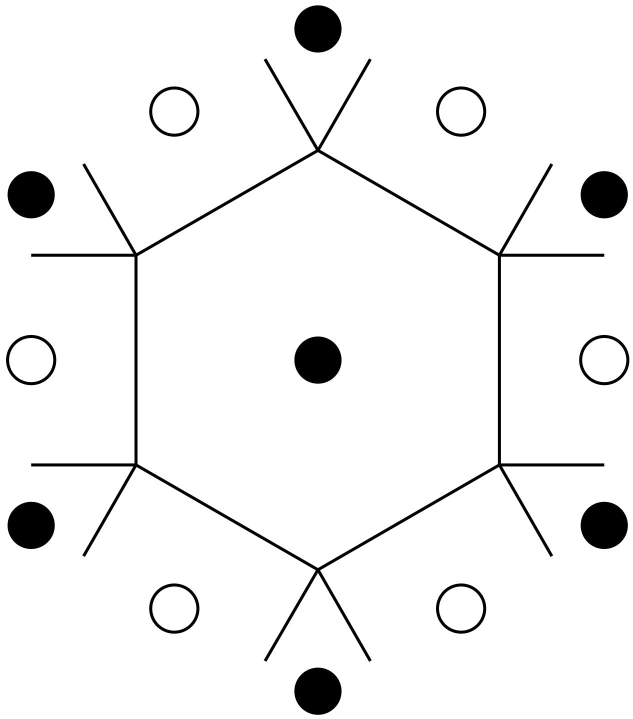
doubly periodic with lattice Λ

$\Rightarrow$ graphs Γ_Λ, $\Gamma_\Lambda^\vee$ on torus $\mathbb{R}^2/\Lambda$

$\Rightarrow$ finitely many vertices, edges, polygons

$\Rightarrow$

- adjacency matrices for Γ_Λ, $\Gamma_\Lambda^\vee$ are finite
- superpotential = pair of permutations of the finite set of edges



superpotential:

edge set $\{1, \ldots, 12\}$

$\sigma_0 = (1,3,2,12)(4,5,10,9)(11,6,7,8)$

$\sigma_1 = (1,5,6,2,9,8)(4,3,7)(11,10,12)$

$$\sigma_0 = (1,3,2,12)(4,5,10,9)(11,6,7,8)$$

$$\sigma_1 = (1,5,6,2,9,8)(4,3,7)(11,10,12)$$

source/target maps for Γ_Λ:

e	1	2	3	4	5	6	7	8	9	10	11	12
s	5	2	1	3	1	4	2	3	6	4	5	6
t	1	6	2	1	4	2	3	5	3	6	4	5

e	1	2	3	4	5	6	7	8	9	10	11	12
s	5	2	1	3	1	4	2	3	6	4	5	6
t	1	6	2	1	4	2	3	5	3	6	4	5

the adjacency matrix of Γ_Λ

$$\mathfrak{A}^{\star\star}(X_*) = \begin{pmatrix} 0 & X_3 & 0 & X_5 & 0 & 0 \\ 0 & 0 & X_7 & 0 & 0 & X_2 \\ X_4 & 0 & 0 & 0 & X_8 & 0 \\ 0 & X_6 & 0 & 0 & 0 & X_{10} \\ X_1 & 0 & 0 & X_{11} & 0 & 0 \\ 0 & 0 & X_9 & 0 & X_{12} & 0 \end{pmatrix}$$

$$\sigma_0 = (1,3,2,12)(4,5,10,9)(11,6,7,8)$$

$$\sigma_1 = (1,5,6,2,9,8)(4,3,7)(11,10,12)$$

adjacency matrices of $\Gamma_\Lambda^\vee$:

$$\mathfrak{A}^{\bullet\circ}(U_*) = \begin{pmatrix} U_1 + U_2 & U_5 + U_9 & U_6 + U_8 \\ U_3 & U_4 & U_7 \\ U_{12} & U_{10} & U_{11} \end{pmatrix}$$

$$\mathfrak{A}^{\circ\bullet}(U_*^\dagger) = \begin{pmatrix} U_1^\dagger + U_2^\dagger & U_3^\dagger & U_{12}^\dagger \\ U_5^\dagger + U_9^\dagger & U_4^\dagger & U_{10}^\dagger \\ U_6^\dagger + U_8^\dagger & U_7^\dagger & U_{11}^\dagger \end{pmatrix}$$

an *integer weight function* for the superpotential $(\mathcal{E}_\Lambda, \sigma_0, \sigma_1)$ is a map

$$\nu : \mathcal{E}_\Lambda \longrightarrow \mathbb{Z}_{\geq 0}$$

such that the sum over each cycle of σ_0 and each cycle of σ_1 is equal to an integer $\deg \nu$ (the degree of ν)

dimer covering=
dimer configuration=
perfect matching=
weight function of degree 1

$$\text{perfect matchings} \quad \overset{1:1}{\leftrightarrow} \quad \text{terms of det } \mathfrak{A}^{\bullet\circ}(U_*)$$

$$\det \begin{pmatrix} U_1 + U_2 & U_5 + U_9 & U_6 + U_8 \\ U_3 & U_4 & U_7 \\ U_{12} & U_{10} & U_{11} \end{pmatrix}$$

$$=$$

$$\begin{array}{l} -U_1U_7U_{10} + U_5U_7U_{12} + U_3U_8U_{10} \\ +U_1U_4U_{11} - U_2U_7U_{10} + U_3U_6U_{10} \\ -U_3U_5U_{11} + U_7U_9U_{12} - U_4U_8U_{12} \\ -U_4U_6U_{12} - U_3U_9U_{11} + U_2U_4U_{11} \end{array}$$

$$\sigma_0 = (1,3,2,12)(4,5,10,9)(11,6,7,8)$$

$$\sigma_1 = (1,5,6,2,9,8)(4,3,7)(11,10,12)$$

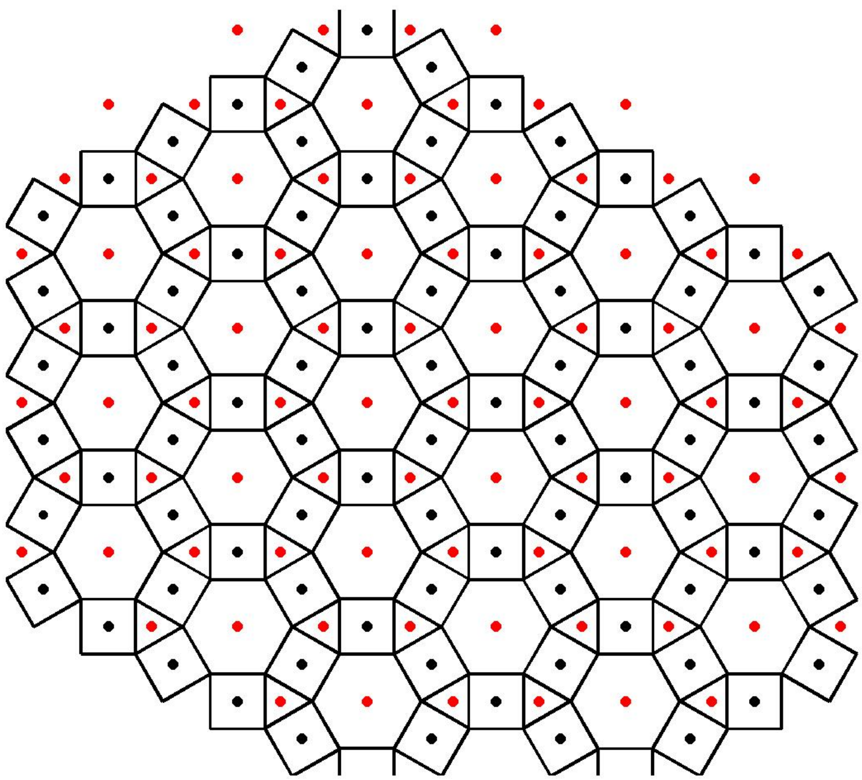
integer weight functions form
a graded semi-group $\mathcal{W}_\Lambda$

dual semi-group $\mathcal{W}_\Lambda^\vee = \mathrm{Hom}(\mathcal{W}_\Lambda, \mathbb{Z}_{\geq 0})$

essential conditions on superpotential:

- perfect matchings exist
- semi-group $\mathcal{W}_\Lambda$ is generated by perfect matchings
- $\mathcal{W}_\Lambda$ spans linear space of rank $2 + \sharp\{\text{vertices}\}$

weight function ν with all $\nu(e) > 0$ selects points in the polygons by taking the convex combination of the midpoints of the polygon's edges e with weights $\frac{1}{\deg \nu}\nu(e)$



$\nu(e) = 1$ if e edge of 6-gon
$\nu(e) = 2$ if e edge of 3-gon

by (slightly) deforming and rescaling one can make from the given tiling a tiling in which the edge vectors lie in $\mathbb{Z}^2$

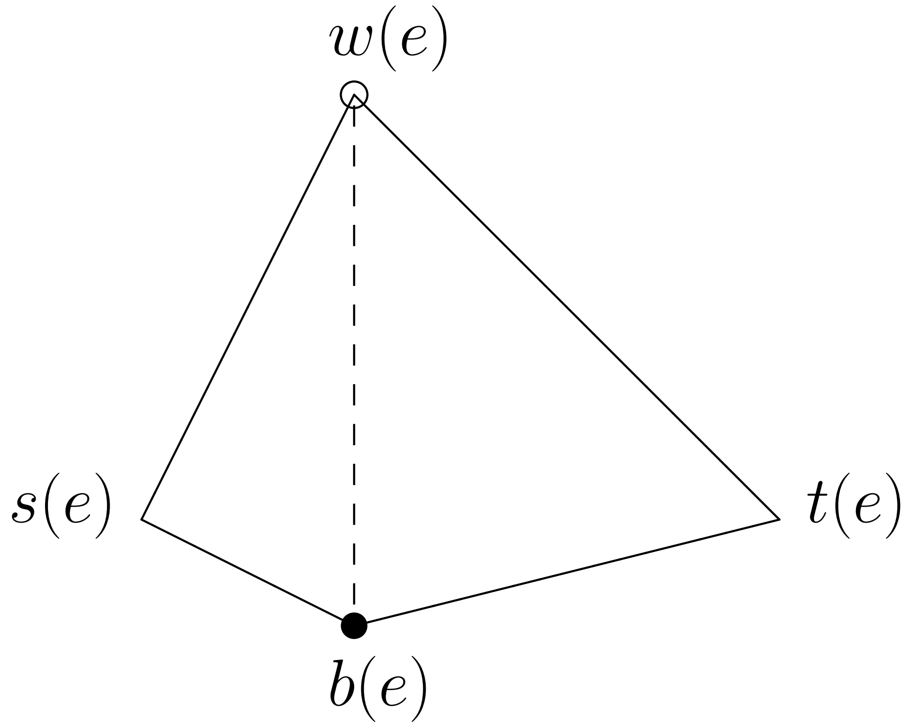
$$\mathsf{vec}(e) = (n_1(e)\,,\, n_2(e))$$

Then
for weight function ν_3 with
all $\nu_3(e)$ sufficiently large

the functions ν_1 and ν_2,

$$\nu_1(e) = \nu_3(e) + n_1(e)$$
$$\nu_2(e) = \nu_3(e) + n_2(e)$$

are also weight functions



quadrangle around edge e

substitution

$$X_e \mapsto y_1^{\nu_1(e)} y_2^{\nu_2(e)} y_3^{\nu_3(e)}$$

in adjacency matrix $\mathfrak{A}^{\star\star}(X_*)$
of quiver Γ_Λ gives matrix

$$\mathfrak{A}^{\star\star}(y_1^{\nu_1} y_2^{\nu_2} y_3^{\nu_3})$$

over polynomial ring $\mathbb{Z}[y_1, y_2, y_3]$

The *Jacobi algebra* of the superpotential (σ_0, σ_1) is the algebra

$$\mathsf{Jac}(\sigma_0, \sigma_1) = {}^{\mathsf{Path}(\Gamma_\Lambda)}\big/_{\langle \mathsf{D}^\circ(e) \,|\, e \in \mathcal{E}_\Lambda \rangle}$$

where $\langle \mathsf{D}^\circ(e) \,|\, e \in \mathcal{E}_\Lambda \rangle$ = 2-sided ideal generated by the elements

$$\mathsf{D}^\circ(e) = \prod_{e' \neq e : w(e') = w(e)}^{\circlearrowleft} e' - \prod_{e' \neq e : b(e') = b(e)}^{\circlearrowright} e' \, .$$

the matrix $\mathfrak{A}^{\star\star}(y_1^{\nu_1} y_2^{\nu_2} y_3^{\nu_3})$
gives a homomorphism

$$\mathsf{Jac}(\sigma_0, \sigma_1) \longrightarrow \mathsf{Mat}(\mathbb{Z}[y_1, y_2, y_3])$$

from the Jacobi algebra $\mathsf{Jac}(\sigma_0, \sigma_1)$
into a matrix algebra over
the polynomial ring $\mathbb{Z}[y_1, y_2, y_3]$

Equivalence relation on $\mathcal{W}_\Lambda$

$$\nu \sim \nu' \qquad \Leftrightarrow$$

$$\exists \alpha : \{\text{vertices}\} \to \mathbb{Z} \quad \text{s.t.} \quad \forall e \in \mathcal{E}_\Lambda :$$

$$\nu(e) - \nu'(e) = \alpha(t(e)) - \alpha(s(e))$$

If $\nu_j \sim \nu'_j$ for $j = 1, 2, 3$ then

$$\mathfrak{A}^{\star\star}(y_1^{\nu_1} y_2^{\nu_2} y_3^{\nu_3}) \;=\; D^{-1}\,\mathfrak{A}^{\star\star}(y_1^{\nu'_1} y_2^{\nu'_2} y_3^{\nu'_3})\, D$$

for some diagonal matrix D

i.e. representations of $\mathsf{Jac}(\sigma_0, \sigma_1)$ are isomorphic.

$\mathcal{W}_\Lambda/\sim$ is a semi-group

It spans a linear space of rank 3

The toric scheme

$$\mathsf{Spec}\left(\mathbb{Z}\big[(\mathcal{W}_\Lambda/\sim)^\vee\big]\right)$$

is a 3-dim singularity

Under suitable conditions

$\mathsf{Jac}(\sigma_0, \sigma_1)$ *is a non-commutative crepant resolution of this singularity*

The set of $\sim$-equivalence classes of the perfect matchings is a finite set

$$\mathcal{A} = \{\mathfrak{a}_1, \ldots, \mathfrak{a}_N\}$$

of points in a plane

Can be used for $\mathcal{A}$-hypergeometrics

Fourier transform of $\mathcal{A}$:

$$\sum_{j=1}^{N} e^{2\pi i \langle \mathfrak{t}, \mathfrak{a}_j \rangle}$$

Contour plot of the absolute value of Fourier transform of $\mathcal{A}$: