SHOREFACE CONNECTED RIDGES

TIDAL RIDGES
- Tidal current
- No shore-line influence

SHOREFACE CONNECTED RIDGES
- Storm driven current

SHORE OBLIQUE SAND RIDGES
or ‘shore-face connected ridges’ (SFCR) are observed on storm-driven sandy shelves

Sand ridges (Belgio) L=50Km, L=5Km
**BEDFORM CHARACTERISTICS**

- Lengths of the order of ten kilometres
- Crest to crest distances of the order of kilometres
- Typical heights of the order of several metres
- Crests forming an angle with respect to the direction orthogonal to the coastline (upstream rotated)
- Dynamic bed forms (they migrate with a speed of several metres per year)

Example of Middle Atlantic and South Atlantic Bights

**WHAT ARE THE MOTIVATIONS TO STUDY SFCR?**
Do shore-oblique sand ridges protect the coast?
Do shore-oblique sand ridges create a sheltered area for fish growth?

Can shore-oblique sand ridges be used as sand pits?
Sand extraction from the crest of a shore-oblique sand ridge

Where the sand comes from? Offshore or from the beach face?
Which are the mechanisms controlling shore-oblique sand ridge formation, development and their interaction with human interventions?

THEORETICAL ANALYSIS OF THE APPEARANCE OF SHORE-OBLIQUE SAND RIDGE

Objective: demonstrate that sfcr can form spontaneously as free instabilities of the coupled water bottom system

Methodology: stability analysis
1 step: model formulation
2 step: basic state = longshore uniform morphodynamic equilibrium state (no bedforms)
3 step: introduction of a small bed perturbation
4 step: evaluation of the flow response
5 step: evaluation of the sediment transport rate
6 step: evaluation of the convergence/divergence of the sediment over the crests of the bed perturbation
7 step: evaluation of the perturbation growth of the perturbation
A SIMPLE APPROACH: 2DH MODEL
(Trowbridge 1995, J. Geophys. Res. 100, C8, 16071)

AIM
Investigation of the growth of shore-oblique sand ridges due to storms (there are also similar bedforms generated by tidal currents)

ASSUMPTION

Since the length scale of the bottom forms is much larger than the local water depth a 2DH Model (x-y plane) is used which follows by averaging the three-dimensional shallow water equations over the vertical.

Let us consider a sea bed like that shown in the figure, such that the initial water depth $\bar{D}$ depends on the cross-shore coordinate $x$. Moreover the basic current $V$ forced by storms is supposed to be unidirectional and along-shore directed.
HYDRODYNAMICS (SHALLOW WATER EQUATIONS)

Continuity equation
\[
\frac{\partial D}{\partial t} + \frac{\partial (DU)}{\partial x} + \frac{\partial (DV)}{\partial y} = 0
\]

Momentum equation along the x-direction
\[
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = 0 = g \frac{\partial \zeta}{\partial x} - \frac{\tau_x}{\rho D}
\]

Momentum equation along the y-direction
\[
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + fU = 0 = g \frac{\partial \zeta}{\partial y} - \frac{\tau_y}{\rho D}
\]

Sediment continuity equation
\[
\frac{\partial h}{\partial t} = - \frac{1}{(1-n)} \left[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right]
\]

\[ f = 2\Omega \sin \phi \]

For a unidirectional unsteady current, the bed shear stress is usually written in the form
\[ \tau = \rho C_d U_i |U_i| \]
\[ C_d = 1/C^2 \] being a resistance coefficient and \( U_i \) the total velocity

In the coastal region
\[ U_i = U + u_w \]
\( (u_w \) being the velocity induced by the propagation of wind waves and \( U \) the current velocity)

During storms \( u_w < U \)
Since $U < u_w$, we can write

$$\tau = \rho C_d \left[ u_w |u_w| + U |u_w| + u_w |U| + O(U^2) \right]$$

Then, the time average of $\tau$ turns out to be

$$\bar{\tau} = \rho C_d \left[ u_w |U| + O(U^2) \right]$$

Therefore, for 2D flows, it is reasonable to assume

$$(\tau_x, \tau_y) = \rho r(x)(U, V)$$

where $r(x)$ depends on characteristics of the local wave field.

---

**SPATIAL AND TEMPORAL SCALES FOR THE PERTURBATION PROBLEM**

a) The forcing storms have a duration of the order of hours, i.e. $10^4$ s
b) Coriolis time scale ($f^{-1}$) of the order of $10^4$ s
c) Advection time scale $l/V$ ($l=$bedform length, $V=$depth averaged velocity) of the order of $10^3$ s
d) Adjustment time scale of the velocity field $D/u_r=D/(V\sqrt{C_D})$
   of the order of $10^2$ s ($C_D=1/C^2$)

\[
\frac{\partial \bar{u}}{\partial t} = \frac{\partial}{\partial z} \left[ v_T \frac{\partial \bar{u}}{\partial z} \right]
\]

\[
\frac{U}{T} \approx \frac{V}{v_T} \frac{u_r D}{D^2} \approx \frac{D^2}{u_r D} \approx \frac{D}{V \sqrt{C_D}} \approx \frac{DC}{V}
\]
SPATIAL AND TEMPORAL SCALES FOR THE PERTURBATION PROBLEM

a) The forcing storms have a duration of the order of hours, i.e. $10^4$ s
b) Coriolis time scale $(f^{-1})$ of the order of $10^4$ s
c) Advection time scale $l/V$ ($l=$bedform length, $V=$depth averaged velocity) of the order of $10^3$ s
d) Adjustment time scale of the velocity field $D/u = D/(V\sqrt{C_D})$ of the order of $10^2$ s ($C_D=1/C^2$)
e) Adjustment time scale for sediment concentration $D/w_s$ of the order of $10^2$ s

\[ F_D = f_D \rho w_s^2 A = f_D \rho w_s^2 \pi \frac{d^2}{4} \]

\[ G = (\rho_s - \rho) g \pi \frac{d^3}{6} \]

SPATIAL AND TEMPORAL SCALES FOR THE PERTURBATION PROBLEM

a) The forcing storms have a duration of the order of hours, i.e. $10^4$ s
b) Coriolis time scale $(f^{-1})$ of the order of $10^4$ s
c) Advection time scale $l/V$ ($l=$bedform length, $V=$depth averaged velocity) of the order of $10^3$ s
d) Adjustment time scale of the velocity field $D/u = D/(V\sqrt{C_D})$ of the order of $10^2$ s ($C_D=1/C^2$)
e) Adjustment time scale for sediment concentration $D/w_s$ of the order of $10^2$ s
f) Morphodynamic time scale much larger than $10^4$ s

\[ (1-n) \frac{\partial h}{\partial t} = - \frac{\partial Q}{\partial y} \]

\[ \frac{D}{\sqrt{\frac{(\rho_s - \rho) g d^3}{l}}} \Rightarrow T \approx \frac{lD}{\sqrt{\frac{(\rho_s - \rho) g d^3}{l}}} \]
SPATIAL AND TEMPORAL SCALES FOR THE PERTURBATION PROBLEM

a) The forcing storms have a duration of the order of hours, i.e. 10^4 s
b) Coriolis time scale \((f^{-1})\) of the order of 10^4 s
c) Advection time scale \(l/V\) \((l=\text{bedform length}, V=\text{depth averaged velocity})\) of the order of 10^3 s
d) Adjustment time scale of the velocity field \(D/u_r=D/(V\sqrt{C_D})\) of the order of 10^2 s \((C_D=1/C^2)\)
e) Adjustment time scale for sediment concentration \(D/w_s\) of the order of 10^2 s
f) Morphodynamic time scale much larger than 10^4 s

From b)-a) and b)-f), it follows the validity of a quasi-steady approach (the flow reacts instantaneously to new bed levels).

From b), c) it appears that it is reasonable to neglect Coriolis effects.

A brief consideration of the spatial scales is also useful.

a) The longshore scale of the bottom topography is \(l=O(1000m)\)
b) The cross-shore scale \(L\) over which the undisturbed water depth changes significantly \(L=D/(dD/dx)=O(5000m)\)

It is possible to introduce the approximation \(l << L\)
Let us assume that a basic state exists which is uniform in the longshore direction. This basic state is given by a given cross-shore profile $\tilde{D}(x)$ and a uniform long-shore current $V(x)$.

Let us consider a bottom perturbation superimposed to the basic bottom configuration

$$z = h(x) + h'(x, y, t)$$

$(h' \ll \tilde{D})$

The water depth is

$$D + d' = D - h'$$

It follows that the basic flow is perturbed by the bottom waviness and the velocity can be written in the form

$$\left( u'(x, y, t), V(x) + v'(x, y, t) \right)$$

$(u', v' \ll V)$

'a' denotes the perturbed quantities
THE LINEARIZED EQUATIONS

Continuity equation
\[
\frac{\partial d'}{\partial t} + \frac{\partial D}{\partial x} u' + D \frac{\partial u'}{\partial x} + \frac{\partial D}{\partial y} v' + D \frac{\partial v'}{\partial y} = 0
\]

Momentum equation along the x-direction
\[
\frac{\partial u'}{\partial t} + V \frac{\partial u'}{\partial y} = -g \frac{\partial \zeta'}{\partial x} + \frac{ru'}{D} + f v'^2
\]

Momentum equation along the y-direction
\[
\frac{\partial v'}{\partial t} + u' \frac{\partial v'}{\partial x} + V \frac{\partial v'}{\partial y} = -g \frac{\partial \zeta'}{\partial y} - \frac{rv'}{D} + \frac{rV}{D^2} d' - fu'
\]

Sediment continuity equation
\[
\frac{\partial h'}{\partial t} = - \frac{1}{(1-n)} \left[ \frac{\partial Q_x'}{\partial x} + \frac{\partial Q_y'}{\partial y} \right]
\]

Because of previous assumptions

Then, it is convenient to describe the perturbed flow introducing the potential and stream functions
\[
(u'(x, y, t), v'(x, y, t)) = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \right) + \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)
\]

(What is the stream function? What is the potential function?)

Continuity equation (taking into account that \(d' = -h')\)
\[
\frac{1}{D} \frac{\partial D}{\partial x} u' + \frac{\partial u'}{\partial x} + \frac{\partial D}{\partial y} v' + \frac{\partial v'}{\partial y} = 0
\]

provides
\[
\frac{1}{D} \frac{dD}{dx} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) + \nabla^2 \phi = \frac{V}{D} \frac{\partial h'}{\partial y}
\]
We are missing the constraint forced by the dynamics of the flow over the ridges, described using the depth averaged vorticity

$$\omega = \frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}$$

(What is vorticity?)

The vorticity has only one component which should satisfies the following equation

$$\frac{\partial \omega}{\partial t} + (U, V) \cdot \left( \frac{\partial \omega}{\partial x}, \frac{\partial \omega}{\partial y} \right) + \frac{r \omega}{D} = -\omega \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) + U \frac{\partial}{\partial y} \left( \frac{r}{D} \right) - V \frac{\partial}{\partial x} \left( \frac{r}{D} \right)$$

You should combine momentum equation along the x-direction
derived with respect to y and momentum equation along the y-direction derived with respect to x

$$\frac{\partial^2 V}{\partial x \partial t} + \frac{\partial U}{\partial x} \frac{\partial V}{\partial y} + \frac{U}{D} \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial x \partial y} + \frac{V}{D} \frac{\partial^2 V}{\partial x \partial y} + \frac{\partial^2 U}{\partial x \partial y} - \frac{\partial U}{\partial x} \frac{\partial^2 U}{\partial y^2} - \frac{\partial U}{\partial x} \frac{\partial^2 U}{\partial y \partial t} - U \frac{\partial^2 U}{\partial y^2} - \frac{\partial V}{\partial y} \frac{\partial^2 U}{\partial y \partial x} - V \frac{\partial^2 U}{\partial y \partial x}$$

$$= -g \left( \frac{\partial^2 \zeta}{\partial x \partial y} \right) - \frac{\partial}{\partial y} \left( \frac{r V}{D} \right) + \frac{g}{D} \frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial}{\partial y} \left( \frac{r u}{D} \right) + \frac{f}{\partial y} \frac{\partial V}{\partial y}$$

Neglecting Coriolis terms, we get

$$\frac{\partial \omega}{\partial t} + U \frac{\partial \omega}{\partial x} + V \frac{\partial \omega}{\partial y} = - \omega \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) + \frac{\partial U}{\partial x} \left( \frac{\partial U}{\partial y} + \frac{\partial V}{\partial y} \right)$$

$$- V \frac{\partial}{\partial x} \left( \frac{r}{D} \right) - \frac{r}{D} \frac{\partial V}{\partial y} + U \frac{\partial}{\partial y} \left( \frac{r}{D} \right) + \frac{r}{D} \frac{\partial U}{\partial y}$$

and finally

$$\frac{\partial \omega}{\partial t} + U \frac{\partial \omega}{\partial x} + V \frac{\partial \omega}{\partial y} + \omega \frac{r}{D} = -\omega \left( \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \right) - V \frac{\partial}{\partial x} \left( \frac{r}{D} \right) + U \frac{\partial}{\partial y} \left( \frac{r}{D} \right)$$
By linearizing the previous equation with respect to velocity perturbations and neglecting the terms due to $h'$, one obtains

$$
\left( V \frac{\partial}{\partial y} + \frac{r}{D} \right) \nabla^2 \psi = \frac{\partial V}{\partial x} \nabla^2 \phi + \frac{\partial^2 V}{\partial x^2} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left( \frac{r}{D} \right) \left( \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \right)
$$

where

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial \psi}{\partial y}$$

is the vertical component of the vorticity associated to the perturbed flow.

Because of the order of magnitude analysis of the horizontal scales, it follows that

$$\frac{\partial \phi}{\partial x} \approx \frac{\phi}{l}; \quad \frac{\partial \psi}{\partial x} \approx \frac{\psi}{l}; \quad \frac{\partial V}{\partial x} \approx \frac{V}{L}; \quad \frac{\partial}{\partial x} \left( \frac{r}{D} \right) \approx \frac{r}{DL}$$

Then, by considering vorticity equation

$$
\left( V \frac{\partial}{\partial y} + \frac{r}{D} \right) \nabla^2 \psi = \frac{\partial V}{\partial x} \nabla^2 \phi + \frac{\partial^2 V}{\partial x^2} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) + \frac{\partial}{\partial x} \left( \frac{r}{D} \right) \left( \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \right)
$$

it follows that, depending on the values of the parameters, there are different cases

i) frictionally dominated case

$$\frac{r}{D} \nabla^2 \psi \ldots = \ldots \frac{\partial}{\partial x} \left( \frac{r}{D} \right) \left( \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x} \right)$$

$$\frac{r}{D} \frac{\psi}{l^2} \approx \frac{1}{L} \frac{r \phi}{D \ l} \text{ or } \frac{1}{L} \frac{r \psi}{D \ l} \quad \Rightarrow \quad \frac{\psi}{\phi} = O \left( \frac{l}{L} \right) \ll 1$$

negligible
ii) inertially dominated case \( V \frac{\partial}{\partial y} \nabla^2 \psi \ldots = \frac{\partial V}{\partial x} \nabla^2 \phi + \frac{\partial^2 V}{\partial x^2} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \ldots \)

\[
\frac{\nu \psi}{l^3} \approx \frac{V \phi}{L^2 L} \text{ or } \frac{V \phi}{L^2 l} \text{ or } \frac{V \psi}{L^2 l} \implies \frac{\psi}{\phi} = O \left( \frac{l}{L} \right) \ll 1
\]

CONCLUSION: the perturbed flow is essentially irrotational

Finally, it is necessary to consider the mass conservation for the sediment which reads

\[
\frac{\partial h'}{\partial t} = -\nabla \cdot (q'_x, q'_y)
\]

where \((q'_x, q'_y)\) denote the sediment transport rate components due to the bottom perturbation. The factor due to the sediment porosity has been absorbed in the definition of the sediment transport rate per unit width
It is reasonable to assume that the sediment transport rate is proportional to some power \((n + 1)\) of the local velocity field which is the sum of the oscillatory velocity \(v_w\) induced by the waves and the velocity \(V\) due to the steady currents.

\[ Q \approx |v_w + V|^n (v_w + V) \quad \text{where} \quad v_w >> V \]

Then

\[ Q \approx \left[ |v_w|^n + |v_w|^{n-1} \frac{v_w}{v_w} V + O(V^2) \right] (v_w + V) \]

\[ Q \approx |v_w|^n v_w + |v_w|^n V + |v_w|^{n-1} \frac{v_w^2}{v_w^2} V + O(V^2) \approx |v_w|^n v_w + 2|v_w|^n V \]

We are interested in the sediment transport rate averaged over the wave period. Hence, taking into account that the time average of \(v_w\) vanishes we get

\[ Q \approx \int_0^T 2|v_w|^n V dt = V \int_0^T 2|v_w|^n dt = K(x) V \]
In other words, the previous formula takes into account that, in coastal environments, the sediment is mobilized and put into suspension by the strong action of sea waves and then the sediment is transported by the action of the steady current even though the latter is much weaker than the former. This model is consistent with the empirical formula

\[ (Q_x, Q_y) = K(x, t)(u'(x, y, t), V(x) + v'(x, y, t)) \]

By assumption, \( K \) depends on the local properties of the wave field and is a prescribed function of \( x \) and \( t \).

Then, the sediment continuity equation gives rise to

\[
\frac{\partial h'}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial K}{\partial x} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \right) + K \left( \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \psi}{\partial x \partial y} \right) + K \left( \frac{\partial^2 \phi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x \partial y} \right) = 0
\]

Then, using the continuity equation which provides

\[
K \nabla^2 \phi = K \left[ \frac{V}{D} \frac{\partial h'}{\partial y} - \frac{1}{D} \frac{dD}{dx} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \right]
\]

it follows

\[
\frac{\partial h'}{\partial t} + \frac{\partial}{\partial x} \left( \frac{\partial K}{\partial x} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) \right) + K V \frac{\partial h'}{\partial y} - K \frac{dD}{dx} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) = 0
\]
Finally, the sediment continuity equation gives rise to
\[
\frac{\partial h'}{\partial t} + \frac{KV}{D} \frac{\partial h'}{\partial y} = -D \frac{\partial}{\partial x} \left( K \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) = -D \frac{\partial}{\partial x} \left( \frac{K}{D} \right) u'
\]

propagation of the bottom waviness
growth/decay of the bottom waviness

(see the following slides)

Indeed, if the terms on the r.h.s of the equation are neglected, it is easy to obtain
\[
\frac{\partial h'}{\partial t} + \frac{KV}{D} \frac{\partial h'}{\partial y} = 0
\]
which describes the alongshore propagation of an initial perturbation at a velocity \( \frac{KV}{D} \) without growth or decay. The quantities on the r.h.s. represent effects that produce growth or decay.

The previous equation needs the following b.c. which follows from the sediment continuity equation and the vanishing of \( u' \) at the coastline
\[
\frac{\partial h'}{\partial t} + \frac{KV}{D} \frac{\partial h'}{\partial y} = 0 \quad \text{at} \quad x = 0
\]
Moreover,
\( h' \to 0 \) for \( x \to \infty \) and \( y \to \pm \infty \)
THE PHYSICS OF THE INSTABILITY

Ridge growth requires the convergence of sediment flux at ridge crests and divergence at ridge troughs. The divergence of the sediment flux turns out to be

\[
\frac{\partial h'}{\partial t} = - \frac{KV}{D} \frac{\partial h'}{\partial y} - D \frac{\partial}{\partial x} \left( \frac{K}{D} \right) h'
\]

The first term \( - \frac{KV}{D} \frac{\partial h'}{\partial y} \) vanishes at both the crests and at the troughs and cannot lead to the ridge growth.

This term represents erosion on the upstream side of the sand ridge and deposition on the downstream side, which leads to alongshore propagation at a velocity \( KV/D \) without growth or decay.

\[
\left( \frac{\partial h'}{\partial t} + \frac{KV}{D} \frac{\partial h'}{\partial y} = \frac{\partial h'}{\partial t} + c \frac{\partial h'}{\partial y} \Rightarrow h' = h'(y - ct) \right)
\]
The second term can be written

\[-D \frac{\partial}{\partial x} \left( \frac{K}{D} \right) u' = \frac{K}{D} \frac{dD}{dx} u' - \frac{\partial K}{\partial x} u'\]

It represents a divergence of sediment flux due to a cross-shore velocity in the presence of gradients in either or both of \( D \) and \( K \).

\[
\frac{K}{D} \frac{dD}{dx} u' - \frac{\partial K}{\partial x} u'
\]

If \( K \) is fixed, the first term on the r.h.s. represents a convergence of sediment flux for offshore velocities because the increase in undisturbed depth slows the flow and decreases its carrying capacity of the sediment.
\[
\frac{K}{D} \frac{dD}{dx} u' = \frac{\partial K}{\partial x} u'
\]

If \( D \) is fixed, the second term represents a divergence of flux for flow in the direction of increasing \( K \) simply because of increase in sediment load.

MATHEMATICAL DETAILS

By combining all the linearized equations it is possible to obtain a single equation for \( h' \) (The algebra is long but straightforward)

\[
\nabla^2 \left( \frac{\partial h'}{\partial t} + \frac{KV}{D} \frac{\partial h'}{\partial y} \right) = -Q_0 - Q_1 - Q_2 - Q_3
\]

where

\[
Q_0 = V \left\{ \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \frac{\partial^2 h'}{\partial x \partial y} \right\};
\]

\[
Q_1 = D \left\{ \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \frac{\partial}{\partial x} \left( \frac{V}{D} \right) \frac{\partial h'}{\partial y} + 2 \sqrt{D} \frac{\partial^2 \phi}{\partial x^2} \frac{\partial}{\partial x} \left[ \sqrt{D} \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \right] + D \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \frac{\partial \psi^2}{\partial y} \right\};
\]

\[
Q_2 = \left\{ \frac{\partial}{\partial x} \left[ D^2 \frac{\partial^2}{\partial x^2} \left( \frac{K}{D} \right) \right] + \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \left( \frac{dD}{dx} \right)^2 \right\} \frac{1}{2} \frac{\partial^2 \phi}{D \partial x} + 2 \sqrt{D} \frac{\partial}{\partial x} \left[ \sqrt{D} \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \right] \frac{\partial \psi^2}{\partial x \partial y} \right\};
\]

\[
Q_3 = \left\{ \frac{\partial}{\partial x} \left[ D^2 \frac{\partial^2}{\partial x^2} \left( \frac{K}{D} \right) \right] + \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \left( \frac{dD}{dx} \right)^2 \right\} \frac{1}{2} \frac{\partial \psi}{D \partial y} \right\}.
\]
CONTINUITY EQUATION

\[
\frac{1}{D} \frac{dD}{dx} \left( \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y} \right) + \nabla^2 \phi = \frac{V}{D} \frac{\partial h'}{\partial y} \]

\[O\left( \frac{\phi}{Ll} \right) \quad O\left( \frac{\psi}{Ll} \right) \quad O\left( \frac{\phi}{l^2} \right) \quad O\left( \frac{Vh'}{Dl} \right)\]

The second term is negligible with respect to the first, the first term is negligible with respect to the third. Hence

\[\phi = O\left( \frac{Vh'l}{D} \right)\]

\[
O(Q_0) = O\left( V \frac{\partial}{\partial x} \left( \frac{K}{D} \frac{\partial^2 h'}{\partial x \partial y} \right) \right) = VK \frac{h'}{DL \ l^2} \]

\[O(Q_1)\]

1st term - \[O\left( D \frac{\partial}{\partial x} \left( \frac{K}{D} \frac{\partial \psi}{\partial x} \frac{\partial h'}{\partial y} \right) \right) = \frac{KV \ h'}{DL^2 \ l}\]

2nd term - \[O\left( 2\sqrt{D} \frac{\partial^2 \phi}{\partial x^2} \frac{\partial}{\partial x} \left( \sqrt{D} \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \right) \right) = \frac{K \ \phi}{L^2 \ l^2} = \frac{K \ h'Vl}{L^2 \ l^2 \ D} = \frac{KV \ h'}{DL^2 \ l}\]

3rd term - \[O\left( D \frac{\partial}{\partial x} \left( \frac{K}{D} \frac{\partial \psi}{\partial y} \phi \right) \right) = \frac{K \ \psi}{L \ l^2} = \frac{K \ lh'Vl}{L \ LDi^3} = \frac{KV \ h'}{DL^2 \ l}\]
By using \( \frac{\psi}{\phi} = O\left( \frac{l}{L} \right) \) and the estimate \( \phi = O\left( \frac{k'Vl}{D} \right) \), which follows from continuity equation (**), it can be verified that

\[
Q_i = O\left( \left( \frac{l}{L} \right)^i Q_0 \right) \text{ for } i = 1,2,3 \quad (**).
\]

Therefore, we should consider

\[
\nabla^2 \left( \frac{\partial h'}{\partial t} + \frac{K}{D} \nabla h' \right) + V \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \frac{\partial^2 h'}{\partial x \partial y} = 0
\]

In storm dominated systems, it is convenient to average over many storms. Let us introduce a temporal filter

\[
\bar{a}(x,y,t) = \frac{1}{T} \int_0^T a(x,y,t') dt'
\]

It follows that \( a = \bar{a} + \bar{\bar{a}} \).

Let us assume that \( \bar{a} \bar{b} \) can be neglected with respect to \( \bar{a} \bar{b} \).
It follows that
\[
\nabla^2 \left( \frac{\partial \tilde{h}'}{\partial t} + \frac{KV}{D} \frac{\partial \tilde{h}'}{\partial y} \right) + V \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \frac{\partial^2 \tilde{h}'}{\partial x \partial y} = 0
\]
with the boundary condition at the coast
\[
\frac{\partial \tilde{h}'}{\partial t} + \frac{KV}{D} \frac{\partial \tilde{h}'}{\partial y} = 0
\]
Let us consider
\[
\tilde{F}' = \Re \{ A(x) \exp[iky + \Omega t] \} = \Re \{ A(x) \exp[i(k - c)t] \}
\]
where \( c = i \frac{\Omega}{k} \) is a complex quantity.
\[
\Re(\Omega) > 0 \text{ growing perturbation} \quad \text{instability}
\]
\[
\Re(\Omega) < 0 \text{ decaying perturbation} \quad \text{stability}
\]
Since \( \nabla^2 = \frac{d^2}{dx^2} - k^2 \) we have
\[
\left( \frac{d^2}{dx^2} - k^2 \right) \left[ \left( -ikc + \frac{KV}{D} - ik \right) A \right] + V \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \left[ \frac{dA}{dx} \right] = 0
\]
with the boundary condition at the coast
\[
\left( \frac{KV}{D} - c \right) A = 0 \quad \text{at} \quad x = 0
\]
\[
A \to 0 \quad \text{for} \quad x \to \infty
\]
\[
\left( \frac{d^2}{dx^2} - k^2 \right) \left[ \frac{KV}{D} - c \right] A + V \frac{\partial}{\partial x} \left( \frac{K}{D} \right) dA = 0
\]

with the boundary condition at the coast

\[
\left( \frac{KV}{D} - c \right) A = 0 \quad \text{at} \quad x = 0
\]

\[
A \to 0 \quad \text{for} \quad x \to \infty
\]

The problem just formulated is an eigenvalue problem.

It admits a non-trivial solution only when an eigenrelation \( c = c(k) \) is satisfied.

The condition for instability is \( c_i > 0 \).

A necessary condition for instability is that \( \frac{\partial}{\partial x} \left( \frac{K}{D} \right) \) must be nonzero somewhere.

Indeed, if \( \frac{K}{D} \) does not depend on \( x \), \( c = \frac{KV}{D} \). Hence, it follows \( c_i = 0 \)

The problem can be solved by numerical means

If \( c \) and \( A \) are an eigenvalue and an eigenfunction, then \( c^* \) and \( A^* \) \((c^* = \text{conjg}(c), \ A^* = \text{conjg}(A))\) are also a solution. Therefore for any stable solution there is an unstable solution.
Since we have no detail information, let us assume that $K = K(t)$. Then the problem becomes
\[
\left( \frac{d^2}{dx^2} - k^2 \right) \left[ (U - c)A \right] = \frac{U}{D} \frac{dD}{dx} \frac{dA}{dx}
\]
with the boundary conditions
\[
A = 0 \quad \text{at} \quad x = 0
\]
\[
A \to 0 \quad \text{for} \quad x \to \infty
\]
where $U = \frac{KV}{D}$.

Indeed,
\[
\frac{U}{D} \frac{dD}{dx} \frac{dA}{dx} = \frac{KV}{D^2} \frac{dD}{dx} \frac{dA}{dx} = -V \frac{d}{dx} \left( \frac{K}{D} \right) \frac{dA}{dx} = \frac{KV}{D^2} \frac{dD}{dx} \frac{dA}{dx} \quad \text{c.v.d}
\]

It is convenient to solve a dimensionless problem by introducing a horizontal length scale, a vertical length scale and a typical velocity scale.
Example of the results
\[ D = 1 + x \quad \text{for} \quad 0 \leq x \leq 1 \]
\[ D = 2 \quad \text{for} \quad x \geq 1 \]
The horizontal length scale is the width of the shelf which is some 5 km in the real system.

\[ U = -1 \]
The model does not indicate the existence of a most unstable mode of finite wavelength, in fact the growth rate \( kc \) increases monotonically as \( k \) increases.
By fixing $k = 5$ for the most unstable mode, as suggested by field observations (the longshore wavelength of the periodic bottom forms is about 6 km), the bottom topography computed by the model is shown in the figure.

![Figure 3: Topography corresponding to the most unstable mode for $U = -1$ and $k = 1$. (left) Contours of depth computed from $h = -[c(x,y) + h_0(x,y)]$, (right) Quantity $x = \int_0^y h(x,y) dy$, which is the projection onto the x-axis of the bottom elevation that crosses along ridge crests. (bottom) Orientation of the top place of lines of constant phase (i.e., ridge crests and troughs), which are defined by $y = -(1/2)\log(1/\alpha)$.

The effects of non constant $U$

![Figure 6: (top) Propagation velocity $c_p$ and (bottom) growth rate $\omega_0$ as functions of wavenumber $k$ corresponding to the most unstable mode for $U = -1 + x/5$ (circles) and $U = -1 - x/5$ (pluses).]
The effects of non constant $U$

$U = -1 + x/5$

$U = -1 - x/5$

Figure 5. Depth contours corresponding to the most unstable mode for $k = 5$ with (left) $U = -1 - x/5$ and (right) $U = -1 + x/5$.

The effects of a complex velocity profile $U$

$U = -1 + x/4(1 - x)$

$U = -1 - x/4(1 - x)$

Figure 8. (top) Propagation velocity $c$, and (bottom) growth rate $kx$, as functions of wavenumber $k$, corresponding to the most unstable mode for $U = -1 - x/4(1 - x)$ (circles) and $U = -1 + x/4(1 - x)$ (pluses).
The effects of a complex velocity profile $U$

$U = -1 - x/4(1-x)$  \hspace{2cm} $U = -1 + x/4(1-x)$

**A REFINED APPROACH**

*Calvete et al. 2001, J. Fluid Mech. 441*

**AIM**

Investigation of the growth of shore-oblique sand ridges due to storms and tides

**ASSUMPTION**

2DH Model ($x$-$y$ plane) which follows by averaging the three-dimensional shallow water equations over the vertical

The long-shore current is due to storms

Coriolis effects are taken into account

The sediment transport is evaluated by means of an improved empirical relationship
Let us consider a sea bed like that shown in the figure, such that the unperturbed water depth \( h \) depends on the cross-shore coordinate \( x \) only. Moreover the basic current \( V \) forced by tides and storms is supposed to be along-shore directed.

![Figure 1. Sketch of the geometry and the coordinate system.](image)

### THE PROBLEM EQUATIONS

**Continuity equation**

\[
\frac{\partial D}{\partial t} + \frac{\partial (DU)}{\partial x} + \frac{\partial (DV)}{\partial y} = 0
\]

**Momentum equations**

\[
\begin{align*}
\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} - fV &= -g \frac{\partial \zeta}{\partial x} + \frac{\tau_{sx} - \tau_{hx}}{\rho D} \\
\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} + fU &= -g \frac{\partial \zeta}{\partial y} + \frac{\tau_{sy} - \tau_{hy}}{\rho D}
\end{align*}
\]

**Sediment continuity equation**

\[
\frac{\partial h}{\partial t} = -\frac{1}{(1-p)} \left[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} \right]
\]
Bed shear stress and sediment transport during storms

\[
\left( \tau_{bx}, \tau_{by} \right) = pr(U, V) \quad r = c_d \left\langle \left| \bar{u}_w \right| \right\rangle; \\
u_w = u_{w0} \left( \frac{H_0}{H} \right)^{\frac{m}{2}} \quad \text{with} \quad m = 1.6
\]

\[
\left( Q_x, Q_y \right) = \left( Q_{bx}, Q_{by} \right) + \left( Q_{sx}, Q_{sy} \right)
\]

\[
\left( Q_{sx}, Q_{sy} \right) = q_b \left( \left\langle \left| \bar{u}_w \right| \right\rangle (U, V) - \lambda_b \left\langle \left| \bar{u}_w \right| \right\rangle \left( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right) \right)
\]

\[
\left( Q_{sx}, Q_{sy} \right) = C(U, V) - \lambda_b \left\langle \left| \bar{u}_w \right| \right\rangle \left( \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y} \right)
\]

\[
q_b \approx 4 \times 10^{-5} \text{ m}^2 \text{ s}^{-1}, \quad \lambda_b \approx 0.4,
\]

\[
\lambda_b \approx 0.3 \text{ m} \text{ s}^{-1}
\]

The value of \( C \) (depth integrated relative volume concentration) is computed by integrating

\[
\frac{\partial C}{\partial t} + \frac{\partial (UC)}{\partial x} + \frac{\partial (VC)}{\partial y} = w_s (c_a - c_b)
\]

\( \hat{u} \) is the reference concentration near the bed

\( \hat{u} \) is a critical velocity for erosion of the sediment

\( c_b = \frac{C}{\delta} \) is the actual concentration near the bed

\( \delta \) is the ratio of the characteristic thickness of the suspended-load sediment layer and the actual water depth

(diffusive effects have been neglected)
THE BASIC FLOW

\[ V = V_0(x) \]

THE LINEAR STABILITY ANALYSIS

the perturbed velocity field \[ (u', V + v') \]
the perturbed water depth \[ -\bar{D}(x) + h' \]

\[ (u', v') = \text{Re} \left\{ \hat{u}(x, t), \hat{v}(x, t) e^{i\omega t} \right\} \]

\[ h' = \text{Re} \left\{ \hat{h}(x) e^{i\omega t} \right\} \]

The most unstable mode has a wavelength of about 4.1 Km which agrees with field observations. The migration speed turns out to be about -2.1m/year in the down-stream direction and it is in good agreement with field data.
Figure 3. Contour plots of the three bottom modes with the largest growth rate. Shoals and pools are indicated by continuous and dashed lines, respectively. In each plot the y-axis represents the shoreface and the x > 0 axis the inner shelf. The direction of the basic current is shown by an arrow. Note the upcurrent rotation of the ridges.

Figure 4. Greyscale plot of the first bottom mode in figure 3 with the associated perturbation of the current indicated by arrows. Shoals and pools are indicated by light and dark shades, respectively. Note the offshore current deflection over the crest.
PERTURBED SEDIMENT TRANSPORT

Figure 5. Grey-scale plot of the first bottom mode in figure 3 with the associated perturbations of the sediment flux indicated by arrows. Shores and pools are indicated by light and dark shades, respectively. The contour lines refer to the perturbation in the concentration (solid lines: positive, dashed lines: negative values).

Figure 6. Relative contribution of suspended-load and bed-load fluxes to (a) the growth rate and (b) migration speed of the first mode of figure 2.