A variation of mixed Hodge structure for a special case of Appell's F_4

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dedicated to Mr Taniguchi on the occasion of his ninetieth birthday

Abstract

We present a variation of mixed Hodge structure of geometric origin for Appell's system of differential equations of type F_4 with parameters $0, \frac{1}{2}, 1, 1$ and variables τ_1^2, τ_2^2.

1 Appell's F_4

In [1] Appell introduced the hypergeometric series of type F_4 with parameters $\alpha, \beta, \gamma, \gamma'$ and variables u_1, u_2:

$$F_4(\alpha, \beta, \gamma_1, \gamma_2, u_1, u_2) \;=\; \sum_{m,n \geq 0} \frac{(\alpha, m+n)(\beta, m+n)}{(\gamma_1, m)(\gamma_2, n)(1, m)(1, n)}\, u_1^m\, u_2^n \;.$$

Here $(\lambda, k) = \lambda \cdot (\lambda+1) \cdots (\lambda+k-1)$. It is one solution for the system of hypergeometric differential equations, with $D_i = u_i \frac{\partial}{\partial u_i}$, $i = 1, 2$,

$$D_i(D_i + \gamma_i - 1)\, Z = u_i\, (D_1 + D_2 + \alpha)(D_1 + D_2 + \beta)\, Z \;.$$

In this paper we want to study Appell's system of differential equations for $(\alpha, \beta, \gamma, \gamma') = (0, \frac{1}{2}, 1, 1)$ and $u_1 = \tau_1^2$, $u_2 = \tau_2^2$: with $\mathcal{D}_i = \tau_i \frac{\partial}{\partial \tau_i}$, $i = 1, 2$,

$$\mathcal{D}_i^2\, Z \;=\; \tau_i^2\, (\mathcal{D}_1 + \mathcal{D}_2)(\mathcal{D}_1 + \mathcal{D}_2 + 1)\, Z \;. \tag{1}$$

This system turns out to be quite interesting, although from the point of view of Appell's function F_4 it looks somewhat degenerate. Set

$$\begin{aligned}
\Delta &= (1+\tau_1+\tau_2)(1+\tau_1-\tau_2)(1-\tau_1+\tau_2)(1-\tau_1-\tau_2) \\
\Gamma &= 1 - \tau_1^2 - \tau_2^2 \\
\Lambda_\pm &= ((2\tau_1^2 \pm \Gamma)\, \tau_2\, d\tau_1 \pm (2\tau_2^2 \pm \Gamma)\, \tau_1\, d\tau_2)/(2\tau_1\tau_2)
\end{aligned}$$

Let $Z_\pm = (\mathcal{D}_1 Z \pm \mathcal{D}_2 Z)/2$ and $Z_3 = (\mathcal{D}_1 Z + \mathcal{D}_2 Z + 2\mathcal{D}_1\mathcal{D}_2 Z)/2\Gamma$. Noticing that the differential equations can be rewritten as $\mathcal{D}_i^2 Z = 2\tau_i^2 Z_3$, one easily deduces the following Pfaffian system equivalent to (1)

$$d\begin{bmatrix} Z \\ Z_- \\ Z_+ \\ Z_3 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{d(\tau_1/\tau_2)}{\tau_1/\tau_2} & \dfrac{d(\tau_1\tau_2)}{\tau_1\tau_2} & 0 \\ 0 & 0 & \dfrac{d(\tau_1/\tau_2)}{2\tau_1/\tau_2} & \Lambda_- \\ 0 & 0 & -\dfrac{d(\tau_1\tau_2)}{2\tau_1\tau_2} & \Lambda_+ \\ 0 & 0 & -\dfrac{\Lambda_+}{\Delta} & \dfrac{d(\tau_1\tau_2)}{2\tau_1\tau_2} - \dfrac{d\Delta}{\Delta} \end{bmatrix} \begin{bmatrix} Z \\ Z_- \\ Z_+ \\ Z_3 \end{bmatrix} \tag{2}$$

2 Variation of mixed Hodge structure

We will associate with (2) a variation of mixed Hodge structure $(\mathbf{V}_\mathbf{Z}, W, \mathcal{F})$ of geometric origin [6]. In this section we describe this VMHS abstractly. In the subsequent sections it will be realized in geometry.

The base space $\mathcal{B}$ is the complement in the projective plane $\mathbf{P}^2$ of the complete quadrilateral with equation

$$t_1 t_2 (t_0 + t_1 + t_2)(t_0 + t_1 - t_2)(t_0 - t_1 + t_2)(t_0 - t_1 - t_2) \,=\, 0 \tag{3}$$

The construction of the vector bundle with connection works completely within the algebraic geometry of schemes over $\mathbf{Z}[\frac{1}{2}]$. The vector bundle $\mathcal{V}$ underlying the VMHS has rank 4 and is generated by global sections $|\omega_1|$, $|\omega_2|$, $|\omega_3|$, $|\omega_4|$. The weight filtration $W_0\mathcal{V} \subset W_1\mathcal{V} \subset W_2\mathcal{V} \subset W_3\mathcal{V} \subset W_4\mathcal{V}$ is

$$0 \,\subset\, \mathcal{O}_\mathcal{B}\,|\omega_3| \oplus \mathcal{O}_\mathcal{B}\,|\omega_4| \,\subset\, W_1\mathcal{V} \oplus \mathcal{O}_\mathcal{B}\,|\omega_2| \,=\, W_3\mathcal{V} \subset \mathcal{V}\,.$$

The Hodge filtration $\mathcal{F}^0\mathcal{V} \supset \mathcal{F}^1\mathcal{V} \supset \mathcal{F}^2\mathcal{V} \supset \mathcal{F}^3\mathcal{V}$ is given by

$$\mathcal{V} \,\supset\, \mathcal{O}_\mathcal{B}|\omega_1| \oplus \mathcal{O}_\mathcal{B}|\omega_2| \oplus \mathcal{O}_\mathcal{B}|\omega_3| \,\supset\, \mathcal{O}_\mathcal{B}|\omega_1| \,\supset\, 0\,.$$

The vector bundle carries a connection compatible with the weight filtration

$$\nabla : \mathcal{V} \to \Omega^1_{\mathcal{B}/\mathbf{Z}} \otimes \mathcal{V}\,.$$

With respect to the basis $|\omega_1|, |\omega_2|, |\omega_3|, |\omega_4|$ the connection is given by

$$\nabla\,\underline{\omega} \,=\, L\,\underline{\omega} \tag{4}$$

where L is the connection matrix from (2) and $\underline{\omega} = (|\omega_1|, |\omega_2|, |\omega_3|, |\omega_4|)^t$.

For the local system on $\mathcal{B}$ we must pass to analytic geometry and view $\mathcal{B}$ as a complex analytic space. The local system $\mathbf{V_Z}$ on $\mathcal{B}$ and its relation with $\mathcal{V}$ are most easily described over the universal covering space $\tilde{\mathcal{B}}$ of $\mathcal{B}$. Let $q : \tilde{\mathcal{B}} \to \mathcal{B}$ be the covering projection. Then $\tilde{\mathbf{V}}_\mathbf{Z} = q^*\mathbf{V_Z}$ is a constant sheaf on $\tilde{\mathcal{B}}$, equal to a free $\mathbf{Z}$-module with basis e_1, e_2, e_3, e_4. The weight filtration $W_0\tilde{\mathbf{V}}_\mathbf{Z} \subset W_1\tilde{\mathbf{V}}_\mathbf{Z} \subset W_2\tilde{\mathbf{V}}_\mathbf{Z} \subset W_3\tilde{\mathbf{V}}_\mathbf{Z} \subset W_4\tilde{\mathbf{V}}_\mathbf{Z}$ is

$$0 \subset \mathbf{Z}\,\mathsf{e}_3 \oplus \mathbf{Z}\,\mathsf{e}_4 \subset W_1\tilde{\mathbf{V}}_\mathbf{Z} \oplus \mathbf{Z}\,\mathsf{e}_2 = W_3\tilde{\mathbf{V}}_\mathbf{Z} \subset \tilde{\mathbf{V}}_\mathbf{Z}\,.$$

There is an isomorphism which is compatible with the weight filtrations

$$\mathcal{I} :\ \mathcal{V} \otimes_{\mathcal{O}_\mathcal{B}} \mathcal{O}_{\tilde{\mathcal{B}}} \ \stackrel{\simeq}{\to}\ \tilde{\mathbf{V}}_\mathbf{Z} \otimes_\mathbf{Z} \mathcal{O}_{\tilde{\mathcal{B}}}\,.$$

With respect to the bases $\underline{\omega}$ and $\underline{\mathsf{e}}$ it is described by a matrix $\mathcal{P} = (\mathcal{P}_{ij})$ of analytic functions on $\tilde{\mathcal{B}}$:

$$\mathcal{I}\,\underline{\omega} = \mathcal{P}\,\underline{\mathsf{e}}\,. \tag{5}$$

Here $\underline{\mathsf{e}}$ is the column vector $(\mathsf{e}_1, \mathsf{e}_2, \mathsf{e}_3, \mathsf{e}_4)^t$. The frame $\underline{\mathsf{e}}$ is horizontal:

$$\nabla\,\mathsf{e}_i = 0\,, \qquad i = 1, 2, 3, 4. \tag{6}$$

From (4)-(6) one sees that the columns of $\mathcal{P}$ satisfy the differential equation (2) and that $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, $\mathcal{P}_{13}$, $\mathcal{P}_{14}$ span the solution space of Appell's system (1) over $\mathbf{C}$. This only involves the $\mathbf{C}$-vector space $\tilde{\mathbf{V}}_\mathbf{Z} \otimes \mathbf{C}$.

Fix a base point $b \in \mathcal{B}$. The natural action of the fundamental group $\pi_1 = \pi_1(\mathcal{B}, b)$ on $\mathcal{B}$ induces a representation ν of π_1 on the sheaf of functions $\mathcal{O}_{\tilde{\mathcal{B}}}$. Let $\mathbf{1}$ denote the trivial representation of π_1 on $\mathcal{V}$. The monodromy representation μ of π_1 on $\tilde{\mathbf{V}}_\mathbf{Z}$ is such that $\mathcal{I}$ is actually an isomorphism of representations $\mathcal{I} : \mathbf{1} \otimes \nu \simeq \mu \otimes \nu$. The integral structure $\tilde{\mathbf{V}}_\mathbf{Z}$ comes in if one wants the monodromy representation to take values in $GL_4(\mathbf{Z})$.

3 A family of elliptic curves minus four points

This section realizes $(W_1\mathcal{V} \subset W_2\mathcal{V},\ \nabla)$ in the de Rham cohomology of a family of elliptic curves minus four points.

Consider in $\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1$ over $\mathbf{Z}[\frac{1}{2}]$ the subscheme $\mathcal{E}$ defined by

$$2t_0\ x_1y_1\ x_2y_2 +\ t_1\ (x_1^2 + y_1^2)\ x_2y_2 +\ t_2\ x_1y_1\ (x_2^2 + y_2^2) = 0\,.$$

Take in $\mathbf{P}^2$ the complete quadrilateral $\boxed{\times}$ defined by (3) and in $\mathbf{P}^1 \times \mathbf{P}^1$ the quadrilateral $\square$: $x_1 y_1 x_2 y_2 = 0$. Let p and p' be the projections from $\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1$ onto $\mathbf{P}^2$ and $\mathbf{P}^1 \times \mathbf{P}^1$, respectively. We define

$$\mathcal{B} = \mathbf{P}^2 \setminus \boxed{\times}\,, \quad \mathcal{A} = \mathcal{E} \cap p^{-1}(\mathcal{B})\,, \quad \mathcal{S} = \mathcal{A} \cap p'^{-1}(\square)\,, \quad \mathcal{A}^\circ = \mathcal{A} \setminus \mathcal{S}\,.$$

Then $p : \mathcal{A} \to \mathcal{B}$ is a family of elliptic curves over the base $\mathcal{B}$ and $\mathcal{S}$ consists of the four sections (00), (0∞), $(\infty 0)$ and $(\infty\infty)$.

On $\mathcal{A}^\circ$ we use the inhomogeneous coordinates $\xi_1 = x_1/y_1$, $\xi_2 = x_2/y_2$. For the computations showing that the Gauss-Manin connection on a subspace of the de Rham cohomology of $\mathcal{A}^\circ/\mathcal{B}$ matches the lower right hand 3×3-block of (2) it suffices to work on $\mathcal{B} \setminus \{t_0 = 0\}$ and use the inhomogeneous coordinates $\tau_1 = t_1/t_0$, $\tau_2 = t_2/t_0$. In these coordinates $\mathcal{A}^\circ$ is described by the equation

$$\mathcal{A}^\circ : \qquad 2 + \tau_1(\xi_1 + \xi_1^{-1}) + \tau_2(\xi_2 + \xi_2^{-1}) \;=\; 0\,.$$

Consider the three differential forms on $\mathcal{A}/\mathcal{B}$:

$$\omega_3 = \frac{-1}{\tau_1(\xi_1 - \xi_1^{-1})}\,\frac{d\xi_2}{\xi_2}$$

$$\omega_4 = \frac{2(\xi_1 + \xi_1^{-1})}{\tau_1^2(\xi_1 - \xi_1^{-1})^3}\,\frac{d\xi_2}{\xi_2}$$

$$\omega_2 = \frac{\tau_1(\xi_1 + \xi_1^{-1}) \;-\; \tau_2(\xi_2 + \xi_2^{-1})}{2\tau_1(\xi_1 - \xi_1^{-1})}\quad \frac{d\xi_2}{\xi_2}\,.$$

These are forms of the first, second and third kind respectively. Their cohomology classes $|\omega_3|$, $|\omega_4|$, $|\omega_2|$ are elements of the de Rham cohomology group $\mathbf{H}^1(\mathcal{A}, \Omega^\bullet_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S})))$, hypercohomology of the complex of differential forms on $\mathcal{A}$ relative to $\mathcal{B}$ with at most logarithmic singularities along $\mathcal{S}$. The 3-dimensional subspace spanned by these classes can be characterized as the $+1$ eigenspace of the involution σ on $\mathbf{H}^1(\mathcal{A}, \Omega^\bullet_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S})))$ induced by the involution σ of $\mathcal{A}^\circ/\mathcal{B}$ defined by

$$\sigma(\xi_1) = \xi_1^{-1}\,, \quad \sigma(\xi_2) = \xi_2^{-1}\,.$$

$\mathbf{H}^1(\mathcal{A}, \Omega^\bullet_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S})))^{\sigma=1}$ carries a Gauss-Manin connection ∇ [3, §4]. One can show (the proofs will appear elsewhere)

$$\nabla \begin{pmatrix} |\omega_2| \\ |\omega_3| \\ |\omega_4| \end{pmatrix} = L \begin{pmatrix} |\omega_2| \\ |\omega_3| \\ |\omega_4| \end{pmatrix}$$

where L is the lower right hand 3×3 block of the matrix in (2). It is well known that the vector bundle with connection $(\mathbf{H}^1(\mathcal{A}, \Omega^\bullet_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S})))^{\sigma=1}, \nabla)$ underlies a variation of mixed Hodge structure on $\mathcal{B}$ [6]. The local system of this VMHS is $\mathbf{R}^1 p^\circ_* \mathbf{Z}_{\mathcal{A}^\circ}$, where $p^\circ : \mathcal{A}^\circ \to \mathcal{B}$ is the restriction of p and $\mathbf{Z}_{\mathcal{A}^\circ}$ is the constant sheaf $\mathbf{Z}$ on $\mathcal{A}^\circ$.

We use the following notation emphasizing the weight filtration

$$\begin{array}{lllllll} W_1\mathcal{V} & = & \mathbf{H}^1(\mathcal{A}, \Omega^\bullet_{\mathcal{A}/\mathcal{B}})\,, & \qquad & W_2\mathcal{V} & = & \mathbf{H}^1(\mathcal{A}, \Omega^\bullet_{\mathcal{A}/\mathcal{B}}(\log \mathcal{S}))^{\sigma=1} \\ W_1\mathbf{V}_\mathbf{Z} & = & \mathbf{R}^1 p_* \mathbf{Z}_\mathcal{A}\,, & \qquad & W_2\mathbf{V}_\mathbf{Z} & = & (\mathbf{R}^1 p^\circ_* \mathbf{Z}_{\mathcal{A}^\circ})^{\sigma=1} \end{array}$$

This VMHS is an extension of the Tate Hodge structure $\mathbf{Z}(-1)$, viewed as a constant VHS on $\mathcal{B}$, by the VHS associated with the family $\mathcal{A} \to \mathcal{B}$ of complete elliptic curves.

4 A rank 4 VMHS; Bloch's regulator

There is an obvious and simple way to extend the rank 3 bundle $W_2\mathcal{V}$ to a rank 4 bundle $\mathcal{V}$ with connection giving the full 4×4 connection matrix of (2); namely

$$\mathcal{V} = W_2\mathcal{V} \oplus \mathcal{O}_\mathcal{B} \cdot |\omega_1|$$

with connection ∇ inducing the Gauss-Manin connection on $W_2\mathcal{V}$ and such that for the global section $|\omega_1|$ of $\mathcal{V}$

$$\nabla|\omega_1| \;=\; \frac{d(\tau_1/\tau_2)}{\tau_1/\tau_2}\,|\omega_2| + \frac{d(\tau_1\tau_2)}{\tau_1\tau_2}\,|\omega_3|\,.$$

In order to get a rank 4 VMHS we must also extend the local system. For this we use Bloch's regulator map and some K-theory.

Algebraic K-theory [5] constructs for every scheme $\mathcal{X}$ abelian groups $K_i(\mathcal{X})$, $i \geq 0$. For a commutative ring R with 1 one sets $K_i(R) = K_i(\mathrm{Spec} R)$. Two invertible elements u, v of R determine an element $\{u, v\}$ of $K_2(R)$ called Steinberg symbol. It satisfies $\{u^{-1}, v^{-1}\} = \{u, v\}$ [4]. In particular one has

$$\{\xi_1, \xi_2\} \in K_2(\mathcal{A}^\circ)^{\sigma=1}\,.$$

Bloch's regulator map [2] for an open Riemann surface $\mathcal{X}$ is defined on the subgroup of $K_2(\mathcal{X})$ generated by the Steinberg symbols and takes values in $\mathbf{H}^1(\mathcal{X}, \mathbf{C}^*)$. The construction applies to Steinberg symbols in K_2 of the fibres of the family $\mathcal{A}^\circ/\mathcal{B}$, but the formulas in [2] can easily be re-interpretated so that they work for $\{\xi_1, \xi_2\}$ in $K_2(\mathcal{A}^\circ)$. For this we need the multiplicative de Rham complex on $\mathcal{A}$ relative to $\mathcal{B}$ with at most logarithmic singularities along $\mathcal{S}$:

$$\Omega^{\times\bullet}_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S})) \;=\; [\, j_*\mathcal{O}^*_{\mathcal{A}^\circ} \xrightarrow{dlog} \Omega^1_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S}))\,]\,.$$

Here j is the inclusion $\mathcal{A}^\circ \hookrightarrow \mathcal{A}$. Let $\mathcal{A}^\circ{}_+$ resp. $\mathcal{A}^\circ{}_- \subset \mathcal{A}^\circ$ be the inverse image of $\mathbf{C}\setminus(-\infty,0]$ resp. $\mathbf{C}\setminus[0,\infty)$ with respect to the function ξ_1. Let $\log_+$ resp. $\log_-$ be a branch of the logarithm on $\mathbf{C}\setminus(-\infty,0]$ resp. $\mathbf{C}\setminus[0,\infty)$. Then

$$\begin{array}{ll} \xi_2^{(\log_-(\xi_1)-\log_+(\xi_1))/2\pi i} & \text{on}\quad \mathcal{A}^\circ{}_+\cap\mathcal{A}^\circ{}_- \\ \frac{-1}{2\pi i}\log_\pm(\xi_1)\frac{d\xi_2}{\xi_2} & \text{on}\quad \mathcal{A}^\circ{}_\pm \end{array} \tag{7}$$

gives a Čech cocycle for $\Omega^{\times\bullet}_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S}))$ and hence represents an element $r\{\xi_1,\xi_2\}$ of its $\mathbf{H}^1$. Formula (7) shows that $r\{\xi_1,\xi_2\}$ is invariant under the involution σ :

$$r\{\xi_1,\xi_2\} \;\in\; \mathbf{H}^1(\mathcal{A},\Omega^{\times\bullet}_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S})))^{\sigma=1}\,.$$

One can immitate the Katz-Oda construction for the Gauss-Manin connection and construct a map

$$\mathbf{H}^1(\mathcal{A},\Omega^{\times\bullet}_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S}))) \xrightarrow{\nabla} \Omega^1_{\mathcal{B}/\mathbf{C}}\otimes\mathbf{H}^1(\mathcal{A},\Omega^{\bullet}_{\mathcal{A}/\mathcal{B}}(\log(\mathcal{S}))) \tag{8}$$

which composes with the map, coming from the exponential sequence,

$$\mathbf{H}^1(\mathcal{A},\Omega^{\bullet}_{\mathcal{A}/\mathcal{B}}(\log\mathcal{S}))) \to \mathbf{H}^1(\mathcal{A},\Omega^{\times\bullet}_{\mathcal{A}/\mathcal{B}}(\log\mathcal{S}))\,, \tag{9}$$

to the usual Gauss-Manin connection on $\mathbf{H}^1(\mathcal{A},\Omega^{\bullet}_{\mathcal{A}/\mathcal{B}}(\log\mathcal{S})))$. One can easily compute $\nabla r\{\xi_1,\xi_2\}$ in (8):

$$2\pi i\nabla r\{\xi_1,\xi_2\} \;=\; -\frac{d\xi_1}{\xi_1}\frac{d\xi_2}{\xi_2} \;=\; \frac{d(\tau_1/\tau_2)}{\tau_1/\tau_2}\,\omega_2 + \frac{d(\tau_1\tau_2)}{\tau_1\tau_2}\,\omega_3 \;=\; \nabla|\omega_1|\,. \tag{10}$$

Now we need the universal covering space $\tilde{\mathcal{B}}$ of $\mathcal{B}$. Let $\tilde{\mathcal{A}}$, $\tilde{\mathcal{S}}$, $\tilde{\mathcal{A}}^\circ$ be the fibre products over $\mathcal{B}$ of $\tilde{\mathcal{B}}$ with $\mathcal{A}$, $\mathcal{S}$, $\mathcal{A}^\circ$, respectively. The involution σ lifts to $\tilde{\mathcal{A}}^\circ/\tilde{\mathcal{B}}$. The obvious analogue of (7) defines an element $\tilde{r}\{\xi_1,\xi_2\}$ in $\mathbf{H}^1(\tilde{\mathcal{A}},\Omega^{\times\bullet}_{\tilde{\mathcal{A}}/\tilde{\mathcal{B}}}(\log(\tilde{\mathcal{S}})))^{\sigma=1}$. It can be shown (details will appear elsewhere) that in the analogue of (9) $\tilde{r}\{\xi_1,\xi_2\}$ can be lifted to an element R in $\mathbf{H}^1(\tilde{\mathcal{A}}_{\tilde{\mathcal{B}}},\Omega^{\bullet}_{\tilde{\mathcal{A}}/\tilde{\mathcal{B}}}(\log(\tilde{\mathcal{S}})))^{\sigma=1}$. Define

$$\mathsf{e}_1 = \frac{1}{(2\pi i)^2}|\omega_1| - \frac{1}{2\pi i}\,R \tag{11}$$

This is a section of the vector bundle $\tilde{\mathcal{V}} = \mathcal{V} \otimes \mathcal{O}_{\tilde{\mathcal{B}}}$. Because of (10) e_1 is a horizontal section:

$$\nabla \mathsf{e}_1 \; = \; 0 \, .$$

The vector bundles $W_1\mathcal{V}$, $W_2\mathcal{V}$ pull back to the vector bundles $W_1\mathcal{V} \otimes \mathcal{O}_{\tilde{\mathcal{B}}}$, $W_2\mathcal{V} \otimes \mathcal{O}_{\tilde{\mathcal{B}}}$ on $\tilde{\mathcal{B}}$. The local systems $W_1\mathbf{V}_{\mathbf{Z}}$, $W_2\mathbf{V}_{\mathbf{Z}}$ pull back to constant sheaves $W_1\tilde{\mathbf{V}}_{\mathbf{Z}}$, $W_2\tilde{\mathbf{V}}_{\mathbf{Z}}$ on $\tilde{\mathcal{B}}$. Choosing a base point $b \in \mathcal{B}$ we have

$$\begin{array}{rcl} W_1\tilde{\mathbf{V}}_{\mathbf{Z}} & = & H^1(\mathcal{A}_b, \mathbf{Z}) \\ W_2\tilde{\mathbf{V}}_{\mathbf{Z}} & = & H^1(\mathcal{A}^{\circ}{}_b, \mathbf{Z})^{\sigma=1} \end{array}$$

where $\mathcal{A}_b$ and $\mathcal{A}^{\circ}{}_b$ are the fibres over b. We can choose any basis e_3, e_4 for the free rank 2 $\mathbf{Z}$-module $W_1\tilde{\mathbf{V}}_{\mathbf{Z}}$ and extend to a basis e_2, e_3, e_4 for the free rank 3 $\mathbf{Z}$-module $W_2\tilde{\mathbf{V}}_{\mathbf{Z}}$ so that under the period isomorphism from Hodge theory $W_2\mathcal{V} \otimes \mathcal{O}_{\tilde{\mathcal{B}}} \simeq W_2\tilde{\mathbf{V}}_{\mathbf{Z}} \otimes \mathcal{O}_{\tilde{\mathcal{B}}}$ we have

$$|\omega_2| \equiv 2\pi i \mathsf{e}_2 \bmod W_1\mathcal{V} \otimes \mathcal{O}_{\tilde{\mathcal{B}}}$$

Define

$$\tilde{\mathbf{V}}_{\mathbf{Z}} \; = \; \mathbf{Z}\, \mathsf{e}_1 \oplus W_3\tilde{\mathbf{V}}_{\mathbf{Z}} \; \subset \; \tilde{\mathcal{V}}$$

The fundamental group $\pi_1 \; = \; \pi_1(\mathcal{B}, b)$ acts on $\tilde{\mathcal{B}}$, $\tilde{\mathcal{A}}$, $\tilde{\mathcal{A}}^{\circ}$, the various de Rham complexes and their cohomology. $\tilde{r}\{\xi_1, \xi_2\}$ is the image of $r\{\xi_1, \xi_2\}$ under the map on cohomology induced by the covering map $q : \tilde{\mathcal{B}} \to \mathcal{B}$. So it is invariant under the action of π_1. Note also that $|\omega_1|$ is π_1-invariant. Therefore, for $\phi \in \pi_1$,

$$\phi\, \mathsf{e}_1 \; - \; \mathsf{e}_1 \; = \; \frac{1}{2\pi i}(R \; - \; \phi\, R) \; \in W_3\tilde{\mathbf{V}}_{\mathbf{Z}}$$

We see that

for the given choice of e_1, e_2, e_3, e_4 *and* $\tilde{\mathbf{V}}_{\mathbf{Z}}$ *the monodromy representation takes values in* $GL_4(\mathbf{Z})$*:*

$$\pi_1(\mathcal{B}, b) \to \mathrm{Aut}_{\mathbf{Z}}(\tilde{\mathcal{V}}) \simeq GL_4(\mathbf{Z}) \, .$$

This completes the construction of the rank 4 variation of mixed Hodge structure for Appell's system of differential equations of type F_4 with parameters $0, \frac{1}{2}, 1, 1$ and variables τ_1^2, τ_2^2.

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