Motives and Strings

Jan Stienstra

Colloquium Universität Hannover, 9 November 2004

Physicists want a unified theory for
relativistic, gravitational and quantum effects.

a Theory of Everything

String Theory and M-theory seem good candidates.

♠♠♠♠

Mathematicians want a theory for patterns
– named *Motives* –
recurring throughout number theory and geometry.

♠♠♠♠

Occasionally they hit on the same examples.

'Is there a more unifying theory?'
'Does Everything include Numbers?'

'The *M* in *M-theory* seems to stand for
magic or *mystery* or *mother of all strings*,
but could it also mean *Motive*?'

What is .... a Motive ?

Barry Mazur in Notices of the AMS october 2004:

Algebraic topology: one cohomology theory, characterized by the Eilenberg-Steenrod axioms, representable by Eilenberg-MacLane spaces and Postnikov towers.

Algebraic geometry: a profusion of different cohomology theories, no axioms, no representing objects:

- Hodge cohomology
- algebraic De Rham cohomology
- crystalline cohomology (for every prime number p)
- étale ℓ-adic cohomology (for every prime number ℓ)

and comparison isomorphisms between these.

Cohomology theories assign to varieties vector spaces (some with additional structure).
Inverse problem: *given a collection of vector spaces (+ additional structure, comparison isomorphisms), is this produced by the various cohomology theories from one common source?*
The common source is called a MOTIVE.

More precisely:
a variety X is defined over some field K,
often with positive transcendence degree over $\mathbb{Q}$.

Every embedding σ of K into $\mathbb{C}$ leads to a complex variety to which algebraic topology and differential geometry with their cohomology theories apply.

Every embedding σ of K into a p-adic field $\overline{\mathbb{Q}}_p$ leads to crystalline cohomology spaces.

As the embeddings σ vary, with fixed target $\mathbb{C}$ or $\overline{\mathbb{Q}}_p$, the variation in the cohomology spaces is described partly by the action of Galois groups and partly by Picard-Fuchs differential equations and Gauss-Manin connections.

As yet no theory to mix the targets $\mathbb{C}$ *and/or the various* $\overline{\mathbb{Q}}_p$*!*

There are at least five String Theories:

- Type I
- Type IIa
- Type IIb
- Heterotic $SO(32)$
- Heterotic $E_8 \times E_8$

plus $11D$ supergravity.

These are interrelated by *dualities*
e.g.
Mirror Symmetry between IIa and IIb

These are limits of one 11-dimensional theory: M-THEORY

M-theory,
the theory formerly known as
Strings

Why look for MOTIVE-STRING relation ?

Computations in Type IIb string theory proceed by manipulating solutions of certain differential equations. During the computations there are many denominators. In the end these drop out and true integers remain.

Many differential equations in Type IIb string can be recognized as Picard-Fuchs equations in De Rham cohomology of families of varieties.

The integrality statements can be recognized as consequences of theorems about crystalline cohomology.

Challenge for Motive people:
Crystalline cohomology deals with only one prime p at a time and puts out statements about p-adic integrality.

What mechanism synchronizes the primes
and leads to true integers?

Challenge for String people:
Crystalline cohomology implies extra symmetries in the differential equations.

Where are these extra symmetries in Nature?

Line & Additive Group

arclength $\int_a^b dx \; = \; b - a$

invariant differential form dx
invariant derivation $\frac{d}{dx}$

$$dx = dz \Rightarrow z = x + y \,, \qquad y \text{ constant}$$

Circle

equations for the unit circle in the plane:

$$\begin{aligned} r &= 1 && \text{polar coordinates} \\ x^2 + y^2 &= 1 && \text{Cartesian coordinates} \end{aligned}$$

arclength $\quad \int_a^b \frac{dx}{\sqrt{1-x^2}} = \arcsin(b) - \arcsin(a)$

$$\begin{aligned} \arcsin(x) &= \sum_{n \geq 0} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{2n+1}}{2n+1} \\ &= 2 \sum_{n \geq 0} \frac{(2n)!}{n!^2} \frac{(x/2)^{2n+1}}{2n+1} \end{aligned}$$

better

$$\frac{1}{2} \arcsin(2x) = \sum_{n \geq 0} \frac{(2n)!}{n!^2} \frac{x^{2n+1}}{2n+1}$$

invariant differential form $\dfrac{dx}{\sqrt{1-x^2}}$

invariant derivation $\sqrt{1-x^2}\,\dfrac{d}{dx}$

Addition law for trigonometric functions:

$$\frac{dx}{\sqrt{1-x^2}} = \frac{dz}{\sqrt{1-z^2}} \qquad \Rightarrow$$

$$\begin{aligned} z &= \sin(\arcsin(x) + \arcsin(y)) \qquad\qquad y \text{ constant} \\ &= x\sqrt{1-y^2} + y\sqrt{1-x^2} \\ &= x + y + \sum_{n\geq 1} (-1)^n \binom{\frac{1}{2}}{n} (x\,y^{2n} + y\,x^{2n}) \end{aligned}$$

Better, with $\qquad z = 2w,\ x = 2u,\ y = 2v,$

$$w = u + v - 4 \sum_{n\geq 1} \frac{(2n-3)!}{n!(n-2)!} (u\,v^{2n} + v\,u^{2n})$$

The coefficients are integers:

$$-4\frac{(2n-3)!}{n!(n-2)!} = 2\binom{2n-2}{n} - 4\binom{2n-3}{n-1}$$

So, this is a formal group law over $\mathbb{Z}$!

Multiplicative Group

arclength $\int_a^b \frac{dx}{x} \; = \; \log(b) - \log(a)$

invariant differential form $\dfrac{dx}{x}$

invariant derivation $x \, \dfrac{d}{dx}$

$$\frac{dx}{x} = \frac{dz}{z} \quad \Rightarrow \quad z = xy \,, \quad y \text{ constant}$$

Coordinate change $\quad z = 1 + w, \; x = 1 + u, \; y = 1 + v$
gives

$$w = u + v + uv \,,$$

the standard multiplicative formal group law over $\mathbb{Z}$.

Circle & Multiplicative Group

Substitution $\qquad x \to ix,\ y \to iy,\ z \to iz$
transforms the addition law for trigonometric sine:

$$z = x\sqrt{1-y^2} + y\sqrt{1-x^2}$$

into the addition law for hyperbolic sine:

$$z = x\sqrt{1+y^2} + y\sqrt{1+x^2}$$

The series

$$\begin{aligned} \sinh(\log(1+u)) &= \frac{1}{2}\left((1+u)-(1+u)^{-1}\right) \\ &= u - \frac{1}{2}\sum_{n\geq 2}(-u)^n \end{aligned}$$

establishes an isomorphism, defined over $\mathbb{Z}[\frac{1}{2}]$, between the multiplicative group law and the addition law for the hyperbolic sine.

Lemniscate & its group law

equations Bernouilli's Lemniscate in the plane:

$$\begin{aligned} r^2 &= \cos(2\phi) && \text{polar coordinates} \\ (x^2+y^2)^2 &= x^2 - y^2 && \text{Cartesian coordinates} \end{aligned}$$

arclength $\displaystyle \int_a^b \frac{dx}{\sqrt{1-x^4}}$

$$\begin{aligned} \int \frac{dx}{\sqrt{1-x^4}} &= \sum_{n\geq 0} (-1)^n \binom{-\frac{1}{2}}{n} \frac{x^{4n+1}}{4n+1} \\ &= \sqrt{2} \sum_{n\geq 0} \frac{(2n)!}{n!^2} \frac{(x/\sqrt{2})^{4n+1}}{4n+1} \end{aligned}$$

Coefficients can be made integral by substitution $x \to x\sqrt{2}$

Euler

$$\frac{dx}{\sqrt{1-x^4}} = \frac{dz}{\sqrt{1-z^4}} \quad \Rightarrow$$

$$z \;=\; \frac{x\sqrt{1-y^4}+y\sqrt{1-x^4}}{1+x^2y^2}\,, \qquad\qquad y \text{ constant}$$

Euler's result was the first example of an addition law for elliptic integrals.

This marked the beginning of the theory of elliptic curves !

The elliptic curve in this case is, in homogeneous coordinates in the weighted projective plane $\mathbb{P}^{[1,1,2]}$,

$$X^4 + Y^4 + Z^2 \;=\; 0$$

$$X^4 + Y^4 + Z^2 - tXYZ \;=\; 0$$

with t a complex (deformation) parameter.

Viewed in $\mathbb{C}^3$ this equation has for $t^4 \neq 64$ only one singular point, namely $(0, 0, 0)$.

This is a so-called simple elliptic singularity known as the $\tilde{E}_7$ singularity.

♠♠♠♠

Viewed in $\mathbb{P}^{[1,1,2]}$ this equation describes for $t^4 \neq 64$ a smooth elliptic curve.

We have a family of elliptic curves with singular fibres at $t = \pm 2\sqrt{2}\,,\ \pm 2\sqrt{-2}\,,\ \infty$

The elliptic curve

$$X^4 + Y^4 + Z^2 - tXYZ \; = \; 0$$

a rank 2 *period lattice.*

One period (of a suitably normalized 1-form along a suitable closed path)
can be computed via the residue theorem:
with $u = t^{-1}$ and $|u|$ sufficiently small:

$$\begin{aligned}
f_0 \;&=\; \frac{1}{2\pi i} \oint_{\gamma_0} \frac{-t\, dx}{2z - tx}\bigg|_{x^4+1+z^2=txz\,,\; |z|<1} \\
&=\; \left(\frac{1}{2\pi i}\right)^2 \oint\oint \frac{-t\, dx\, dz}{x^4+1+z^2-txz} \\
&=\; \left(\frac{1}{2\pi i}\right)^2 \oint\oint \frac{1}{1-ux^{-1}z^{-1}(x^4+1+z^2)} \frac{dx}{x} \frac{dz}{z} \\
&=\; \sum_{n\geq 0} u^n \left(\frac{1}{2\pi i}\right)^2 \oint\oint \left(\frac{x^4+1+z^2}{xz}\right)^n \frac{dx}{x} \frac{dz}{z} \\
&=\; \sum_{m\geq 0} \frac{(4m)!}{m!^2(2m)!} u^{4m}
\end{aligned}$$

A second period

$$f_1 = \frac{1}{2\pi i} \oint_{\gamma_1} \frac{-t\,dx}{2z - tx}\Big|_{x^4+1+z^2=txz\,,\ |z|<1}$$

can be determined from the formula (with $\epsilon^2 = 0$)

$$f_0 + f_1\epsilon \ \equiv \ \sum_{m\geq 0} \frac{(1+4\epsilon)_{4m}}{(1+\epsilon)_m^2\,(1+2\epsilon)_{2m}} u^{4m+4\epsilon}$$

using the rising Pochhammer symbol: for $k \geq 0$

$$(a)_k := a(a+1)\cdot\cdots\cdot(a+k-1)$$

(so $(1)_k = k!$)

Note

$$f_1 = 4f_0 \log u + g_1$$

where g_1 is a power series in u^4 with constant term 0.

Thus if we define τ and q by

$$\tau := \frac{f_1}{4f_0}\,, \qquad q \ := \ \exp(\tau) \ = \ u\exp(\frac{g_1}{4f_0})$$

then q is another local coordinate on the u-line near $u = 0$.

More formal group laws.

Put

$$\ell(u) = \int f_0\, du = \sum_{m\geq 0} \frac{(4m)!}{m!^2(2m)!}\frac{u^{4m+1}}{4m+1}$$

and

$$L(x) = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{1-4x^4}} = \sum_{n\geq 0} \frac{(2n)!}{n!^2}\,\frac{x^{4n+1}}{4n+1}$$

Then

$$\begin{aligned}&\ell^{-1}(\ell(u)+\ell(v))\\ &L^{-1}(L(x)+L(y))\end{aligned}$$

are two formal group laws over $\mathbb{Z}$ and they are isomorphic over $\mathbb{Z}$.

Moreover $L^{-1}(L(x)+L(y))$ is the integer version of the addition law for the lemniscate.

i.e. the base of the elliptic pencil

$$X^4 + Y^4 + Z^2 - tXYZ \;=\; 0$$

carries in the neighborhood of $t = \infty\,,\; u = 0$ a formal group law over $\mathbb{Z}$, in the coordinate u, which is over $\mathbb{Z}$ isomorphic to the formal group law of the fiber at
$t = 0\,,\; u = \infty$

Seiberg-Witten
'Electric-Magnetic Duality, Monopole Condensation, and Confinement in $N = 2$ Supersymmetric Yang-Mills Theory'

illustrate their general theory with a first example starring the functions

$$\begin{aligned} \mathfrak{a} &= \int f_0 \, dt \\ \mathfrak{a}_D &= \int f_1 \, dt \end{aligned}$$

They show that $\mathfrak{a}$ and $\mathfrak{a}_D$ give the periods of some
meromorphic 1-*form without residues*
on the elliptic curve $X^4 + Y^4 + Z^2 - tXYZ \;=\; 0\,.$

The functions $\mathfrak{a}$ and $\mathfrak{a}_D$ are used to construct the potential

$$K := \frac{1}{2i}(\overline{\mathfrak{a}}\,\mathfrak{a}_D - \mathfrak{a}\,\overline{\mathfrak{a}}_D)$$

for a so-called *rigid special Kähler metric* on the base space (Moduli space) of the pencil; the metric is

$$\begin{aligned}
\frac{\partial^2 K}{\partial\mathfrak{a}\,\partial\overline{\mathfrak{a}}}\,d\mathfrak{a}\,d\overline{\mathfrak{a}} &= \frac{1}{2i}\left(\frac{\partial\mathfrak{a}_D}{\partial\mathfrak{a}} - \frac{\partial\overline{\mathfrak{a}}_D}{\partial\overline{\mathfrak{a}}}\right)\,d\mathfrak{a}\,d\overline{\mathfrak{a}} \\
&= \frac{1}{2i}\left(\frac{f_1}{f_0} - \frac{\overline{f_1}}{\overline{f_0}}\right) f_0\overline{f_0}\,dt\,d\overline{t} \\
&= -2i(\tau - \overline{\tau}) f_0\overline{f_0}\,dt\,d\overline{t}
\end{aligned}$$

A Glimpse of Mirror Symmetry

Golyshev looked at the system of differential equations

$$\frac{d}{dt}\Phi = (t - A)^{-1} P \Phi$$

where

$$A = \begin{pmatrix} 12 & 552 & 7488 \\ 1 & 40 & 552 \\ 0 & 1 & 12 \end{pmatrix}, \qquad P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

He showed that it has a solution

$$\Phi = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ \mathfrak{a}_D & \mathfrak{a} & 1 \end{pmatrix}$$

and he showed that the entries of A count how many curves of certain kinds there exist on the DelPezzo surface dP_7, i.e. the blow up of $\mathbb{P}^2$ at seven points.

Note: classical theory relates this DelPezzo to the root system E_7.

More formal group laws.

Recall

$$\mathfrak{a} = \int f_0 \, dt = -\int f_0 \frac{du}{u^2} = -\sum_{m\geq 0} \frac{(4m)!}{m!^2(2m)!}\frac{u^{4m-1}}{4m-1}$$

Note that the coefficients

$$\frac{(4m)!}{m!^2(2m)!}\frac{1}{4m-1} \;=\; 4\frac{(4m-3)!}{m!^2(2m-2)!}$$

$$=\; -2\frac{(4m-2)!}{m!^2(2m-2)!} + 8\frac{(4m-3)!}{m!(m-1)!(2m-2)!}$$

are sums of multinomial coefficients
and, hence, are integers.

This can also be stated as: *The function*

$$\frac{1}{\mathfrak{a}} = u + 4u^5 + 76u^9 + 2224u^{13} + \ldots$$

is the logarithm of a formal group law over $\mathbb{Z}$ *which is over* $\mathbb{Z}$ *isomorphic to the additive group law.*

Lerche, Mayr, Warner

'Non-critical Strings, Del Pezzo Singularities and Seiberg-Witten curves'

work in one of their examples with the functions

$$\begin{aligned} \mathfrak{b} &= \int f_0 \frac{du}{u} \\ \mathfrak{b}_D &= \int f_1 \frac{du}{u} \end{aligned}$$

They relate these to periods of *multivalued* 1-*forms* on the elliptic curve $X^4 + Y^4 + Z^2 - tXYZ = 0$.

They also define a function $\mathcal{F}$ so that

$$\mathfrak{b}_D = \frac{d\mathcal{F}}{d\mathfrak{b}}.$$

The functions $\mathfrak{b}$ and $\mathcal{F}$ are then used to construct the potential

$$K = -\log\left(\mathrm{Re}\mathcal{F} + \mathrm{Im}\mathfrak{b}\ \mathrm{Im}\mathfrak{b}_D\right)$$

for a so-called *local special Kähler metric* on the base space (Moduli space) of the pencil.

A Glimpse of Mirror Symmetry

The function $\mathcal{F} - \frac{\mathfrak{b}^3}{6}$ is a function of $e^{\mathfrak{b}}$. It can be written as a series

$$\mathcal{F} \;=\; \frac{1}{6}\mathfrak{b}^3 + \sum_{k\geq 1} a_k \mathrm{Li}_3(e^{k\mathfrak{b}})$$

with *integers* a_k;

here Li_3 is the *trilogarithm function*:

$$\mathrm{Li}_3(x) := \sum_{n\geq 1} \frac{x^n}{n^3}$$

The integers a_k count *instantons*.

More formal group laws.

Recall

$$\mathfrak{b} = \int f_0 \frac{du}{u} = \log u + \sum_{m\geq 1} \frac{(4m)!}{m!^2(2m)!} \frac{u^{4m}}{4m}$$

It can be shown that the series

$$\sum_{m\geq 1} \frac{(4m)!}{m!^2(2m)!} \frac{x^m}{m}$$

is the logarithm of a formal group law over $\mathbb{Z}$ which over $\mathbb{Z}$ is isomorphic to the multiplicative formal group law.

From curves to surfaces.

The total space of our elliptic pencil can be embedded as the surface

$$\mathcal{S}: \qquad (X^4 + Y^4 + Z^2)\,U - XYZ\,T \;=\; 0$$

of bidegree $(4, 1)$ in $\mathbb{P}^{[1,1,2]} \times \mathbb{P}^{[1,1]}$.

The elliptic pencil arises via the projection from $\mathcal{S}$ onto the projective line $\mathbb{P}^{[1,1]}$.

On the other hand, the projection onto the weighted projective plane $\mathbb{P}^{[1,1,2]}$ shows $\mathcal{S}$ as the blow up of this plane in 8 points.

From 1-forms to 2-forms.

In the coordinate patch $Y \neq 0\,,\ U \neq 0$ with affine coordinates $x = \frac{X}{Y},\ z = \frac{Z}{Y}, t = \frac{T}{U}$ one has

$$(x^4 + 1 + z^2) \;-\; xz\,t \;=\; 0$$

and thus

$$(4x^3 - zt)\,dx \;-\; (2z - xt)\,dz \;-\; xz\,dt \;=\; 0$$

and

$$\frac{-\,dx\,dt}{2z - tx} \;=\; \frac{dx}{x}\,\frac{dz}{z}$$

So we have on $\mathcal{S}$ a meromorphic 2-form ω, which on the above coordinate patch is

$$\omega \;=\; \frac{dx}{x}\,\frac{dz}{z}$$

Our previously defined functions ℓ, $\mathfrak{a}$ and $\mathfrak{b}$ can now be written as integrals over some disc $\mathcal{D}$ in $\mathcal{S}$:

$$\ell(u) \;=\; \int f_0\,du \;=\; -\int_{\mathcal{D}} t^{-1}\,\omega$$

$$\mathfrak{a} \;=\; \int f_0\,dt \;=\; \int_{\mathcal{D}} t\,\omega$$

$$\mathfrak{b} \;=\; \int f_0\,\frac{du}{u} \;=\; -\int_{\mathcal{D}} \omega$$

Recall

$$
\begin{aligned}
\mathfrak{b} &= \int f_0 \,\frac{du}{u} \\
&= \int \sum_{n\geq 0} u^n \left(\frac{1}{2\pi i}\right)^2 \oint \oint \left(\frac{x^4+1+z^2}{xz}\right)^n \frac{dx}{x}\,\frac{dz}{z}\,\frac{du}{u} \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint \oint \left[\log u + \sum_{n\geq 1} \frac{1}{n} u^n \left(\frac{x^4+1+z^2}{xz}\right)^n\right] \frac{dx}{x}\,\frac{dz}{z} \\
&= -\left(\frac{1}{2\pi i}\right)^2 \oint \oint \log\left[\frac{txz-(x^4+1+z^2)}{xz}\right] \frac{dx}{x}\,\frac{dz}{z}
\end{aligned}
$$

So $-\mathrm{Re}\,\mathfrak{b}$ is the *logarithmic Mahler measure* of the Laurent polynomial

$$
\frac{txz-(x^4+1+z^2)}{xz}
$$

The *Mahler measure* $M(F)$ of a Laurent polynomial $F(x,y)$ with complex coefficients is defined as:

$$M(F) := \exp(m(F))$$

where $m(F)$ is the *logarithmic Mahler measure*:

$$\begin{aligned} m(F) &:= \int_0^1 \int_0^1 \log |F(e^{2\pi i u}, e^{2\pi i v})| du\, dv \\ &= \frac{1}{(2\pi i)^2} \iint_{|x|=|y|=1} \log |F(x,y)| \frac{dx}{x}\frac{dy}{y}. \end{aligned}$$

C. Smyth, D. Boyd and others found many examples of Laurent polynomials F for which the (logarithmic) Mahler measure equals up to a rational factor and to many decimal places the value at $s=0$ of the derivative of an L-function of the zero locus of F (suitably compactified)

$$\frac{txz - (x^4 + 1 + z^2)}{xz}$$

with integer values for the parameter t, is not in their lists; probably one did not look for it.

Thus, special values of L-functions, the main enumerative problem about Motives, appear alongside with instanton counts, the main enumerative problem about Strings