DIFFRACTION

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Talk at workshop 'Coherent Sheaves and Mirror Symmetry', Cambridge, May 20-22, 2005

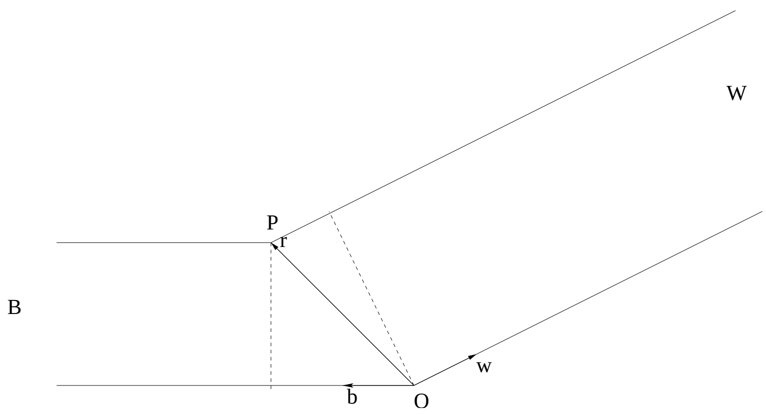
Talk at workshop 'Variations on Mahler's measure', Luminy, May 30 - June 3, 2005

related papers:

1. *Motives from Diffraction.*
   submitted for Proceedings of EAGER conference in honour of Jaap Murre (for his 75th birthday) september 2004
2. *Mahler Measure, Eisenstein Series and Dimers.*
   arXiv:math.NT/0502197
3. *Mahler Measure Variations, Eisenstein Series and Instanton Expansions.*
   arXiv:math.NT/0502193

DIFFRACTION AS MIRROR SYMMETRY:

Frauenhofer model for diffraction



distance BW via P =
distance BW via O $-$ $\langle \mathsf{b}+\mathsf{w}, \mathsf{r}\rangle$

$\rightsquigarrow$ phase difference
$\rightsquigarrow$ multiply by factor $e^{-i\omega\langle \mathsf{b}+\mathsf{w},\mathsf{r}\rangle}$
$\rightsquigarrow$ Sum over $P \approx$ Fouriertransform

General set up

finite $\mathfrak{A} \subset \mathbb{Z}^n$ + weights $c : \mathfrak{A} \to \mathbb{N}$

$$=$$

distribution $\mathcal{D} = \sum_{\mathsf{a}\in\mathfrak{A}} c_{\mathsf{a}}\, \delta_{\mathsf{a}}$

$\Longrightarrow$ Fourier transform $\Longrightarrow$

function on $\mathbb{R}^n$

$\widehat{\mathcal{D}}(\mathsf{t}) = \sum_{\mathsf{a}\in\mathfrak{A}} c_{\mathsf{a}} e^{-2\pi i\langle \mathsf{t},\mathsf{a}\rangle}$

diffraction pattern = level sets $|\widehat{\mathcal{D}}(\mathsf{t})|^2$

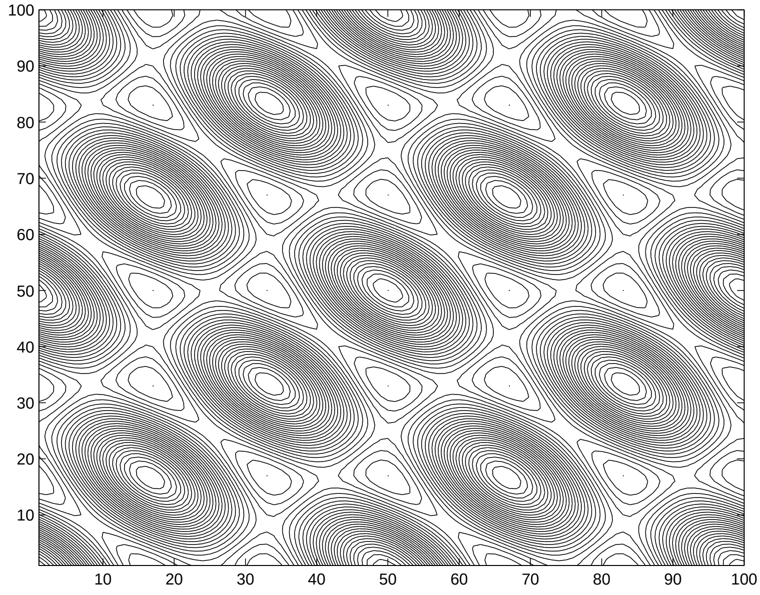
$$\begin{aligned} |\widehat{\mathcal{D}}(\mathsf{t})|^2 &= \sum_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} c_{\mathsf{a}} c_{\mathsf{b}} e^{2\pi i\langle \mathsf{t},\mathsf{a}-\mathsf{b}\rangle} \\ &= \sum_{\mathsf{a},\mathsf{b}\in\mathfrak{A}} c_{\mathsf{a}} c_{\mathsf{b}} \cos(2\pi\langle \mathsf{t},\mathsf{a}-\mathsf{b}\rangle) \end{aligned}$$

EXAMPLE

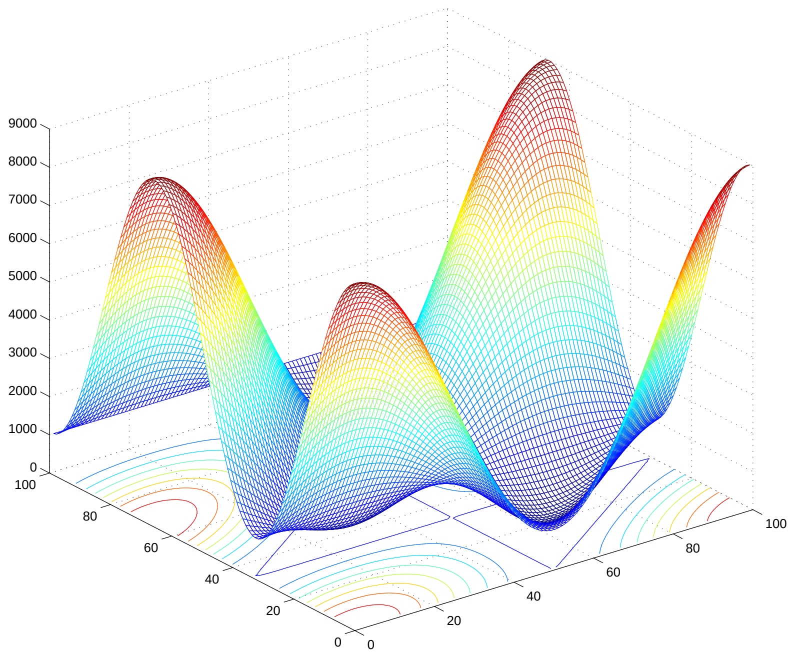
$\mathfrak{A} = \{(1,0),\ (0,1),\ (-1,-1)\} \subset \mathbb{Z}^2,$

$c_{(1,0)} = c_{(0,1)} = c_{(-1,-1)} = 1.$

$$|\widehat{\mathcal{D}}(t_1,t_2)|^2 \;=\; 3 + 2\cos(2\pi(t_1 - t_2)) + \\ +2\cos(2\pi(2t_1 + t_2)) + 2\cos(2\pi(t_1 + 2t_2))$$



Diffraction pattern


9000
8000
7000
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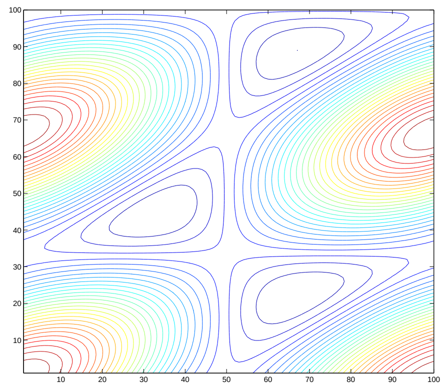
100
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$|\widehat{\mathcal{D}}(\mathsf{t})|^2$ is $\Lambda^\vee$-periodic,

$$\begin{aligned} \Lambda &:= \mathbb{Z}\text{–Span}\{\mathsf{a}-\mathsf{b} \mid \mathsf{a},\mathsf{b} \in \mathfrak{A}\} \\ \Lambda^\vee &:= \{\mathsf{t} \in \mathbb{R}^n \mid \langle \mathsf{t}, \lambda \rangle \in \mathbb{Z}\,,\ \forall \lambda \in \Lambda\} \end{aligned}$$

Assume Λ has rank n.

Introduce enumerative data for $N \in \mathbb{N}$, $r \in \mathbb{R}$

$$\mathrm{mult}_N(r) := \sharp\{\mathsf{t} \in \tfrac{1}{N}\Lambda^\vee \,/\, \Lambda^\vee \mid |\widehat{\mathcal{D}}(\mathsf{t})|^2 = r\,\}$$

$N^{-n}\,\mathrm{mult}_N(r)$ is a density on $\mathbb{R}$ with *moments*

$$\mathsf{m}_k^{(N)} := N^{-n} \sum_r r^k \,\mathrm{mult}_N(r)\,.$$

Encode data $\mathrm{mult}_N(r)$ as

$$\mathsf{B}_N(z) := \prod_{r \in \mathbb{R}} (z - r)^{\mathrm{mult}_N(r)} .$$

$\mathsf{B}_N(z)$ is a polynomial in $\mathbb{Z}[z]$.

Mahler measure will appear in limit

$$\boxed{\lim_{N \to \infty} |\mathsf{B}_N(z)|^{N^{-n}}}$$

EXAMPLE

$\mathfrak{A}=\{(1,0),\ (0,1),\ (-1,-1)\}, \forall c_{\mathsf{a}}=1.$

$$
\begin{aligned}
\mathsf{B}_1 &= (z-9) \\
\mathsf{B}_2 &= (z-9)(z-1)^3 \\
\mathsf{B}_3 &= (z-9)(z-3)^6 z^2 \\
\mathsf{B}_4 &= (z-9)(z-5)^6(z-1)^9 \\
\mathsf{B}_5 &= (z-9)(z^2-8z+11)^6 \times \\
&\quad \times(z^2-3z+1)^6 \\
\mathsf{B}_6 &= (z-9)(z-7)^6(z-4)^6 \times \\
&\quad \times(z-3)^6(z-1)^{15} z^2 \\
\mathsf{B}_7 &= (z-9)(z^3-6z^2+5z-1)^6 \times \\
&\times(z-2)^{12}(z^3-13z^2+47z-43)^6
\end{aligned}
$$

Substitution

$$x_j = e^{2\pi i t_j}$$

$$|\widehat{\mathcal{D}}(\mathsf{t})|^2 \quad \rightsquigarrow \quad \text{Laurent polynomial}$$

$$W(x_1, \ldots, x_n) = \sum_{\mathsf{a},\mathsf{b} \in \mathfrak{A}} c_{\mathsf{a}} c_{\mathsf{b}} \mathsf{x}^{\mathsf{a}-\mathsf{b}}$$

$$\in \mathbb{Z}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$$

Laurent polynomials are familiar objects
in
toric mirror symmetry,
Landau-Ginzburg models,
algebraic dynamical systems,
Mahler measures,.....

Special feature of above W:

$$W(x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1})$$
$$= \ W(x_1, x_2, \ldots, x_n)$$

implies that W is real function on real torus

$$\mathbb{U}^{\Lambda} := \mathrm{Hom}(\Lambda, \mathbb{U})$$

$\mathbb{U} := \{z \in \mathbb{C} \mid |z| = 1\}$ unit circle

$W(\mathbb{U}^{\Lambda})$ is compact interval $\mathcal{I} \ \subset \ \mathbb{R}$.

EXAMPLE

$\mathfrak{A} = \{(1,0),\ (0,1),\ (-1,-1)\},\ \forall c_{\mathsf{a}} = 1$

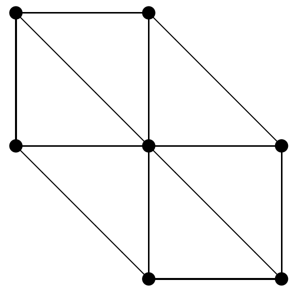
$$W(x_1, x_2) \ = \ (x_1 + x_2 + x_1^{-1}x_2^{-1})(x_1^{-1} + x_2^{-1} + x_1x_2)$$

Basis $\{(2,1),\ (-1,-2)\}$ for Λ gives coordinates $u_1 = x_1^2x_2,\ u_2 = x_1^{-1}x_2^{-2}$ on $\mathbb{U}^{\Lambda}$.

$$W(u_1, u_2) \ = \ (u_1+u_2+1)(u_1^{-1}+u_2^{-1}+1)$$

$$W(u_1, u_2) \;=$$
$$u_1 + u_1^{-1} + u_2 + u_2^{-1} + u_1^{-1}u_2 + u_1u_2^{-1} + 3$$

Triangulated Newton polygon of W:



is toric diagram for Del Pezzo surface dP_3.

Using the Laurent polynomial W

$$\mathsf{B}_N(z) = \prod_{\mathsf{x}\in\mu_N^\Lambda} (z - W(\mathsf{x}))\,,$$

with

$$\mu_N^\Lambda := \mathrm{Hom}(\Lambda, \mu_N)$$

$(\mu_N := \{z \in \mathbb{C} \mid z^N = 1\})\,.$

This shows that $\mathsf{B}_N(z) \in \mathbb{Z}[z]$.

Consider function $\mathsf{Q} : \mathbb{C} \setminus \mathcal{I} \longrightarrow \mathbb{R}_{>0}$

$$\mathsf{Q}(z) = \exp\left(\frac{-1}{(2\pi i)^n}\int_{\mathbb{U}^n} \log|z - W(\mathsf{x})|\,\frac{d\mathsf{x}}{\mathsf{x}}\right)$$

$\mathsf{Q}(z)^{-1}$ is the *Mahler measure* of the Laurent polynomial $z - W(\mathsf{x})$.

THEOREM: For real $z > \max \mathcal{I}$

$$\boxed{\lim_{N\to\infty} |\mathsf{B}_N(z)|^{-N^{-n}} = \mathsf{Q}(z)}$$

PROOF:

$$\log |\mathsf{B}_N(z)|^{-N^{-n}}$$
$$= -\log z + \sum_{k\geq 1} \mathsf{m}_k^{(N)} \frac{z^{-k}}{k}$$

with

$$\mathsf{m}_k^{(N)} = N^{-n} \sum_{\mathsf{x}\in\mu_N^{\Lambda}} W(\mathsf{x})^k$$

So: $\mathsf{m}_k^{(N)}$ is the sum of the coefficients of those monomials in W^k for which the exponent is in $N\Lambda$.

$$\log \mathsf{Q}(z) \;=\; -\log z \;+\; \sum_{k\geq 1} \mathsf{m}_k \frac{z^{-k}}{k}$$

with

$$\mathsf{m}_k := \frac{1}{(2\pi i)^n} \int_{\mathbb{U}^n} W(\mathsf{x})^k \, \frac{d\mathsf{x}}{\mathsf{x}}$$

So: m_k is the constant term of W^k.

$$W(\mathsf{1})^k \;\geq\; \mathsf{m}_k^{(N)} \;\geq\; \mathsf{m}_k \;\geq\; 0$$

for all N, k,

$$\mathsf{m}_k^{(N)} \;=\; \mathsf{m}_k$$

if $N > k \max_{\mathsf{a},\mathsf{b}\in\mathfrak{A},\, 1\leq j\leq n} |a_j - b_j|$.

■

COROLLARY:

For $n = 1$:

$$\lim_{N\to\infty} \frac{\mathsf{B}_{N+1}(z)}{\mathsf{B}_N(z)} = \mathsf{Q}(z)^{-1}$$

For $n = 2$:

$$\lim_{N\to\infty} \frac{\mathsf{B}_{N-1}(z)\mathsf{B}_{N+1}(z)}{\mathsf{B}_N(z)^2} = \mathsf{Q}(z)^{-2}$$

■

EXAMPLE

$\mathfrak{A} = \{(1,0),\ (0,1),\ (-1,-1)\}, \forall c_{\mathsf{a}} = 1$

$$\mathsf{m}_k \ = \ \sum_{j=0}^{k} \binom{k}{j}^2 \binom{2j}{j}$$

$$\mathsf{m}_k^{(N)} \ =$$

$$\sum \binom{k}{i_1} \binom{k-i_1}{j_1} \binom{k}{i_2} \binom{k-i_2}{j_2}$$

sum over all $(i_1,\ j_1, i_2, j_2)$ s.t.
$(i_1,\ j_1) \equiv (i_2, j_2) \bmod N$

Confirms $\mathsf{m}_k^{(N)} = \mathsf{m}_k$ for $k < N$.

The numbers m_k $(k \geq 0)$ satisfy the recurrence

$$(k+1)^2 \mathsf{m}_{k+1} = (10k^2 + 10k + 3)\mathsf{m}_k - 9k^2 \mathsf{m}_{k-1}\,.$$

This recurrence is equivalent to the Picard-Fuchs differential equation for

$$z \frac{d}{dz} \log \mathsf{Q}(z)\,.$$

and, thus, yields quantum cohomology and instanton counts for dP_3.

Laurent polynomial $F(x_1, \ldots, x_n)$
logarithmic Mahler measure

$$\mathsf{m}(F) := \frac{1}{(2\pi i)^n} \oint_{\mathbb{U}^n} \log |F(\mathsf{x})| \frac{d\mathsf{x}}{\mathsf{x}}$$

Mahler measure

$$\mathsf{M}(F) := \exp(\mathsf{m}(F)) .$$

Boyd found many (two-variable) Laurent polynomials for which

$$\mathsf{m}(F) \cdot \mathbb{Q}^* \;=\; \mathrm{L}'(Z_F, 0) \cdot \mathbb{Q}^* .$$

Z_F curve with equation $F = 0$

Boyd's table 2 deals with

$$X^2Y + XY^2 + X^2Z + XZ^2 + \\ +Y^2Z + YZ^2 - kXYZ$$

for $k \in \mathbb{Z}$.

This is the homogeneous form of

$$(u_1 + u_2 + 1)(u_1^{-1} + u_2^{-1} + 1) - 3 - k$$

Recall

$$-\log \mathsf{Q}(z) \;=\; \lim_{N\to\infty} \; N^{-n}\log|\mathsf{B}_N(z)|$$

$-\log \mathsf{Q}(z)$ is *log. Mahler measure* of Laurent polynomial $F(\mathsf{x}) = z - W(\mathsf{x})$

$$-\log \mathsf{Q}(z) \;=\; \mathsf{m}(F)$$

Boyd's numerically verified results

$$\mathsf{m}(F) \;\equiv\; \mathrm{L}'(Z_F, 0) \bmod \mathbb{Q}^*\,.$$

This suggests a link between enumerative data on the level sets of

- W as a function on $\bigcup_N \mu_N^\Lambda \subset \mathbb{U}^\Lambda$

- W as a function on $\bigcup_{p,\nu} \mathbb{F}_{p^\nu}^n$

WINDOW TO FINITE FIELDS

p prime and $\nu \in \mathbb{Z}_{>0}$.

$\mathbb{W}(\mathbb{F}_{p^\nu})$ = Witt vectors of $\mathbb{F}_{p^\nu}$

is complete DVR
maximal ideal $p\mathbb{W}(\mathbb{F}_{p^\nu})$
residue field $\mathbb{F}_{p^\nu}$

Teichmüller lift $\tau : \mathbb{F}_{p^\nu} \longrightarrow \mathbb{W}(\mathbb{F}_{p^\nu})$
$x \equiv \tau(x) \bmod p$
$\tau(xy) = \tau(x)\tau(y) \qquad \forall x, y \in \mathbb{F}_{p^\nu}$.

$$x^{p^\nu - 1} = 1 \qquad \forall x \in \mathbb{F}^*_{p^\nu}$$

Injective homomorphism

$$j : \ \mu_{p^\nu - 1} \simeq \mathbb{F}^*_{p^\nu} \overset{\tau}{\hookrightarrow} \mathbb{W}(\mathbb{F}_{p^\nu})$$

Thus

$$\mathsf{x} \in \mu^{\Lambda}_{p^\nu - 1} \quad \Rightarrow \quad W(j(\mathsf{x})) \in \mathbb{W}(\mathbb{F}_{p^\nu})$$

$$\mathsf{B}_{p^\nu - 1}(z) = \prod_{\mathsf{x} \in \mu^{\Lambda}_{p^\nu - 1}} (z - W(\mathsf{x}))\,,$$

p-adic valuation on $\mathbb{Z}$ and $\mathbb{W}(\mathbb{F}_{p^\nu})$

$$v_p(k) \ := \ \max\{v \in \mathbb{Z} \mid p^v \text{ divides } k\}\,.$$

Proposition: p, ν as above , $z \in \mathbb{Z}$

$v_p(\mathsf{B}_{p^\nu - 1}(z)) \geq$

$\sharp\{\xi \in (\mathbb{F}^*_{p^\nu})^n \mid W(\xi) = z \text{ in } \mathbb{F}_{p^\nu}\ \}\,.$